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RECONSTRUCTION OF DISCRETE DATA TRANSMISSIONS: A WORST CASE OPTIMAL APPROACH

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Abstract

In this paper we present a deterministic worst-case framework for accurate reconstruction of discrete (source) data as an alternative to the traditional probabilistic approaches in the communications area. This framework can be explored based on robust control ideas and formulations. Some of the particular problems touched upon are: (i) necessary and sufficient conditions for causal (no delay) and noncausal (with delay) reconstruction under deterministic magnitude bounded noise for SISO and MIMO channels, (ii) reconstruction based on linear estimation, (iii) performance optimization under channel fading and (iv) combined precoding and estimation optimization under power constraints. The ℓ^1 control theory is proposed as a natural key player in this approach. **Keywords:** communications, equalization, ℓ^1 optimal

1 Introduction

In the communications area, the topic of data transmission and reconstruction is based almost entirely on a stochastic formulation of the various problems involved (e.g., [1]). In this paper we present a deterministic worst-case framework for accurate reconstruction of discrete (source) data as an alternative that can be explored based on robust control ideas and formulations. Some of the particular problems touched upon are: (i) necessary and sufficient conditions for causal (no delay) and noncausal (with delay) reconstruction under deterministic magnitude bounded noise for single-input single-output (SISO) and multi-input multi-output (MIMO) channels, (ii) reconstruction based on linear estimation, (iii) performance optimization under channel uncertainty and (iv) combined precoding and estimation optimization under power constraints. All these topics are relevant to standard themes in communications such as receiver design and equalization, multiple antenna systems and code division multiple access (e.g., ch. 10, 11, 14, 15 in [1]) which are traditionally dealt from a stochastic point of view. The proposed framework in the paper mainly addresses the question of when perfect reconstruction of a sequence of source symbols (e.g., +1 or -1) is possible if the magnitude of the noise is allowed to be anything as long as it is bounded by an apriori known bound. In other words it is a worst case, deterministic approach that provides conditions that, if violated, an error will occur. Certain constructions of optimal algorithms are provided some of which tie to ℓ^1 optimal control problems. The notation is as follows: $\|x\| := \sup_k |x(k)|$ is the ℓ^∞ norm of a sequence $\{x(k)\}_{k=0}^{\infty}$, $\|T\|_1 := \sum_{k=0}^{\infty} |t(k)|$ is the ℓ^1 norm of the linear time invariant (LTI) system T having a pulse response $\{t(k)\}_{k=0}^{\infty}$, $\hat{T}(\lambda) := \sum_{k=0}^{\infty} t(k)\lambda^k$ is the λ transform of T, $\|S\|_{\infty-\infty} := \sup_x \frac{\|Tx\|}{\|x\|}$ is the ℓ^∞-induced norm of a possibly time varying and/or nonlinear system S (note $\|S\|_{\infty-\infty} = \|S\|_1$ if S is LTI.)

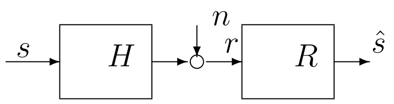

Figure 1: Basic set-up

2 Problem Definition

The setup of the main problem we are concerned with is depicted in Figure 1 where s is a binary signal to be transmitted with for all $k = 0, 1, \ldots$ $s(k) \in \{-1, 1\}$ for all $k = 0, 1, \ldots$; n is noise with $|n(k)| \leq b$ where b is known; $H = \{h_0, h_1, \ldots\}$ represents the channel dynamics which are assumed known apriori. We want to accurately reconstruct s via R, i.e., $\hat{s} = s$ causally in time. Thus we are after the necessary and sufficient conditions for this to happen.

3 Problem Solution

Definition 3.1 *The sequences s_1, s_2 are indistinguishable at t if $s_1(m) \neq s_2(m)$ for some $0 \leq m \leq t$ and there exists n_1, n_2 with $\|n_1\| \leq b$, $\|n_2\| \leq b$ such that $r_1(k) = r_2(k)$ all $k = 0, 1, \ldots, t$ where $r_1 = Hs_1 + n_1$, $r_2 = Hs_2 + n_2$.*

Clearly, the problem has a solution if and only if there are no s_1, s_2 which are indistinguishable at some $t = 0, 1, \ldots$. The following can be proved [6]

