ECE 359

Supplementary notes on
Signal detection in noise & Matched filtering
Draft

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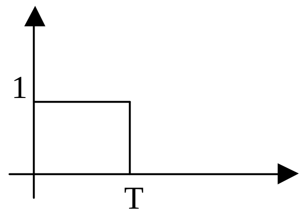
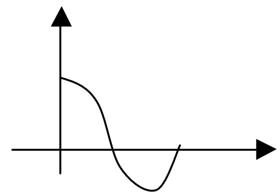
Based on notes by Prof. Christoforos N. Hadjicostis

Suppose that we would like to transmit a single bit ("0" or "1") using the following approach:

"1": s(t) = the pulse p(t) or more generally p(t)

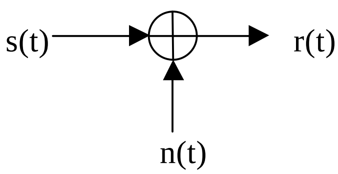




"0": s(t) = 0

A priori, Pr("0") = Pr("1") = ½

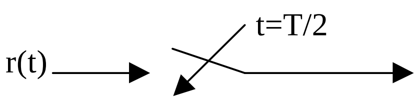
s(t) gets corrupted additively by WGN



Receiver looks at r(t), $0 \le t \le T$ and needs to decide whether "0" or "1" was sent. (Note that r(t) outside [0,T] is irrelevant; this is all only true if n(t) is WGN!)

WGN
$R_{NN}(t) = d(t) N_0/2$

Idea: at the receiving end, sample r(t) and make a decision based on the sampled value.



R = r(T/2)

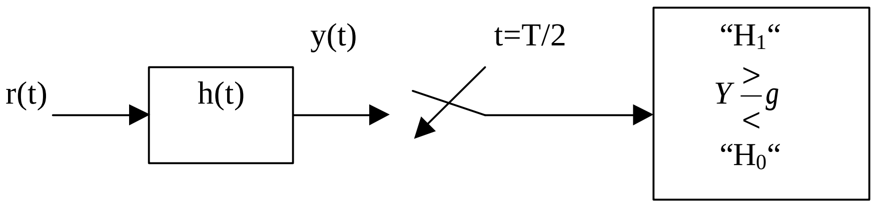
If p(t) is not a pulse, perhaps we can sample at its peak. Based on R, how do we minimize the Pr(error) ?

H_1: R = r(T/2) = s(T/2) + n(T/2)
= 1 + n(T/2)
↓
$N(0, s^2)$

H_0: R = r(T/2) = s(T/2) + n(T/2)
= 0 + n(T/2)
↓
$N(0, s^2)$

Problem: $s^2 = \infty$

Solution: One way to avoid this problem is to filter first.
g is a threshold to be determined.



Questions:

1. How do we pick h(t)?
2. How do we pick g ?
3. How good is this approach?

$$Y(t) = \underbrace{h(t) * s(t)} \quad + \quad \underbrace{w(t)}$$

Deterministic — Gaussian R.P. with zero mean

Variance given by $R_{ww}(0) = \int_{-\infty}^{\infty} Sww(f)df$

$$= \int_{-\infty}^{\infty} |H(f)|^2 No/2df = No/2\int_{-\infty}^{\infty} h^2(u)du < \infty$$

(assuming h(t) is real, $|h(t)|^2 = h(t)^2$)

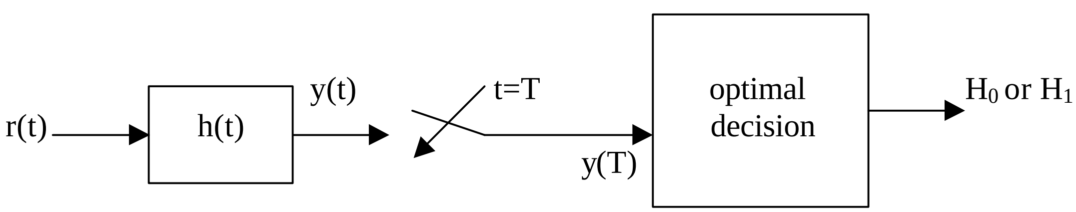
H_1: Y $= y(T) = \{h(t) * s(t)\}|_{t=T} + w(T)$

$$= \underbrace{\int_{-\infty}^{\infty} h(T-t)s(t)dt} \quad + \quad \underbrace{N(0, s^2)})$$

