University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: Communications I

Fall 2001

Problem Set 8

WSS Random Processes through LTI Systems, Power Spectral Density

Issued: Thursday, November 1st. **Due:** Never.

Reading from Haykin: Chapter 1, Sections 1.1–1.7.

Announcement: The second Mid-Semester Exam will be held on Thursday, November 8th, from 1:30pm to 2:50pm in 161 Everitt. The exam will cover all material from the beginning of the term *up to and including* the lecture on Thursday, November 1st. This includes the material through Problem Set 8 and, in particular, Problem Sets 5 through 8, and Chapter 2 (Sections 2.6–2.9), Appendix 1 and Chapter 1 (Sections 1.1–1.7).
For the exam, you can bring *two* 8.5 × 11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 8.1

Let $X[n]$ be a discrete-time random process defined for $n = \ldots, -1, 0, 1, 2, \ldots$. The samples of the process (i.e., ..., $X[-1]$, $X[0]$, $X[1]$, ...) are independent, identically distributed (i.i.d.) random variables and each $X[i]$ is uniformly distributed in the interval $[-1, 1]$. Define the random process $Y[n]$ to be

$$Y[n] = \frac{2}{3}X[n] + \frac{1}{3}X[n-1] .$$

(a) Are $Y[n_1]$ and $Y[n_2]$ for $n_1 \neq n_2$ independent?

(b) Find $E[Y[n]]$ and $R_{YY}[n_1, n_2]$.

(c) Is $Y[n]$ a wide-sense stationary random process?

(d) Find the LMMSE of $Y[5]$ given that $Y[4] = y$.

Problem 8.2

Let X and Y be independent, identically distributed (i.i.d.) Gaussian random variables with zero mean and unit variance, and define the Gaussian random process $Z(t)$ as

$$Z(t) = X\cos(2\pi t) + Y\sin(2\pi t) .$$

Determine the joint probability density function $f_{Z(t_1),Z(t_2)}(z_1, z_2)$ of random variables $Z(t_1)$ and $Z(t_2)$ (obtained by observing the random process $Z(t)$ at times t_1 and t_2). Is $Z(t)$ strict sense stationary?

Problem 8.3

A communication channel has an input signal $S(t)$ which can be modeled as a modulated sinusoidal wave with random phase and random, time-varying amplitude at any given time, i.e.,

$$S(t) = X(t)\sin(2\pi f_c t + \Theta) ,$$

where f_c is a constant, Θ is a random variable that is uniformly distributed in $[0, 2\pi]$, and $X(t)$ is a wide-sense stationary random process that is independent from the phase and satisfies

$$\mu_X(t) = 0 , \qquad -\infty < t < +\infty ,$$

$$R_{XX}(t+\tau, t) \equiv R_{XX}(\tau) = Ae^{-|\tau|} , \qquad -\infty < \tau < +\infty .$$

Find the autocorrelation function $R_{SS}(t_1, t_2)$ for the signal $S(t)$. Is $S(t)$ a wide-sense stationary random process?

Problem 8.4

Consider a random process $X(t) = A$, where A is a random variable that is uniformly distributed in the interval $[-1, 1]$.

(a) Determine $\mu_X(t)$ and $R_{XX}(t_1, t_2)$. Is $X(t)$ a wide-sense stationary random process?

(b) Is $X(t)$ ergodic? Justify your answer.

Problem 8.5

Determine which of the following functions could be the power spectral density of a real random process:

(a) $\dfrac{f^2}{f^2 + 16}$,

(b) $\dfrac{1}{f^2 - 2}$.

Problem 8.6

A zero-mean Gaussian random process $X(t)$ has power spectral density

$$S_{XX}(f) = \frac{4}{1 + (2\pi f)^2} , \qquad -\infty < f < +\infty .$$

(a) Determine $R_{XX}(\tau)$, the autocorrelation function of the random process $X(t)$.

(b) The random process $X(t)$ is passed through a stable LTI system with frequency response

$$H(f) = \begin{cases} 1\,, & |f| < 1/\pi\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Determine the average power $E[Y^2(t)]$ of the output random process $Y(t)$.

Problem 8.7

(a) Let random processes $X(t)$ and $Y(t)$ be the input and output respectively of a stable LTI system with frequency response $H(f)$. Assume $X(t)$ is wide-sense stationary and define random process $Z(t)$ to be $Z(t) = Y(t) - X(t)$. Find $S_{ZZ}(f)$, the power spectral density of $Z(t)$, in terms of $H(f)$ and $S_{XX}(f)$.