Proposition 3.1 *There exists s_1, s_2 indistinguishable at some t if and only if*

$$|h_0| \leq b.$$

3.1 How to construct optimal R

By the previous analysis it follows that we should have $|h_0| > b$. In this case the construction is as follows: From $r(0) = h_0 s(0) + n(0)$ set $\hat{s}(0) = \text{sgn}[r(0)]$ where sgn $[x] = 1$ if $x \geq 0$ and sgn $[x] = -1$ otherwise. Then as $|h(0)| > b$ $\hat{s}(0) = s(0)$. Moving to $r(1) = h_1 s(0) + h_0 s(1) + n(1)$ obtain $\tilde{r}(1) := r(1) - h_1 s(0) = h_0 s(1) + n(1)$ but $\tilde{r}(1)$ is known as $s(0)$ is accurately estimated from previous step. Since $|h_0| > b$ then $s(1)$ can be estimated accurately as $\hat{s}(1) =$ sgn $[\tilde{r}(1)]$ and the procedure can be similarly extended to any $k = 0, 1, 2, \ldots$. A block diagram interpretation of optimal R is given below in Figure 2 where $\tilde{H} := H - h_0$. Note that this is a structure of a Decision Feedback Equalizer (DFE) (e.g., ch. 10 in [1].) Generalizations of the above setup can be

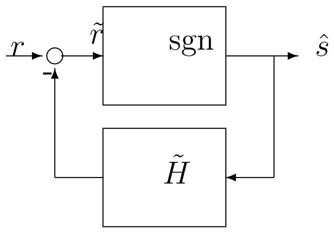

Figure 2: DEF structure

considered where $s(k)$ belongs to a set of equally spaced numbers in $[-1, 1]$. For instance, if $s(k) \in \{j/N,\ j = -N, -N+1, \ldots, 0, \ldots, N-1, N\}$, i.e. there are $2N + 1$ numbers spaced by $2N$ intervals of size $1/N$, the condition for accurate estimation becomes $|h_0| > 2Nb$. The algorithm to obtain accurate estimates is an obvious extension. That is, from $r(0) = h_0 s(0) + n(0)$, $s(0)$ can be accurately estimated as $\hat{s}(0) = \frac{[Nr(0)/h_0]}{N}$ where $[\bullet]$ stands for the closest integer part. Then the procedure can be repeated for $\tilde{r}(k) = r(k) - \sum_{j=0}^{k-1} h_{k-j} s(j) = h_0 s(k) + n(k)$ to obtain accurate estimates for all k as $\hat{s}(k) = \frac{[N\tilde{r}(k)/h_0]}{N}$. Also, the case of $s(k) \in \{-N, -N+1, \ldots, N-1, N\}$ is a scaled version of the instance above.

3.2 Non-causal reconstruction

The case of non-causal reconstruction (smoothing) can also be considered in the same framework. In this case we are allowed to estimate $s(k)$ by incorporating K future receptions $r(k+1), \ldots, r(k+K)$. The necessary and sufficient condition for accurate reconstruction is that there are no sequences s_1 and s_2 such that if they are indistinguishable at any time t they remain so for the next K time steps. Following the same line of argument as in Proposition 3.1 we obtain that the necessary and sufficient condition is

$$\min_{v(0)\neq 0, v(i)\in\{-1,0,1\}} \max\{|a(0)|,|a(1)|,\ldots,|a(K)|\} > b$$

where $\begin{pmatrix} a(0) \\ a(1) \\ \vdots \\ a(K) \end{pmatrix} :=$

$$\begin{pmatrix} h_0 & & & & \\ h_1 & h_0 & & & \\ \vdots & h_1 & h_0 & & \\ h_K & \ldots & \ddots & \ldots & \ddots & h_0 \end{pmatrix} \begin{pmatrix} v(0) \\ v(1) \\ \vdots \\ v(K) \end{pmatrix}$$

3.3 MIMO channels

Generalizations are also possible in the case of MIMO channels. In the case of m transmitters and p receivers the (equivalent) channel dynamics can be represented by a $p \times m$ transfer H with pulse response $H = \{H_0, H_1, \ldots\}$ where each H_i is a $p \times m$ matrix. The motivation for problems of this sort comes from multiple antenna systems designed to combat fading channels and/or in the detection of multiuser code division multiple access (CDMA) signals The following can be obtained along the lines of Proposition 3.1: Let

$$H_0 = \begin{pmatrix} h_{00} & h_{01} & \ldots & h_{0m} \end{pmatrix}$$

where h_{0i}s are $p \times 1$ column vectors and let $v(i) \in \{-1,0,1\}$, $i = 1,\ldots,m$. Then for perfect reconstruction it is necessary and sufficient that

$$\min_{v(i) \text{ not all equal } 0} \max\{|a(0)|,|a(1)|,\ldots,|a(p)|\} > b$$

where $a = \sum_{i=1}^{m} v(i)h_{0i}$, $a := \begin{pmatrix} a(0) \\ a(2) \\ \vdots \\ a(p) \end{pmatrix}$. Similarly to the SISO case one can look at noncausal reconstruction for MIMO channels.