Deterministic $\qquad s^2 = R_{ww}(0) = No/2\int_{-\infty}^{\infty} h^2(u)du$

H_0: Y $= 0 + w(T) = N(0, s^2)$

Consider the following approach:



Based on Y = y(T) we need to make a decision about whether H_0 or H_1 took place. We need to do this in a way that minimizes the probability of error.

Question: Is this going to be the best strategy? Not necessarily. It is the best strategy given that we only know y(T) = Y, in reality we know r(t) for $0 \le t \le T$.

Given $f_{Y|H0}(y|H0)$ and $f_{Y|H1}(y|H1)$ we can decide using the likelihood ratio test.

$$\frac{f_{Y|H0}(y|H_0)}{f_{Y|H1}(y|H_1)} \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} \frac{\Pr(H_1)}{\Pr(H_0)} = 1 \text{ in this case}$$

So how do we find $f_{Y|H0}(y|H_0)$ and $f_{Y|H1}(y|H_1)$?

Under H_0 : $Y = y(T) = 0 + w(T) = N(0, R_{WW}(0))$

Under H_1 : $Y = \underbrace{p(t) * h(t) \,|\, t = T}_{\text{Deterministic}} + \underbrace{w(T)}_{N(0,\, R_{WW}(0))} = N(\mu, R_{WW}(0))$

Where $\mu = \int_{-\infty}^{\infty} p(t)h(T-t)dt$

Now we can apply the MAP rule:

$$\frac{f_{Y|H0}(y|H_0)}{f_{Y|H1}(y|H_1)} \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} \frac{\Pr(H_1)}{\Pr(H_0)} = 1 \text{ in this case}$$

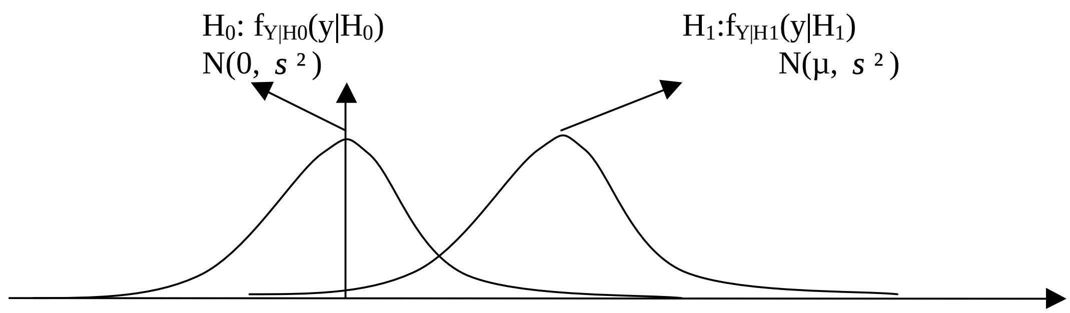
$$\Rightarrow \frac{\frac{1}{\sqrt{2p}\, s}\exp(\frac{-1}{2s^2}y^2)}{\frac{1}{\sqrt{2p}\, s}\exp(\frac{-1}{2s^2}(y-m)^2)} \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} 1$$

$$\Rightarrow \exp(\frac{-1}{2s^2}(y^2 - y^2 + 2ym - m^2)) \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} 1$$

$$\Rightarrow \frac{-1}{2s^2}(2ym - m^2) \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} 0$$

$$\Rightarrow 2ym - m^2 \underset{\text{“}H_0\text{”}}{\overset{\text{“}H_1\text{”}}{\gtrless}} 0$$

$$\Rightarrow \quad y \quad \underset{\text{"}H_0\text{"}}{\overset{\text{"}H_1\text{"}}{\gtrless}} \quad \frac{m}{2} \qquad \text{(assuming } \mu>0\text{)}$$



If $y>\mu/2$ then we guess H_1, otherwise we guess H_0.

