"Semiglobal L_2 Performance Bounds for Disturbance Attenuation in Nonlinear Systems"

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Outline

- Preliminaries: Input-to-State Stability (ISS) and disturbance attenuation

- Changing supply functions for ISS systems

- Problem of disturgance attenuation

- Refinement to zero dynamics

- Linear system example

- $\gamma^\star$ for nonlinear systems

- Rescaling a control Lyapunov function

Preliminaries: ISS

Defn (Sontag): The system

$$\dot{x} = f(x, u) \tag{J.1}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is said to be *input-to-state stable* if there exist $\beta \in \mathcal{KL}_\infty$ and $\alpha \in \mathcal{K}_\infty$ such that for all $T > 0$, the solution $\varrho(t)$ of (J.1) satisfies

$$|\varrho(T)| \le \max\left\{\beta(|\varrho(0)|, T), \gamma(\|u_{[0,T]}\|_\infty)\right\}. \tag{J.2}$$

Sufficient Condition for ISS (Sontag): The system in (J.1) is ISS if $\exists$ "supply functions" $\alpha, \gamma \in \mathcal{K}_\infty$ and "storage function" V that is p. d. and proper (*i.e.*, radially unbounded) such that

$$\frac{\partial V}{\partial x} f(x, u) \le \gamma(|u|) - \alpha(|x|) \tag{J.3}$$

"ISS" as in (J.3) $\Rightarrow$ L_*-type bound on input-to-state gain under zero initial conditions

Example: suppose $\gamma(r) = g^2 r^2$ and $\alpha(r) = r^2$. Integrate

$$V(x(T)) - V(x(0)) \le g^2\|u\|_2^2 - \|x\|_2^2$$
$$\|x\|_2^2 \le g^2\|u\|_2^2$$

Defn (Sontag): (γ, α) are a supply function pair for a system if $\exists$ a storage function V such that (J.3) is satisfied.

Que: Given a system, characterize its possible supply pairs.

- *Consider*: $\dot{x} = x + u$ with $x, u \in \mathbb{R}$
  Suppose (J.3) is satisfied for some (V, γ, α). Necessarily
  $$\left|\frac{\partial V}{\partial x}\right| \geq \frac{\alpha(|x|)}{|f(x)|}$$
  But $\partial V/\partial x(0) = 0$ hence must have $\alpha(|x|) = o(f(x))$ as $x \to 0$.

- *Essentially*: for supply pair (γ, α), α can be modified for large arguments and γ can be modified for small arguments.

Thm 1 (Sontag): Assume that (γ, α) is a supply pair. Suppose that $\tilde{\gamma}$ is a $\mathcal{K}_\infty$ with $\gamma(r) = O(\tilde{\gamma}(r))$ as $r \to \infty$. Then $\exists\ \tilde{\alpha} \in \mathcal{K}_\infty$ such that $(\tilde{\gamma}, \tilde{\alpha})$ is also a supply pair.

Thm 2 (Sontag): Assume that (γ, α) is a supply pair. Suppose that $\tilde{\alpha}$ is a $\mathcal{K}_\infty$ with $\alpha(r) = O(\tilde{\alpha}(r))$ as $r \to 0^+$. Then $\exists\ \tilde{\gamma} \in \mathcal{K}_\infty$ such that $(\tilde{\gamma}, \tilde{\alpha})$ is also a supply pair.

Corr (Sontag): Given two systems which satisfy (J.3), $\exists\ \tilde{\gamma}_1, \tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$ so that $(\tilde{\alpha}_2/2, \tilde{\alpha}_1)$ is a supply pair for the first system and $(\tilde{\gamma}_2, \tilde{\alpha}_2)$ is a supply pair for the second system.

Example: Cascaded systems

$$\dot{z} = f(z, x)$$
$$\dot{x} = g(x, u)$$

both individually ISS.

Apply Corrolary

$$\frac{\partial V_1}{\partial z} f(z, x) \leq \frac{1}{2}\tilde{\alpha}_2(|x|) - \tilde{\alpha}_1(|z|)$$
$$\frac{\partial V_2}{\partial x} g(x, u) \leq \tilde{\gamma}_2(|u|) - \tilde{\alpha}_2(|x|)$$

Using $V(z, x) := V_1(z) + V_2(x)$ for cascaded system

$$\frac{\partial V}{\partial (z, x)} \begin{pmatrix} f(z, x) \\ g(x, u) \end{pmatrix} \leq \tilde{\gamma}_2(|u|) \underbrace{-\frac{1}{2}\tilde{\alpha}_2(|x|) - \tilde{\alpha}_1(|z|)}_{\geq \text{ some } -\alpha(|(z,x)|)}$$

Problem of Disturbance Attenuation (DAP): Consider the system

$$\dot{x} = f(x) + g(x)u + p(x)w$$
$$y = h(x) \tag{I.1}$$

with control input $u \in I\!R$, disturbance $w \in I\!R$, and output $y \in I\!R$. Assume smooth vector fields, $f(0) = 0$, and $h(0) = 0$.