3.4 Some remarks

In the case of noncausal reconstruction and/or MIMO channels the test for perfect reconstructability requires solving a mixed integer linear program. This instance is polynomial in the number of variables and hence easy to solve. The construction however of the optimal R is more complex. The algorithms presented appear to have a strong dependence on the accurate reconstruction in previous steps. Hence, it could be that there may be sensitivity to wrong previous estimates due to noise that is above the allowable bound for perfect reconstruction. This is not dealt in this paper. For some recent work on this subject we refer to [2]. Finally we mention that the case where the noise n enters thru a "filter" (sometimes called whitening filter) F, i.e., $r = Hs + Fn$, can be dealt analogously and the results are similar in flavor.

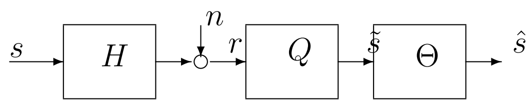

Figure 3: Linear Equalization

4 Reconstruction Based on Linear Estimation

In this case we restrict the structure of R to be of the form in Figure 3 where Q is the linear filter $Q = \{q_0, q_1, \ldots\}$ and Θ is a thresholding operator that produces -1 or 1 depending on which one has the closest distance to $\tilde{s}$. In this particular case $(\Theta\tilde{s})(k) =$ sgn $[\tilde{s}(k)]$. This is the structure of what is called a linear equalizer (e.g., ch.10 in [1]). Given this specific form for R we would like to assure that

$$|s(k) - \tilde{s}(k)| < 1$$

for all k so that perfect reconstruction is possible. As

$$s - \tilde{s} \;=\; [(I\ 0) - Q(H\ I)] \begin{pmatrix} s \\ n \end{pmatrix}$$

it is necessary and sufficient to have a Q such that

$$J \;:=\; \sup_{s,n} \left\| [(I\ 0) - Q(H\ I)] \begin{pmatrix} s \\ n \end{pmatrix} \right\| < 1$$

where $s(k) \in \{-1,1\}$, $|n(k)| \leq b$. Define the relevant ℓ_1-optimization

$$\mu \;:=\; \inf_Q \|(I - 0) - Q(H \quad bI)\|_1$$

then since the "worst" s is such that $s(k) \in \{-1, 1\}$ all k it follows that the problem has a solution if and only if $\mu < 1$. Finding μ and its associated optimizer Q is a standard ℓ_1 problem [3]. The solution procedure involves solving a set of finite dimensional linear programs that allow for approximation of the optimal performance to any apriori specified degree. In general, this type of problems do not admit closed from solutions. However, for the case of first order FIR channel dynamics we provide such a solution in the following subsection. We should also mention that the same holds for MINO channels: to check whether $|s_i(k) - \tilde{s}_i(k)| < 1$ for all of the i source data s_i transmitted, leads to a MIMO ℓ^1 problem [3]. Finally, we should further point out that restricting Q to be FIR of some desired order, poses no serious difficulty to the solution of the ℓ^1 problem. Moreover, in MIMO channels one can solve without significant difficulty decentralized reconstruction problems by imposing structural constraints on Q. For example, the ith receiver-decision maker can be restricted to obtain information only from its neighboring sites in a multiple antenna system.

4.1 ℓ_1-optimization for first order FIR channels

We consider first order FIR channel as $H = \{h_0, h_1, 0, \ldots\}$. This could be representative of channel dynamics in wireless-communications. Let

$$\Phi \ := \ (I \ \ 0) - Q(H \ \ bI)$$

then we have the following [6]

Proposition 4.1 *Let H be a first order channel $\hat{H}(\lambda) = h_0 + h_1\lambda$ and let $\omega := \frac{|h_1|}{|h_0|}$, $\alpha := \frac{b}{|h_0|}$, then the optimal estimator Q° and its associated optimal cost μ is*

$$\begin{array}{lll} Q^\circ = H^{-1}, & \mu = \frac{\alpha}{1-\omega} < 1 & \textit{whenever} \ \ \omega + \alpha < 1 \\ Q^\circ = 0 & \mu = 1 & \textit{whenever} \ \ \omega + \alpha \geq 1 \end{array}$$

