Non-Concurrent Error Detection and Correction in Discrete-Time LTI Dynamic Systems

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Motivation for Fault Tolerance

Fault tolerance describes ability to:

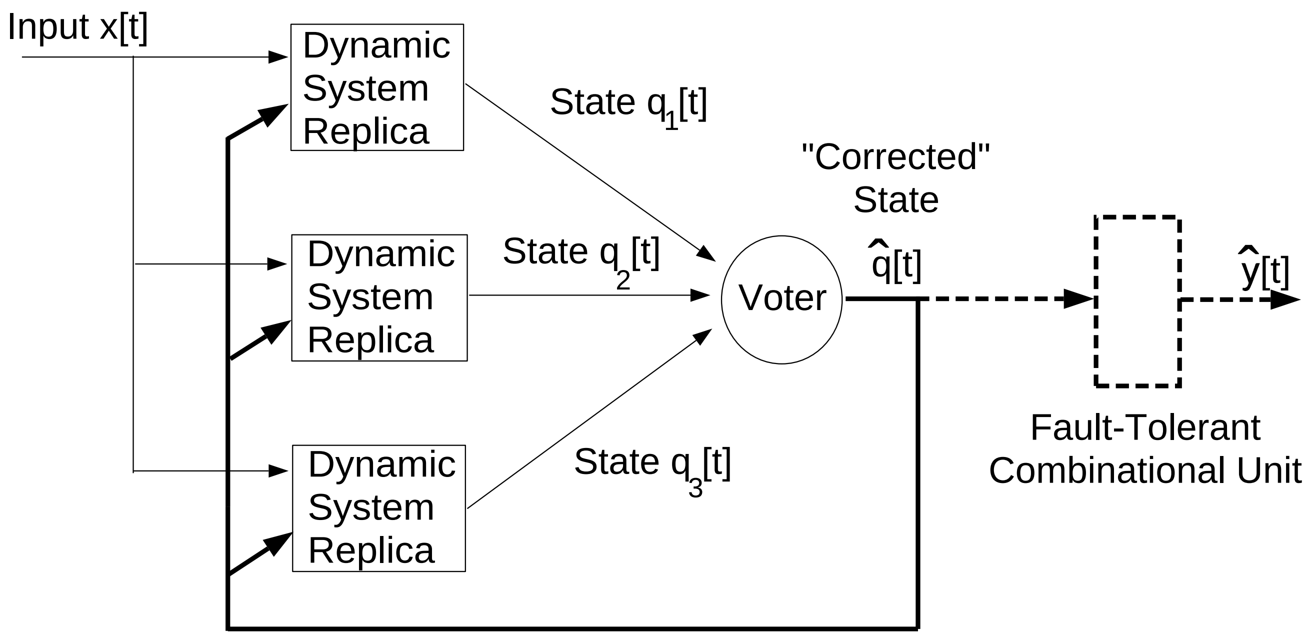
- Withstand internal faults
- Produce desirable overall "behavior" (e.g., correct or acceptable output)

Necessary or desirable in:

- Life-threatening circumstances (military, transportation, medical)
- Systems in inaccessible environments (space missions)
- Reliable systems from unreliable components
  (faster, less expensive, less power)

Previous work includes: Communication systems, computational circuits, special-purpose systems, networked systems

Universal Approach: Modular Redundancy



Examples: Computing systems, digital filters, encoders/decoders

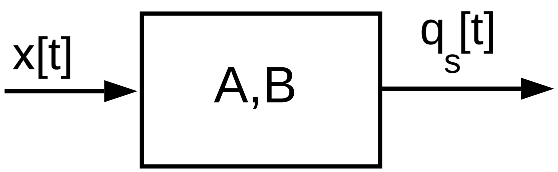
Problems with modular redundancy:

- Replication
- Checking overhead, voter reliability

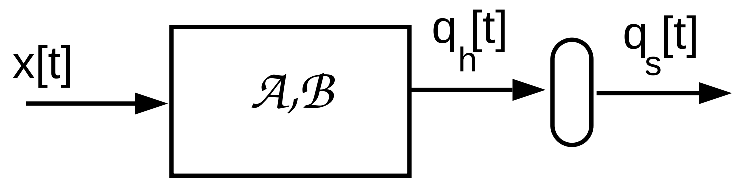
} Will be addressing both of these issues!