The *problem of disturbance attenuation* with no cost on control is: find $u(x)$ such that the origin is g.a.s. and

$$\int_0^t |y(\tau)|^2 d\tau \le \gamma^2 \int_o^t |w(\tau)|^2 d\tau \tag{I.2}$$

Almost disturbance decoupling refers to systems where the DAP can be solved for any $\gamma > 0$, *i.e.*, when

$$\gamma^\star := \inf\{\gamma > 0 : \text{ DAP is solved}\} = 0.$$

Sufficient Condition: The above problem is solved if $\exists\ u(x)$ and a smooth, real-valued $V(x)$ that is p.d. and proper which satisfy the *dissipation inequality*

$$\frac{\partial V}{\partial x}[f(x) + g(x)u(x) + p(x)w] + h(x)^2 \le \gamma^2 w^2 - \alpha(|x|) \tag{I.3}$$

for all x and w, where $\alpha \in \mathcal{K}_\infty$.

Can complete the square in w to obtain HJI

$$L_{f(x)+g(x)u(x)}V(x) + \frac{1}{4\gamma^2}\left(L_pV(x)\right)^2 + h(x)^2 \le -\underset{\uparrow}{\underline{\alpha(|x|)}}. \tag{J.4}$$

$w = \frac{1}{2\gamma^2}L_pV(x)$ is the worst case policy.

Note: The $\alpha(|x|)$ provides g.a.s. by providing state cost and making V a Lyapunov function for $w = 0$.

Suppose system can be put into *strict feedback form*:

$$\begin{aligned}
\dot{z} &= f_0(z, \xi_1) + q_0(z, \xi_1) w \\
\dot{\xi}_1 &= \xi_2 + p_1(z, \xi_1) w \\
\dot{\xi}_2 &= \xi_3 + p_2(z, \xi_1, \xi_2) w \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r + p_{r-1}(z, \xi_1, \dots, \xi_{r-1}) w \\
\dot{\xi}_r &= u + p_r(z, \xi_1, \dots, \xi_r) w \\
y &= \xi_1
\end{aligned} \tag{I.4}$$

Via backstepping-type argument, the DAP can be solved if associated DAP can be solved for *zero dynamics*:

$$\begin{aligned}
\dot{z} &= f_0(z, u) + q_0(z, u) w \\
v &= u
\end{aligned} \tag{I.7}$$

Specifically, if $\exists\ v^\star(z)$ with $v^\star(0) = 0$, $V^\star(z)$ p.d. and proper, and $\alpha^\star \in \mathcal{K}_\infty$ such that

$$\frac{\partial V^\star}{\partial z} f_0(z, v^\star(z)) + \frac{1}{4\gamma^2} \left[\frac{\partial V^\star}{\partial z} q_0(z, v^\star(z)) \right]^2 + [v^\star(z)]^2 \le -\alpha^\star(|z|) \tag{I.5}$$

In this manner, solution of DAP for (4) with no cost on control can be reduced to solution of DAP for (7) *with cost on control*.

Que: What are the fundamental limitations on $\gamma^\star$ for a given problem?

- Consider the linear equivalent of (4)

$$
\begin{aligned}
\dot{z} &= Fz + G\xi_1 + Qw \\
\dot{\xi}_1 &= \xi_2 + P_1 w \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r + P_{r-1} w \\
\dot{\xi}_r &= u + P_r w \\
y &= \xi_1
\end{aligned} \tag{I.8}
$$

- The DAP can be solved iff Sylvester's Inequality

$$FZ + ZF^T + \frac{1}{\gamma^2}QQ^T - GG^T < 0 \tag{I.10}$$

holds for some $Z > 0$. In this case, $\gamma^\star$ can be calculated explicitly.

- wlog, split the z subsystem of (8) into stable and unstable portions

$$
\begin{aligned}
\dot{z}_s &= F_s z_s + G_s \xi_1 + Q_s w \\
\dot{z}_u &= F_u z_u + G_u \xi_1 + Q_u w
\end{aligned} \tag{I.11}
$$

and (I.10) can be solved iff $\exists\ Z_u > 0$ satisfying

$$F_u Z_u + Z_u F_u^T + \frac{1}{\gamma^2}Q_u Q_u^T - G_u G_u^T < 0 \tag{I.12}$$

Hence for a linear system it is the *unstable portion of the forced zero dynamics* which determine the performance limit $\gamma^\star$ for the DAP with no cost on control.

Considering again (I.7), *suppose* that it can be split as

$$\begin{aligned}\dot{z}_1 &= f_1(z_1, z_2, \xi_1) + q_1(z_1, z_2, \xi_1)w \\ \dot{z}_2 &= f_2(z_2, \xi_1) + q_2(z_2, \xi_1)w \end{aligned} \tag{I.17}$$

where $z_1 = 0$ is g.a.s. for $z_2 = 0$ and $\xi_1 = y = 0$, and the z_2 subsystem is "stabilizable" in the sense that for some $v(z_2)$ the subsystem $\dot{z}_2 = f_2(z_2, v(z_2))$ is g.a.s. at $z_2 = 0$.

