University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: COMMUNICATIONS I

Fall 2001

Problem Set 6

Probability Review, Random Variables, Decision Rules

Issued: Thursday, October 18th. **Due:** Thursday, October 25th (beginning of lecture).

Reading from Haykin: Appendix 1.

Problem 6.1 (Optional)

A Gaussian random variable X of zero mean and variance σ_X^2 is transformed to random variable Y via the transformation

$$Y = X^2 \ .$$

Show that the probability density function of Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}\sigma_X} exp\left(-\frac{y}{2\sigma_X^2}\right) \ , & y \geq 0 \\ 0 \ , & y < 0 \ . \end{cases}$$

Problem 6.2

The joint pdf of two random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} A(1-|x-y|), & 0 < x < 1, \ 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find A.

(b) **(optional)** Find the marginal pdf of X and Y.

(c) Find $\Pr(X + Y < 1 \mid X > \frac{1}{2})$.

Problem 6.3 (Optional)

(a) Show that if two random variables X and Y are related by

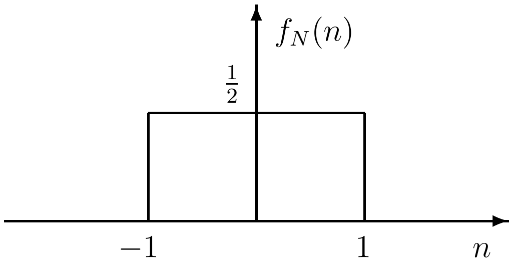
$$y = \alpha x + \beta$$

for arbitrary constants α and β, then their correlation coefficient ρ_{yx} equals 1 if $\alpha > 0$ and equals -1 if $\alpha < 0$.

(b) Given that $X = \cos\Phi$ and $Y = \sin\Phi$, where Φ is a uniform random variable in the interval $[0, 2\pi]$, show that X and Y are uncorrelated but not independent.

Problem 6.4

Consider the following communication scenario: under hypothesis H_0 no signal is transmitted and the receiver observes a (constant) signal $R = N$, whereas under hypothesis H_1 a constant signal is transmitted and the receiver observes $R = 1 + N$. The random variable N models noise in the channel and is assumed to have the pdf shown below.



The prior probabilities for hypotheses H_0 and H_1 are given by

$$\Pr(H_0) = 1/4 \,, \qquad \Pr(H_1) = 3/4 \,.$$

As we discussed in class, given a received signal $R = r$, we can use the MAP rule at the receiving end in order to minimize the probability of error. Furthermore, we know that the MAP rule reduces to the likelihood ratio test

$$\frac{f_{R|H_1}(r|H_1)}{f_{R|H_0}(r|H_0)} \underset{`H_0\text{'}}{\overset{`H_1\text{'}}{\gtrless}} \frac{\Pr(H_0)}{\Pr(H_1)} \,.$$

(a) Find the range or ranges of values of the observation r for which you would decide that the signal was transmitted ("H_1" has taken place).

(b) Find the corresponding (minimal) probability of error.

Problem 6.5

Consider a system for determining whether a certain communication channel is being used or not. Let H_1 denote the hypothesis that the channel is being used, and let H_0 denote the hypothesis that the channel is not being used. The decision between hypotheses H_1 and H_0 is based on a single scalar measurement R (e.g., the output of our antenna at a particular time).

This measurement R is a zero-mean Gaussian random variable that has larger variance if the radio channel is indeed being used. More specifically,

$$H_0 \quad : \quad f_{R|H_0}(r|H_0) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2}$$

$$H_1 \quad : \quad f_{R|H_1}(r|H_1) = \frac{1}{\sqrt{4\pi}} e^{-r^2/4} \ .$$

Assume that the a priory probabilities for these two hypotheses are $\Pr(H_0) = p_0$ and $\Pr(H_1) = p_1$.

(a) Find the decision rule that minimizes the probability of error.

(b) Find the probability of error for the decision rule in part (a). Express your answer as

$$\Pr(\text{error}) = \alpha_1 Q(\gamma_1) + \alpha_2 Q(\gamma_2) + \alpha_3 \ ,$$

where α_1, α_2, α_3, γ_1 and γ_2 are appropriate constants, and $Q(\gamma)$ is defined as

$$Q(\gamma) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\tau^2/2} d\tau \ .$$

(c) Is there a choice of p_0 and p_1 such that the rule of part (b) always decides in favor of one of the hypothesis regardless of the measurement R?

Problem 6.6

A signal X which has Gaussian distribution with mean $\mu_X = 2$ and variance $\sigma_X^2 = 0.5$ is sent via a cable. The receiver needs to form an estimate $\widehat{X}$ for the transmitted signal X based on the received signal $Y = X + N$, where N models the (*additive*) noise. Assume that N also has Gaussian distribution with mean $\mu_N = 0$ and variance $\sigma_N^2 = 2$. Find $\widehat{X}_{\text{MMSE}}$, the minimum mean square error estimator for X given Y.