Note: If the a priori probabilities were not the same then we would use a different threshold.

The probability of error is:

$$\begin{aligned}
\Pr(\text{error}) &= \Pr(\text{ error}, H_0) + \Pr(\text{error}, H_1) \\
&= \Pr(\text{ error}|H_0)\Pr(H_0) + \Pr(\text{eror}|H_1)\Pr(H_1) \\
&= \Pr(\ y>\mu/2|H_0)½ \ + \Pr(y<\mu/2|H_1)½ \\
&= ½Q(½\mu/s\ ) + ½(1\text{-}Q((½\mu\text{-}\mu)/\ s\ ) \\
&= ½Q(\mu/(2s\ )) + ½(1\text{-}Q(\text{-}\mu/(2s\ )) \\
&= ½Q(\mu/(2s\ )) + ½(Q(\mu/(2s\ )) && \text{note } Q(x) = 1\text{-}Q(\text{-}x) \\
&= Q(\mu/(2s\ )) = \quad Q(½\sqrt{SNR} && \text{where } SNR = \mu^2/s^2
\end{aligned}$$

To minimize the Pr(error), maximize $\mu/(2s\ )$ (or equivalently maximize the SNR).

$\mu = h(t) * p(t)|_{t=T}$

$$s^2 = R_{ww}(0) = N/2\int_{-\infty}^{\infty} h^2(u)du$$

Given p(t) we want to choose h(t) so that μ/s is maximized.

$$\frac{m}{2s} = \frac{[p(t)*h(t)]\,|\,t=T}{\sqrt{(2No\int_{-\infty}^{\infty}|h(u)|^2du)}}$$

Without loss of generality we can set $\int_{-\infty}^{\infty}|h(u)|^2du = 1$ because given any h(t) such that $\int_{-\infty}^{\infty}|h(u)|^2du = k$, we can find $h'(t) = {}^{1}\!/\!_{\sqrt{k}}\,h(t)$ such that $\int_{-\infty}^{\infty}|h'(u)|^2du = 1$ and

$$\frac{[p(t)*h(t)]\,|\,t=T}{\sqrt{2No\int_{-\infty}^{\infty}|h(u)|^2du}} = \frac{[p(t)*h'(t)]\,|\,t=T}{\sqrt{2No\int_{-\infty}^{\infty}|h'(u)|^2du}}$$

Therefore, what we are looking for is for an h(t) such that $\int_{-\infty}^{\infty} |\, h(u)\,|^2 du = 1$

and $[h(t) * p(t)]|_{t=T} = \int_{-\infty}^{\infty} h(T-t)\,p(t)\,dt = \int_{-\infty}^{\infty} p(u)\,p(u)\,du = \int_{-\infty}^{\infty} p^2(u)\,du$

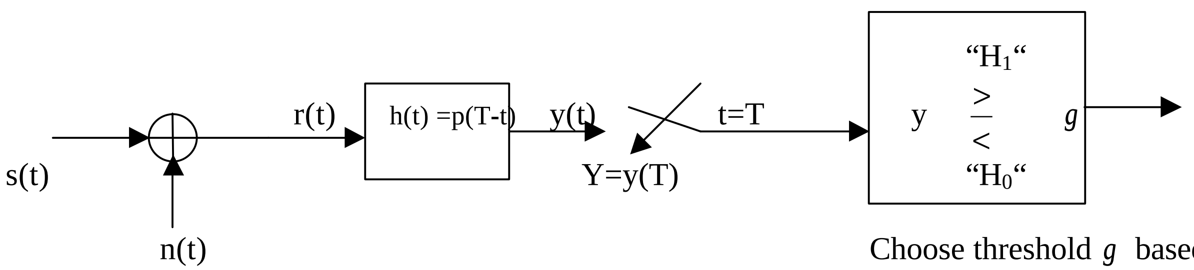
$$\sigma^2 = {No}/{2}\int_{-\infty}^{\infty} |\, h(u)\,|^2 du = {No}/{2}\int_{-\infty}^{\infty} |\, p(T-u)\,|^2 du = {No}/{2}\int_{-\infty}^{\infty} |\, p(u)\,|^2 du = \mathrm{E}$$