Thm 3.1 (Isidori, et al.): Suppose that:

1. $\exists\ V_1(z_1)$, $\alpha_1 \in \mathcal{K}_\infty$, and $\gamma_0 > 0$ s.t.

$$\frac{\partial V_1}{\partial z_1}[f_1(z_1, z_2, \xi_1) + q_1(z_1, z_2, \xi_1)w] \leq -\alpha_1(|z_1|) + \gamma_0^2|w|^2 + \gamma_0^2|z_2|^2 + \gamma_0^2|\xi_1|^2; \tag{I.18}$$

2. $\exists\ V_2(z_2)$ and $v_2(z_2)$ s.t.

$$\frac{\partial V_2}{\partial z_2}[f_2(z_2, v_2(z_2)) + q_2(z_2, v_2(z_2))w] + |v_2(z_2)|^2 \leq -\alpha_2(|z_2|) + \bar{\gamma}^2|w|^2 \tag{I.19}$$

   for some $\bar{\gamma} > 0$ and some $\alpha_2 \in \mathcal{K}_\infty$ satisfying 3) below;

3. for some $r_1 \in [0, \infty)$ and some $a \in \mathbb{R}$

$$\frac{r^2}{\alpha_2(r)} \leq a$$

   for all $r \in [r_1, \infty)$.

Then for every $\hat{\gamma} > \bar{\gamma}$, $\exists\ v^*(z)$ with $v^*(0) = 0$ and V^* p.d. and proper s.t.

$$\frac{\partial V^*}{\partial z} f_0(z, v^*(z)) + \frac{1}{4\hat{\gamma}^2}\left[\frac{\partial V^*}{\partial z} q_0(z, v^*(z))\right] + [v^*(z)]^2 \leq -\alpha^*(|z|)$$

for some $\alpha \in \mathcal{K}_\infty$ and for all z.

Interpretation of Thm 3.1

1. The z_1 subsystem considered as $(w, z_2, \xi_1) \mapsto z_1$ is ISS and g.a.s. at $z_1 = 0$.

2. The z_2 subsystem viewed as $w \mapsto v_2(z_2)$ has disturbance attenuation $\bar{\gamma}$ and is g.a.s. at $z_2 = 0$ for a particular control law v_2.

3. α_2 is at least of order r^2 for r large enough, hence

$$\int_0^T \alpha_2(|z_2(t)|)dt < \infty \quad \Rightarrow \quad \int_0^T |z_2(t)|^2 dt < \infty$$

Then the entire system viewed as $w \mapsto v^*(z)$ has disturbance attenuation $\hat{\gamma}$ for all $\hat{\gamma} > \bar{\gamma}$.

Note: The value of γ_0 has no effect on the result. That is, the performance limit is dictated by the unstable portion of the zero dynamics.

Que: For what values of $\bar{\gamma} > 0$ is (I.19) solvable?

- Make the further assumption that z_2 portion of (I.17) is

$$\dot{z} = f(z) + g(z)\xi + q(z)w$$

- (I.19) $\iff$

$$L_fV(x) + \frac{1}{4}\left[\frac{1}{\gamma^2}|L_pV(x)|^2 - |L_gV(x)|^2\right] + \alpha(|x|) \leq 0 \quad \text{(I.26)}$$

- $V(x)$ satisfying (I.26) must necessarily satisfy

$$\left(|L_gV(x)| \leq \frac{1}{\gamma^2}|L_pV(x)|^2 \quad \Rightarrow \quad L_fV(x) \leq -\alpha(|x|)\right) \quad \text{(I.27)}$$

for all $x \neq 0$. In particular,

$$|L_gV(x)| = 0 \Rightarrow L_fV(x) < 0.$$

Hence V is a control Lyapunov function for L_gV control.

Que: Suppose $V(x)$ satisfies (I.27). Is it possible to find some $U(x)$ which satisfies (I.26)?

Consider rescalings of $V(x)$ according to

$$U(x) := \int_0^{V(x)} q(s)ds$$

for q smooth, positive, and nondecreasing. Then

$$\frac{\partial U}{\partial x} = q(V(x))\frac{\partial V}{\partial x}$$

- Assuming that V satisfies (I.27), define

$$M_1 := \{x : L_f V(x) + \alpha(|x|) \geq 0\} \tag{1}$$

$$M_2 := \{x : \frac{1}{\gamma^2}|L_p V(x)| > |L_g V(x)|\} \tag{2}$$

- on M_1, necessarily $\frac{1}{\gamma^2}|L_p V(x)| - |L_g V(x)| < 0$. Want $U(x)$ such that

$$\frac{1}{\gamma^2}|L_p U(x)|^2 - |L_g U(x)|^2$$

dominates the term

$$L_f U(x) + \tilde{\alpha}(|x|)$$

- on M_2, necessarily $L_f V(x) + \alpha(|x|) < 0$. Want reverse domination.

- Problem: Thm 4.1 requires radial symmetry of M_i. If not true, then condition may not be solvable.