Fault Detection and Correction in LTI Dynamic Systems

Original System



Redundant Implementation



$$\mathbf{q}_s[t+1] = \mathbf{A}\mathbf{q}_s[t] + \mathbf{B}\mathbf{x}[t]$$

$$\mathbf{q}_h[t+1] = \mathcal{A}\mathbf{q}_h[t] + \mathcal{B}\mathbf{x}[t]$$

- **Concurrent simulation:** $\mathbf{q}_s[t] = \mathbf{L}\mathbf{q}_h[t]$
- **Encoding constraints:** $\mathbf{q}_h[t] = \mathbf{G}\mathbf{q}_s[t]$

} **Linear (not necessary)**

- **Fault detection:** If $\mathbf{q}_h[t]$ is *not* in the column space of $\mathbf{G}$, *or*

$$\mathbf{P}\mathbf{q}_h[t] \neq \mathbf{0}\ , \qquad \mathbf{P}\mathbf{G} = \mathbf{0}$$

Characterization of Redundant Implementations

Original System

$$\mathbf{q}_s[t+1] = \mathbf{A}\mathbf{q}_s[t] + \mathbf{B}\mathbf{x}[t]$$

$\mathbf{q}_s$ is d-dimensional

$$\xrightarrow{\mathbf{q}_h[t]=\mathbf{G}\mathbf{q}_s[t]} \qquad \xleftarrow{\mathbf{q}_s[t]=\mathbf{L}\mathbf{q}_h[t]}$$

Redundant Implementation

$$\mathbf{q}_h[t+1] = \mathcal{A}\mathbf{q}_h[t] + \mathcal{B}\mathbf{x}[t]$$

$\mathbf{q}_h$ is η-dimensional ($\eta = d + s$)

Standard redundant implementations (Hadjicostis & Verghese 1999):
$(\mathcal{A}, \mathcal{B})$ is a redundant implementation for $(\mathbf{A}, \mathbf{B})$ iff $(\mathcal{A}, \mathcal{B})$ is similar to the following standard form:

$$\mathbf{q}_\sigma[t+1] = \underbrace{\left[\begin{array}{c|c} \mathbf{A} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right]}_{\mathcal{A}_\sigma} \mathbf{q}_\sigma[t] + \underbrace{\left[\begin{array}{c} \mathbf{B} \\ \hline \mathbf{0} \end{array}\right]}_{\mathcal{B}_\sigma} \mathbf{x}[t]$$

for some matrices $\mathbf{A}_{12}$, $\mathbf{A}_{22}$

Specifically: Invertible $\mathcal{T}$ such that $\mathcal{A}_\sigma = \mathcal{T}^{-1}\mathcal{A}\mathcal{T}$, $\mathcal{B}_\sigma = \mathcal{T}^{-1}\mathcal{B}$

Concurrent Fault Detection and Identification

Fault model: Single fault corrupts ith state variable

$$\mathbf{q}_f[t] = \underbrace{\mathbf{q}_h[t]}_{\text{fault-free}} + v\,\mathbf{e}_i$$

Justification: Constrained interconnections of adders, multipliers and delays
$\Rightarrow$ A single fault corrupts a single state variable

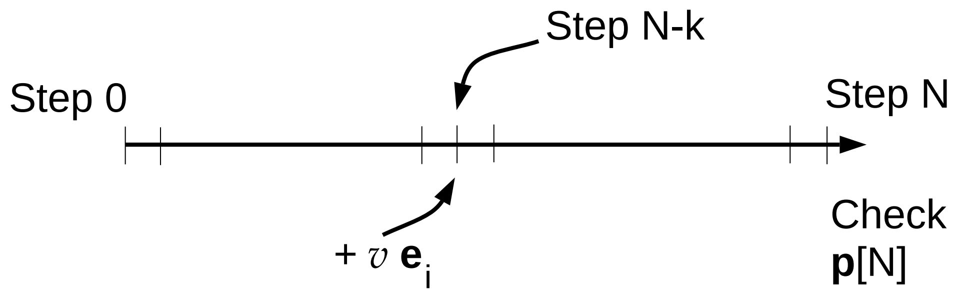
(Class of signal flow graphs, Hadjicostis & Verghese 1999)

Concurrent error detection (Abraham, Chatterjee, Hadjicostis, ...):
At end of *each* time step, perform the *parity check*

$$\mathbf{p}[t] \equiv \mathbf{P}\,\mathbf{q}_f[t] = \mathbf{P}\,v\,\mathbf{e}_i \stackrel{?}{=} \mathbf{0}$$

Error detection/correction capabilities: Constraints on matrix $\mathbf{P}$

Non-Concurrent Checking (1)



Goal: Design redundant implementation so that knowledge of $\mathbf{p}[N]$ allows detection and identification of error(s) in interval $[0, N]$

Motivation: Relax checking requirements (e.g., periodic checking)