(b) The input voltage $X(t)$ to a stable LTI filter with frequency response

$$H(f) = \frac{2}{2 + j2\pi f}$$

can be modeled as a wide-sense stationary random process with zero mean and autocorrelation function $R_{XX}(\tau) = 2e^{-|\tau|}$. Find $S_{ZZ}(f)$, the power spectral density of random process $Z(t) = Y(t) - X(t)$, where $Y(t)$ is the output of the filter.

Problem 8.8

A wide-sense stationary random process $X(t)$ with autocorrelation function $R_{XX}(\tau) = e^{-|\tau|}$ is processed by a stable LTI system with real-valued impulse response $h(t)$.

(a) For this part, assume that the output of the filter $Y(t)$ is a wide-sense stationary random process with autocorrelation function $R_{YY}(\tau) = 3e^{-3|\tau|}$.

(i) Find $|H(f)|$, the magnitude of the frequency response of the filter.

(ii) Suppose that $h(t)$ is causal. Find a possible impulse response $h(t)$ for the LTI system. Is your answer unique?

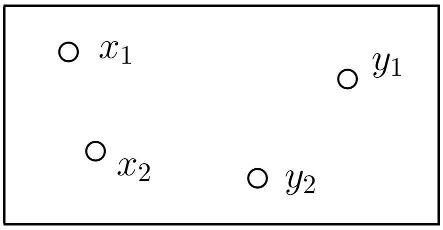
(b) Suppose that the LTI system is known to be stable and that

$$R_{YX}(\tau) = e^{-\tau}u(\tau) - 2e^{-2\tau}u(\tau) + e^{-3\tau}u(\tau)\,.$$

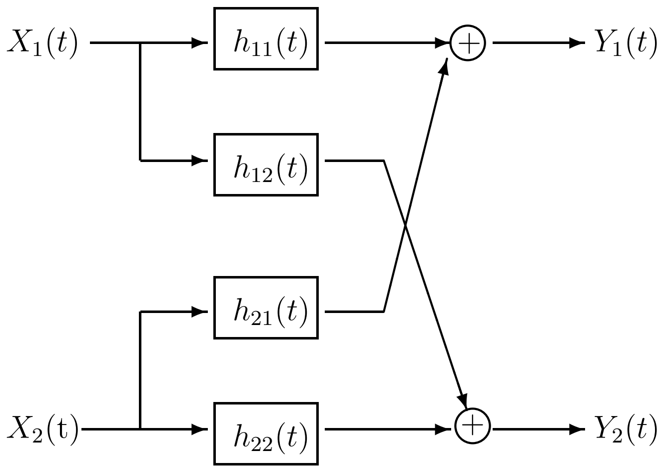
Find a possible impulse response $h(t)$. Is your answer unique?

Problem 8.9

In a variety of real communication situations two (or more) uncorrelated sources are received through channels or systems with crosstalk. Since signals interfere with each other, it is essential that they are separated at the receiving end. One such scenario is shown below, where we have two sources, denoted by x_1 and x_2, and two receivers, denoted by y_1 and y_2.



The above scenario is modeled in terms of LTI systems as shown below. The impulse response from source i to receiver j is denoted by $h_{ij}(t)$.



Approaches for recovering the signals from sources x_1 and x_2 usually involve estimating the impulse responses $h_{ij}(t)$. In this problem we investigate such techniques; to simplify the problem we assume that $h_{11}(t) = h_{22}(t) = \delta(t)$. We also assume that $h_{12}(t)$ and $h_{21}(t)$ are stable and causal and that the sources can be modeled by uncorrelated wide-sense stationary random processes $X_1(t)$ and $X_2(t)$ (with zero mean and known autocorrelation functions $R_{X_1X_1}(\tau)$ and $R_{X_2X_2}(\tau)$).

(a) Show that $Y_1(t)$ and $Y_2(t)$ are jointly wide-sense stationary and determine $R_{Y_1Y_1}(\tau)$, $R_{Y_2Y_2}(\tau)$ and $R_{Y_1Y_2}(\tau)$ in terms of $R_{X_1X_1}(\tau)$, $R_{X_2X_2}(\tau)$, $h_{12}(t)$ and $h_{21}(t)$.

(b) Suppose that $h_{12}(t) = h_{21}(t) = h(t)$ and that you can measure *only one* of $R_{Y_1Y_1}(\tau)$, $R_{Y_2Y_2}(\tau)$ or $R_{Y_1Y_2}(\tau)$. Which one would be most helpful in determining $h_{12}(t)$?