Hence from this scheme we get that the noise level b should be $b < |h_0| - |h_1|$ for perfect reconstruction as opposed to $b < |h_0|$ obtained from the non-restricted nonlinear R. The conservatism is expected due to the restricted structure considered. Also another cause of conservatism can be possibly attributed to the fact that requiring $|s(k) - \tilde{s}(k)| < 1$ is not an exact necessary condition for the scheme to reproduce perfectly s. The exact condition is

$$\begin{array}{ll} s(k) - \tilde{s}(k) < 1 & \text{whenever} \ \ s(k) = 1 \\ \tilde{s}(k) - s(k) < 1 & \text{whenever} \ \ s(k) = -1 \end{array}$$

i.e, no undershoot larger than 1 when $s(k)$ is positive and no overshoot larger than 1 when $s(k)$ is negative. However, the contribution to conservatism of this last factor has not been assessed. In the case where $s(k)$ belongs to a set of equally spaced numbers in $[-1, 1]$, for instance, if $s(k) \in \{j/N, \ j = -N, -N+1, \ldots, 0, \ldots, N-1, N\}$ the same approach (assuming that the thresholding now changes to produce the closest j/N to $\tilde{s}(k)$) leads to the condition $b < \frac{|h_0| - |h_1|}{2N}$ provided $\omega + \alpha < 1$. This is again more conservative than the condition $b < \frac{|h_0|}{2N}$ in the unrestricted case, but the degree of conservatism is decreasing as N grows.

4.2 Noncausal reconstruction

Results for smoothing can also be obtained. For K steps noncausal reconstruction the relevant problem is

$$\mu \ = \ \inf_Q \left\| \Lambda^K (I \ \ 0) - Q(H \ \ bI) \right\|_1$$

where Λ^K is the K-step delay operator.[1] This is again a standard ℓ^1 problem. In the case of first order FIR channel and 1-step noncausal reconstruction i.e., $K = 1$ we obtain the following [6]

Proposition 4.2 *Let H be a first order channel $\hat{H}(\lambda) = h_0 + h_1\lambda$ with $\omega = \frac{|h_1|}{|h_0|}$, $\alpha = \frac{b}{|h_0|}$. Then the optimal Q° and its associated optimal cost μ is $Q^\circ = \Lambda H^{-1}, \mu = \frac{\alpha}{1-\omega} < 1$ whenever $\omega + \alpha < 1$; $Q^\circ = \frac{1}{h_1}, \mu = \frac{1+\alpha}{\omega} < 1$ whenever $\omega + \alpha \geq 1$ and $0 > \omega - \alpha > 1$; $Q^\circ = 0, \mu = 1$ otherwise.*

[1] If Q° is optimal for the above problem then the optimal non-causal estimator is $\Lambda^{-K} Q^\circ$.

4.3 Robustness to channel uncertainty

We now consider the case of uncertain channel dynamics. A model of such a channel can be described as $H + W\Delta$. The uncertainty here is given in terms of an additive weighted block ΔW where Δ is assumed to be an unknown perturbation, possibly time varying and even nonlinear, that has a bounded ℓ^∞ to ℓ^∞ norm $\|\Delta\|_{\infty-\infty} < 1$. The weight W is a known stable LTI system that may reflect magnitude normalizations and partial information on the magnitude of the uncertainty over the frequencies, i.e., it shapes the uncertainty block. For example, if there are uncertain higher order dynamics in a nominally first order FIR channel, then a representation of the uncertainty can be given as ΔW with $\hat{W}(\lambda) = \epsilon\lambda^2$ where ϵ is a scaling and $\hat{\Delta}(\lambda) = \sum_{i=0}^{\infty} \delta_i \lambda^i$ with $\sum_{i=0}^{\infty} |\delta_i| < 1$; if the uncertainty is time varying then δ_i can also be time varying $\delta_i(k)$ with $\sup_k \sum_{i=0}^{k} |\delta_i(k)| < 1$ where the operator Δ is defined by $(\Delta x)(k) = \sum_{i=0}^{k} \delta_i(k) x(k-i)$. We note that this uncertainty formulation is different in nature than what is typically assumed in the stochastic framework (e.g., ch. 14 in [1].) However, we believe that it captures a number of relevant fading phenomena due to time variations and can be used to design reliable reconstruction algorithms. We assume that when no uncertainty is present $\Delta = 0$, $J = \left\| \begin{pmatrix} s \\ n \end{pmatrix} \to s - \tilde{s} \right\|_1 < 1$ and hence perfect reconstruction is possible with Q. What we want is robust performance (RP) in the presence of all $\|\Delta\|_{\infty-\infty} < 1$, i.e.,

$$\left\| \begin{pmatrix} s \\ n \end{pmatrix} \to s - \tilde{s} \right\|_{\infty-\infty} < 1 \quad \text{all} \quad \|\Delta\|_{\infty-\infty} < 1.$$