Need: For each fault (j), identify

- Value (v_j)
- State variable ($\mathbf{e}_{i_j}$)
- Step ($N - k_j$)

} For error correction, one may have to reset past states/outputs

Non-Concurrent Checking (2)

Error model: Initially, fault j at step $N - k_j$ causes

$$\mathbf{q}_f[N - k_j] = \mathbf{q}_h[N - k_j] + v_j \; \mathbf{e}_{i_j}$$

Error propagation: At step N,

$$\mathbf{q}_f[N] = \mathbf{q}_h[N] + \mathcal{A}^{k_j} \; v_j \; \mathbf{e}_{i_j}$$

Parity check: At step N,

$$\mathbf{p}[N] = \mathbf{P} \; \mathbf{q}_f[N] = v_j \; \mathbf{P} \; \mathcal{A}^{k_j} \; \mathbf{e}_{i_j}$$

Multiple errors result in:

$$\mathbf{p}[N] = \sum_{j=1}^{D} v_j \; \mathbf{P} \; \mathcal{A}^{k_j} \; \mathbf{e}_{i_j}$$

Task: Construct redundant implementation so that D errors can be identified

Syndrome Generation

Observation: Syndrome $\mathbf{p}[N]$ is a linear combination of columns of

$$\mathbf{S} = \begin{bmatrix} \mathbf{P} & \mathbf{P}\mathcal{A} & \mathbf{P}\mathcal{A}^2 & \cdots & \mathbf{P}\mathcal{A}^N \end{bmatrix}$$

Lemma 1: Detection of D errors *if and only if* all sets of D columns of $\mathbf{S}$ are linearly independent

$\Rightarrow$ Need at least D additional variables ($s \geq D$)

Lemma 2: Identification of D errors *if and only if* all sets of $2D$ columns of $\mathbf{S}$ are linearly independent

$\Rightarrow$ Need at least $2D$ additional variables ($s \geq 2D$)

Theorem: The syndrome matrix $\mathbf{S}$ can be expressed as

$$\mathbf{S} = \begin{bmatrix} \mathbf{P} & \mathbf{A}_{22}\mathbf{P} & \mathbf{A}_{22}^2\mathbf{P} & \cdots & \mathbf{A}_{22}^N\mathbf{P} \end{bmatrix}$$

Construction for Non-Concurrent Identification of D Errors

Fact: Any $2D$ columns of $\mathbf{V}$ are linearly independent if $x_i \neq x_j$

$$\mathbf{V}(x_1, x_2, ..., x_r) = \begin{bmatrix} 1 & 1 & ... & 1 \\ x_1 & x_2 & ... & x_r \\ x_1^2 & x_2^2 & ... & x_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{2D-1} & x_2^{2D-1} & ... & x_r^{2D-1} \end{bmatrix}$$

Construction of redundant implementation:

- $2D$ additional state variables ($s = 2D$)
- $\mathbf{\Lambda} = \text{diag}(1, x, x^2, x^3, ..., x^{2D-1})$, $\quad \mathbf{M} = \mathbf{V}(x_{d+1}, x_{d+2}, ..., x_\eta)$
- **Step 1:** In standard coordinates, set

$$\mathbf{A}_{22} = \mathbf{M}^{-1}\mathbf{\Lambda}\mathbf{M} \text{ and } \mathbf{C} = -\mathbf{M}^{-1}\mathbf{V}(x_1, x_2, ..., x_d)$$

- **Step 2:** Perform similarity transformation with $\mathcal{T} = \begin{bmatrix} \mathbf{I}_d & \mathbf{0} \\ \mathbf{C} & \mathbf{I}_{2D} \end{bmatrix}$

Theorem and Proof

Theorem: Resulting redundant implementation allows non-concurrent identification of D errors (detection of $2D$ errors)

Why? The syndrome matrix $\mathbf{S}$ can be written as

$$\mathbf{S} \;=\; \mathbf{M}^{-1}\underbrace{\mathbf{V}(x_1, ..., x_\eta, x_1 x, ..., x_\eta x, x_1 x^2, ..., x_\eta x^2, ..., x_1 x^N, ..., x_\eta x^N)}_{\mathbf{Q}}$$

$\mathbf{Q}$ is a *large* Vandermonde matrix ($2D \times (\eta N)$-dimensional)

Requirement: Parameters $x, x_1, x_2, ..., x_\eta$ need to be chosen so that

$x_i x^k$ are unique $\quad\Rightarrow\quad$ any $2D$ columns of $\mathbf{Q}$ are linearly independent