= Energy of pulse p(t)

Therefore Pr(error) = $Q(\mu/(2\sigma)) = Q\left(\dfrac{\mathrm{E}}{2\sqrt{N/2\,\mathrm{E}}}\right) = Q\left(\dfrac{\sqrt{\mathrm{E}}}{\sqrt{2N}}\right) = Q\left(\dfrac{1}{2}\sqrt{SNR}\right)$

where SNR = $\dfrac{\mathrm{E}^2}{\mathrm{No}/2\,\mathrm{E}} = \dfrac{\mathrm{E}}{No/2}$

The higher the energy of p(t), the smaller the probability of error.
To summarize:



WGN
$R_{NN}(\tau) = N_0/2\,\delta(\tau)$

Choose threshold γ based on MAP rule

This is the best strategy to minimize the probability of error.

Example 1:

$$p(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

h(t) = p(T-t)

Then $\mu = h(t) * p(t)|_{t=T} = \int_{-\infty}^{\infty} p^2(u)\,du = \int_0^T 1\,du = T$

$$\sigma^2 = R_{ww}(0) = \int_{-\infty}^{\infty} {No}/{2}\,|\,h(u)\,|^2 du = {No}/{2}\int_0^T 1\,du = {N}/{2}\,T$$

The detection strategy should be

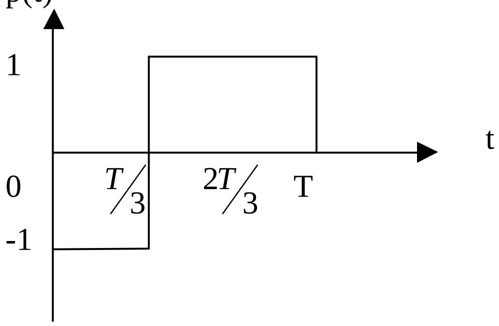
$$y \underset{\text{“}H_0\text{”}}{\overset{\text{“}H_1\text{”}}{\gtrless}} \frac{T}{2}$$

and the Pr(error) = $Q\left(\frac{m}{2s}\right) = Q\left(\frac{T}{2\sqrt{No/2\,T}}\right) = Q\left(\frac{\sqrt{T}}{\sqrt{2No}}\right)$

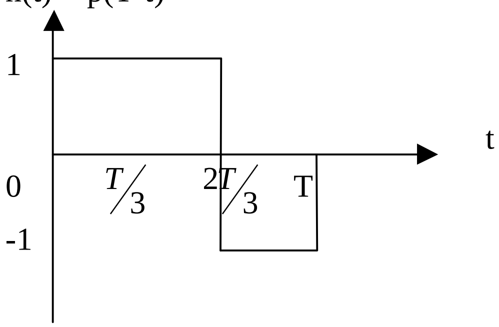
SNR = $\frac{T^2}{No/2\,T}$

Example 2:

p(t)



h(t) = p(T-t)



Pr(error) = Q(μ/(2 s)) = $Q\left(\frac{\sqrt{T}}{\sqrt{2N}}\right)$

$$m = \int_{-\infty}^{\infty} h(T-u)\,p(u)\,du = \int_{-\infty}^{\infty} p^2(u)\,du = T$$

$$s^2 = R_{ww}(0) = \int_{-\infty}^{\infty} No/2\,|h(u)|^2 du = No/2 \int_0^T 1\,du = No/2\,T$$

The result is the same as for the previous example since the energy of the pulse is the same. The shape of the pulse does not matter.