University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: Communications I

Fall 2001

Problem Set 7

Detection and Estimation, Random Processes, Autocorrelation

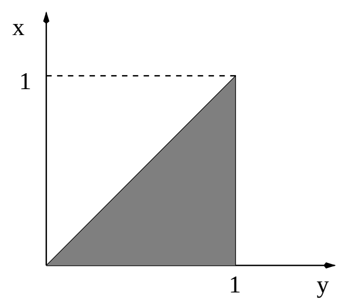
Issued: Thursday, October 25th. **Due:** Thursday, November 1st (beginning of lecture).

Reading from Haykin: Chapter 1, Sections 1.1–1.5.

Announcement: The second Mid-Semester Exam will be held on Thursday, November 8th, from 1:30pm to 2:50pm in 161 Everitt. The exam will cover all material from the beginning of the term *up to and including* the lecture on Thursday, November 1st. This includes the material through Problem Set 8 and, in particular, Problem Sets 5 through 8, and Chapter 2 (Sections 2.6–2.9), Appendix 1 and Chapter 1 (Sections 1.1–1.7).
For the exam, you can bring *two* 8.5×11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 7.1

Random variables X and Y have joint pdf $f_{X,Y}(x,y)$ that is constant in the shaded region (and zero elsewhere).



(a) Make fully labeled sketches of the densities $f_X(x)$ and $f_Y(y)$.

(b) Are X and Y statistically independent? Explain.

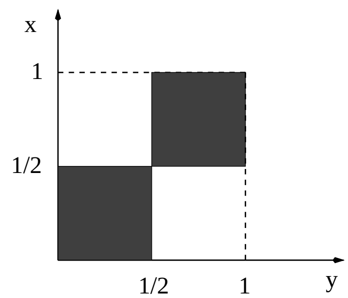
(c) Determine $\widehat{X}_{MMSE}(y)$, the minimum mean square error estimator for X, given the observation $Y = y$.

Problem 7.2

A random process $X(t)$ is given by $X(t) = A\cos(2\pi f_0 t + \Theta)$, where f_0 is a constant and Θ is random variable that is uniformly distributed between 0 and $\pi/2$. Find $E[X(t)]$.

Problem 7.3

Random variables X and Y have joint pdf $f_{X,Y}(x,y)$ that is constant in the shaded region (and zero elsewhere).



(a) Make a fully labeled sketch of the density $f_X(x)$. What is the mean and variance of X?

(b) Are X and Y uncorrelated? Are X and Y statistically independent?

(c) Determine $\widehat{X}_{MMSE}(y)$, the minimum mean square error estimator for X, given the observation $Y = y$.

(d) Determine $\widehat{X}_{LMMSE}(y)$, the *linear* minimum mean square error estimator for X, given the observation $Y = y$.

Problem 7.4

(a) Let $X[n]$ be a discrete-time random process defined for $n = 1, 2, 3, \ldots$. Samples $X[1]$, $X[2]$, $X[3]$, ..., are independent, identically distributed (i.i.d.) random variables and have $\Pr(X[n] = 0) = \Pr(X[n] = 1) = 1/2$. Let the random process $Y[n]$ be defined by

$$Y[n] = X[n] - X[n-1] \ .$$

Find $E[Y[n]]$ and $\mathrm{Var}[Y[n]]$.

(b) Let $X(t)$ be a random process defined by $X(t) = At + B$.

(i) If B is constant and A is a random variable that is uniformly distributed in $[-1, 1]$, sketch a sample function of this process and find $E[X(t)]$.

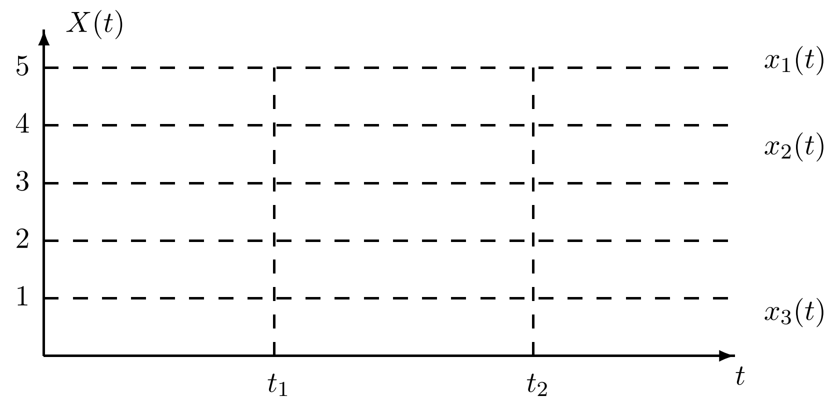
(ii) If A is constant and B is a random variable that is uniformly distributed in $[0, 2]$, sketch a sample function of this process and find $E[X(t)]$.

(iii) If A, B are independent, identically distributed (i.i.d.) Gaussian random variables with mean 0 and variance σ^2, find the joint pdf $f_{X(t_1),X(t_2)}(x_1, x_2)$.

Problem 7.5

A random process $X(t)$ has a sample space with three possible time waveforms as shown below. The probabilities for these three time waveforms are

$$\Pr(x_1(t)) = 1/4\ ,\quad \Pr(x_2(t)) = 1/8\ ,\quad \Pr(x_3(t)) = 5/8\ .$$



(a) Determine the pmf $\Pr[X(t_1) = x]$ for the random variable $X(t_1)$.

(b) Determine the joint pmf $\Pr[X(t_1) = x_1, X(t_2) = x_2]$ for random variables $X(t_1)$ and $X(t_2)$.

(c) Determine the autocorrelation $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$.

Problem 7.6 (Optional)

Consider the random process $X(t) = A\cos(2\pi f_0 t)$, where f_0 is a constant and A is a random variable that is uniformly distributed in $[0, 1]$.

(a) Find the autocorrelation $R_{XX}(t_1, t_2)$ of random process $X(t)$. Recall that $R_{XX}(t_1, t_2)$ is defined as $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$.

(b) Find the autocovariance $C_{XX}(t_1, t_2)$ of random process $X(t)$. Recall that $C_{XX}(t_1, t_2)$ is defined as $C_{XX}(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))]$.

(c) Is $X(t)$ wide-sense stationary?

Problem 7.7

Show that the random process $X(t) = \sin(2\pi F t)$, where F is a random variable uniformly distributed in the interval $[0, W]$, is non-stationary.

Problem 7.8

Let $X(t)$ be a wide-sense stationary random process with

$$R_{XX}(\tau) = 2e^{-|\tau|} \,, \quad -\infty < \tau < +\infty \,.$$

(a) What is the average power in the random process $X(t)$?

(b) Find the value of $E[(X(t+1) - X(t-1))^2]$.

(c) Let $Y(t)$ be a random process defined by

$$Y(t) = 5X(2t) - X(t-1) \,, \quad -\infty < t < +\infty \,.$$

Find $R_{YY}(t_1, t_2)$. Is $Y(t)$ a wide-sense stationary random process?