Summary of Design so Far

- **Jointly choose:**
  - (i) Encoding constraints ($\mathbf{P}$ or $\mathbf{G}$)
  - (ii) Redundant dynamics ($\mathbf{A}_{22}$)
- **Perform one (non-concurrent) parity check:**

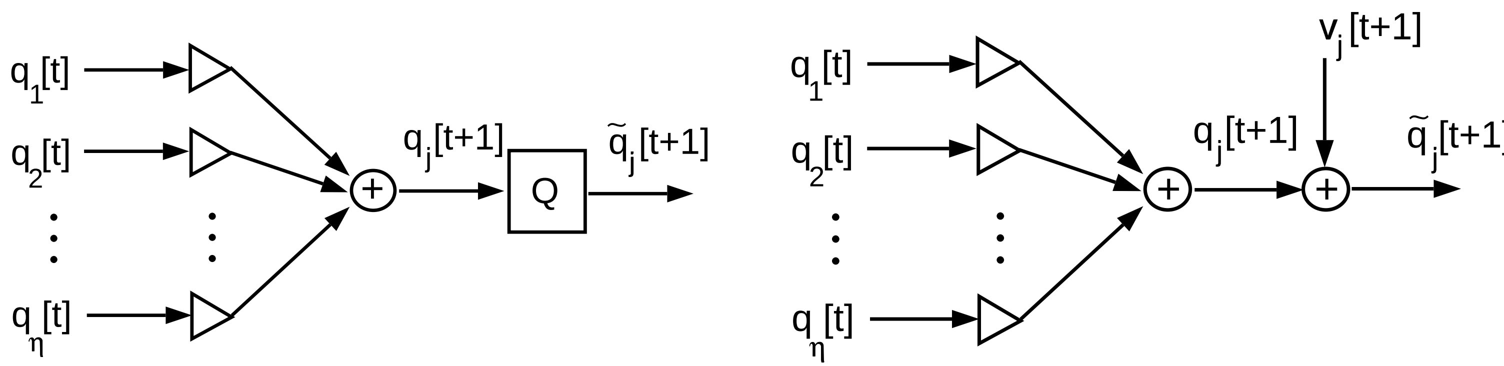
$$\mathbf{p}[N] = \mathbf{P}\ \mathbf{q}_f[N]$$

  - Detect $2D$ faults (on any variable, at any step in $[0, N]$)
  - Identify D faults (on any variable, at any step in $[0, N]$)
- **Advantages:**
  - Only $2D$ additional state variables (optimal)
  - Efficient identification (Peterson-Gorenstein-Ziegler decoding)
- **Limitation:** Finite precision arithmetic (quantization noise)

Effects of Finite Precision Arithmetic

Model as additive quantization noise: Ignoring input

$$\mathbf{q}_j[t+1] = \sum_{i=1}^{\eta} \mathbf{A}(j,i)\mathbf{q}_i[t]$$



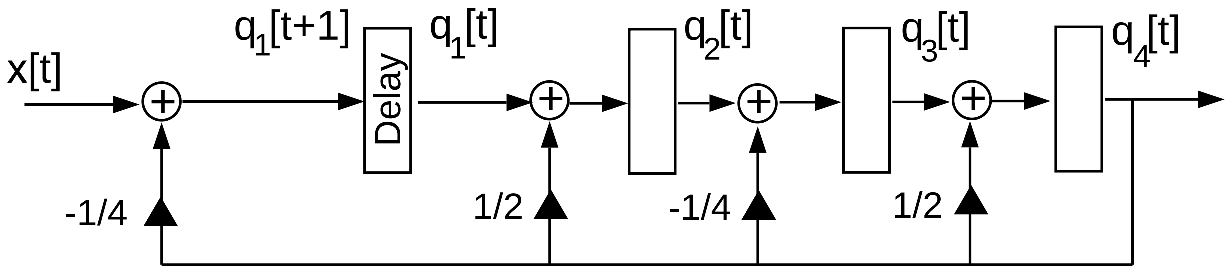
- Fixed point arithmetic with b bits in fractional part

- $\mathbf{v}_j[t]$ uncorrelated, uniformly distributed in $[-\frac{Q}{2}, \frac{Q}{2}]$, where $Q = 2^{-b}$

- Parity check at Nth time step has mean $\mathbf{p}[N]$ and covariance matrix

$$\mathbf{R} = \frac{Q^2}{12} \sum_{t=0}^{N-1} \mathbf{P}\mathcal{A}^{N-1-t}(\mathcal{A}^T)^{N-1-t}\mathbf{P}^T = \frac{Q^2}{12} \sum_{t=0}^{N-1} \mathbf{A}_{22}^{N-1-t}\mathbf{P}\mathbf{P}^T(\mathbf{A}_{22}^T)^{N-1-t}$$