A more general situation is depicted in Figure 4 where $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ where $H_{22} = H$ and H_{ij} can be general (stable) dynamics that connect the nominal channel with the sources of dynamical uncertainty lumped in Δ. For example, consider a channel $H = N_H D_H^{-1}$ where $N_H = N_0 + \Delta_N W_N$ and $D_H = D_0 + \Delta_D W_D$ where D_0, N_0 are the nominal "numerator" and "denominator" respectively,and, Δ_N and Δ_D are normalized perturbations with W_N, W_D shaping

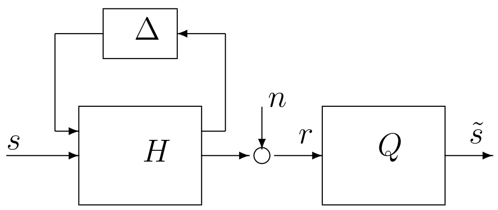

Figure 4: A more general fading model

known weights. Then the system can be brought into the form of Figure 4. Applying the criterion for R.P. (see [6] for details) we have that it is necessary and sufficient that $\|H_{11}\|_1 < 1$ and

$$\left\| (I\ 0\ 0) - Q(H \quad bI \quad \frac{\|H_{12}\|_1}{1 - \|H_{11}\|_1} H_{21}) \right\|_1 < 1.$$

Hence the optimization for R.P. amounts to solve an appropriately modified ℓ_1- problem as above.

4.4 Optimal Precoding-Reconstruction

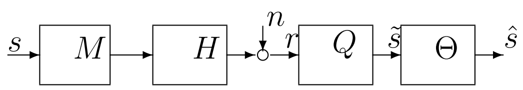

Figure 5: Precoder and estimator structure

In this section we are looking at performance improvements by introducing a precoding block M before data enters the channel H as in Figure 5. The error dynamics therefore become

$$s - \tilde{s} = [(I\ 0) - Q(HM\ I)] \begin{pmatrix} s \\ n \end{pmatrix}$$

Based on our previous analysis the higher the "gain" of the channel the more tolerance to noise and unmodelled dynamics. Here the role of channel is played by HM. So in principle, one can choose a "large" M and make perfect reconstruction possible for any given level of noise and fading. In reality however there is a limit in the available power to be transmitted. Hence we need to consider a "power" constraint on the size of M by requiring that

$$\|M\|_{\mathcal{H}^\infty} := \sup_{0 \le \theta < 2\pi} |\hat{M}(\exp^{j\theta})| \le \gamma$$

where $\gamma > 0$ is a specified power level. The problem of interest then is a cost minimization as follows

$$\inf_{Q,M,\|M\|_{\mathcal{H}^\infty}\leq\gamma} \|(I\ 0) - Q(HM\ I)\|_1 .$$

The above problem is convex in each of the variables Q and M but fails in general to be jointly convex. A "Q-M" procedure can be applied to lead to a (possibly local) minimum. That is, fix M first and design for optimal Q—this is a standard ℓ^1 problem—then, fix Q at what was found and minimize over M—this is a mixed $\ell^1/\mathcal{H}^\infty$ optimization that can be solved via convex programming [4]—once M is found, optimize over Q and so on. The procedure generates a sequence of improved costs (decreasing) and hence it converges to a minimum. Alternative relaxation methods can also be considered to obtain a global minimum [5]

5 Concluding Remarks

We presented a purely deterministic formulation of various communications-relevant problems and provided some solutions based on developments in ℓ^1 optimal and robust control. The approach leads to exact magnitude bounds on the noise level for which perfect reconstruction of the transmitted symbols is possible. It also allows for analysis and synthesis for perfect reconstruction when uncertainty, possibly time varying, is present in the channel. Another interesting feature is the possibility of a combined precoding-estimation optimization in the presence of a power constraint that leads to mixed $\ell^1/\mathcal{H}^\infty$ optimization type of problems. Obviously, a lot remains to be done to validate our approach. At first, a full scale comparison with the current probabilistic methods in communications is in order. This however, along with other developments, will be the subject of future work and publications on the topic. Finally, we should mention that the optimal (linear) solution given for first order FIR channels is, by itself, a contribution to ℓ^1 model-matching theory as it is a closed form solution to a so-called two-block problem for which, to the best of our knowledge, no closed form solutions are available.

6 Aknowledgements

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