Example (1)



State evolution:

$$\mathbf{q}_s[t+1] = \mathbf{A}\mathbf{q}_s[t] + \mathbf{b}x[t]$$
$$= \begin{bmatrix} 0 & 0 & 0 & -1/4 \\ 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 1/2 \end{bmatrix} \mathbf{q}_s[t] + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x[t]$$

Goal: $N = 15$, Detect and identify two errors $\Rightarrow$ Use 4 additional variables

Example (2)

Choose:

$$\mathbf{M} = \mathbf{V}(-2,-1,1,2) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & x^3 \end{bmatrix}, \quad x = \tfrac{4}{5}$$

Set:

$$\mathbf{C} \;=\; -\mathbf{M}^{-1}\mathbf{V}(-4,-3,3,4) = \begin{bmatrix} -7.5 & -3.333 & 0.667 & 2.5 \\ 10 & 3.333 & -1.667 & -6 \\ -6 & -1.667 & 3.333 & 10 \\ 2.5 & 0.667 & -3.333 & -7.5 \end{bmatrix}$$

$$\mathbf{A}_{22} \;=\; \mathbf{M}^{-1}\mathbf{\Lambda}\mathbf{M} = \begin{bmatrix} 0.468 & -0.084 & -0.036 & 0.052 \\ 0.624 & 1.008 & 0.112 & -0.144 \\ -0.144 & 0.112 & 1.008 & 0.624 \\ 0.052 & -0.036 & -0.084 & 0.468 \end{bmatrix}$$

Example (3)

Redundant implementation after transformation:

$$\mathbf{q}_h[t{+}1] = \left[\begin{array}{rrrr|rrrr} 0 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 & 0 & 0 \\ \hline 0.671 & 2.412 & 2.341 & 0.368 & 0.468 & -0.084 & -0.036 & 0.052 \\ -1.035 & -2.664 & -5.589 & -1.129 & 0.624 & 1.008 & 0.112 & -0.144 \\ 0.621 & 3.744 & 9.003 & 0.465 & -0.144 & 0.112 & 1.008 & 0.624 \\ -0.257 & -3.492 & -5.755 & 0.796 & 0.052 & -0.036 & -0.084 & 0.468 \end{array}\right] \mathbf{q}_h[t] + \left[\begin{array}{r} 1 \\ 0 \\ 0 \\ 0 \\ \hline -7.5 \\ 10 \\ 6 \\ 2.5 \end{array}\right] x[t]$$

Syndrome matrix:

$$\mathbf{S} = \mathbf{M}^{-1} \underbrace{\left[\begin{array}{ccccc} \mathbf{S}_0 & \mathbf{S}_1 & \mathbf{S}_2 & \cdots & \mathbf{S}_{15} \end{array}\right]}_{\mathbf{Q}}$$

where $\mathbf{S}_k = \mathbf{V}(-4x^k, -3x^k, 3x^k, 4x^k, -2x^k, -x^k, x^k, 2x^k)\ , \qquad x = \frac{4}{5}$

Example (4)

Assumption: $b = 16$ in fixed-point arithmetic

For $N = 15$, covariance matrix of non-concurrent parity check is

$$\mathbf{R} = 10^{-8} \times \begin{bmatrix} 0.20 & -0.26 & 0.17 & -0.10 \\ -0.26 & 0.44 & -0.24 & 0.17 \\ 0.17 & -0.24 & 0.44 & -0.26 \\ -0.10 & 0.17 & -0.26 & 0.20 \end{bmatrix}$$

For $N = 1000$, covariance matrix of non-concurrent parity check becomes

$$\mathbf{R} = 10^{-6} \times \begin{bmatrix} 0.05 & -0.17 & -0.17 & 0.04 \\ -0.17 & 0.69 & 0.69 & -0.17 \\ -0.17 & 0.69 & 0.69 & -0.17 \\ 0.04 & -0.17 & -0.17 & 0.05 \end{bmatrix}$$

- *Cannot* arbitrarily increase N
- Tradeoffs between period N, number of bits b and minimal allowed error $|v_j|$

Summary and Future Research Directions

- Reflection of design into hardware (fault model)
- Standard form of redundant implementations
- Encoding of LTI systems:
  - (i) Introduces redundancy systematically
  - (ii) Generalizes/combines modular redundancy and checksum schemes
  - (iii) Dynamic error correction & reconfiguration

Future work:

- Quantization analysis for the Peterson-Gorenstein-Ziegler decoding algorithm
- Other pairs of coding schemes and redundant dynamics
- General hardware descriptions
- Analog systems