

# ECE 359: Communications I

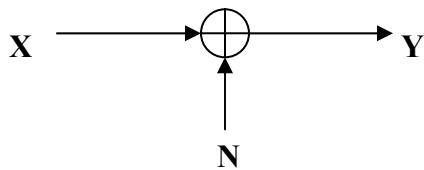
## Supplemental Notes on Estimation

Prepared by: Suneil Hosmane  
Based on Lectures by Prof. Hadjicostis  
University of Illinois at Urbana-Champaign  
Department of Electrical and Computer Engineering  
© Copyright 2002 Christoforos Hadjicostis. All rights reserved.

## Introduction to Estimation

Just like (binary) hypothesis testing is important in digital communication systems, **estimation** is important in analog communications.

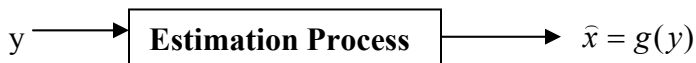
Consider the following simplified “communication channel”:



$\mathbf{X}$  denotes the signal that we would like to transmit over a channel and is modeled as a random variable with some known pdf  $f_x(x)$ .

$\mathbf{N}$  represents noise in the channel and is modeled by as a random variable with normal distribution (Gaussian random variable) with mean =  $\mu_n$  (typically 0) and variance =  $\sigma_n^2$ . (Noise is added to the signal  $\mathbf{X}$  when it is transmitted through the channel.)

The receiver receives  $\mathbf{Y}$  and needs to make an intelligent guess as to the actual signal  $\mathbf{X}$  that was sent. How do we do this? We need a suitable estimator that can map our observation  $\mathbf{Y} = y$  to some estimate about what was transmitted. The following sections discuss how we can choose our estimators.



$y$  is probabilistically related to  $x$  (e.g., via a known joint pdf  $f_{x,y}(x,y)$ ).  
 $\hat{x}$  is an estimate of  $x$  and is a function of  $y$ .

Consider the following example which is not a communication example but still a very relevant application of estimation.

### Example: Class Grade Estimation

An exam was given at UIUC. The exam was out of a total of 80 points. We need to guess what a particular student (student  $\mathbf{X}$ ) scored in this exam. We only need to guess, 0-5, 6-10, 11-15, etc.... (interval guessing). What should our guess be? This is a difficult question because we have no prior information about how hard or easy the exam was.

Suppose that I now reveal to you the histogram for that particular exam. What would be your guess for student's  $\mathbf{X}$  score?

The mean? The mode? The median?

It turns out that each of these (Mean, Mode and Median) are “optimal” estimators for different criteria.

Suppose we play the following game:

- i) I randomly choose a student in the class and I write down the name on a piece of paper.
- ii) Then you “guess” the student score.
- iii) We repeat this many times (say N times).

What should you pick as your guess of a student’s score? It depends on the rules of the game (optimization criterion).

If you want to minimize  $\frac{1}{N} \sum_{i=1}^N (\hat{x}_i - x_i)^2$  you should pick  $\hat{x}_i = E[x | Info]$ .

If you want to minimize  $\frac{1}{N} \sum_{i=1}^N |\hat{x}_i - x_i|$  you should pick  $\hat{x}_i = Median(x | Info)$ .

If you want to minimize  $\frac{1}{N} \sum_{i=1}^N cost(\hat{x}_i - x_i)$ , where

$$\begin{aligned} cost(\hat{x}_i - x_i) &= 1 \text{ if } \hat{x}_i \neq x_i \\ &= 0 \text{ if } \hat{x}_i = x_i, \end{aligned}$$

you should pick  $\hat{x}_i = Mode(x | Info)$ .

(Note that the last calculation is very similar to the MAP rule (the mode(x | Info) is what has the maximum a posteriori probability.) Also note that in all of the above we are conditioning on the available information, i.e., the histogram that I revealed to you. Next we will focus on Minimum Mean Square Error (MMSE) estimation.

## Minimum Mean Square Error (MMSE) Estimation

Suppose we want to choose  $\hat{x}$  so that we minimize the following quantity:

$$E[(\hat{x} - x)^2] = E[(\hat{x}(y) - x)^2], \text{ or}$$

$$E[(\hat{x}(y) - x)^2] = \iint (\hat{x}(y) - x)^2 \cdot f_{x,y}(x, y) dx dy$$

If we make the substitution  $f_{x,y}(x, y) dx dy = f_{x|y}(x | Y = y) \cdot f_y(y) dx dy$ , we get

$$\min(\hat{x}(y)) \iint (\hat{x}(y) - x)^2 \cdot f_{x|y}(x | Y = y) \cdot f_y(y) dx dy$$

$$\min(\hat{x}(y)) \int \left[ \int (\hat{x}(y) - x)^2 \cdot f_{x|y}(x | Y = y) dx \right] \cdot f_y(y) dy$$

Since  $\hat{x}(y)$  is fixed for a given  $\mathbf{Y} = y$  and since all quantities are positive, what we need to do is:

$$\min \alpha \int (\alpha - x)^2 \cdot f_{x|y}(x | Y = y) dx \text{ where we substituted } \alpha = \hat{x}(y)$$

To minimize the expression we take the derivative with respect to  $\alpha$ .

$$\frac{\partial}{\partial \alpha} \int (\alpha - x)^2 \cdot f_{x|y}(x | Y = y) dx = 0$$

$$\int 2 \cdot (\alpha - x) \cdot f_{x|y}(x | Y = y) dx = 0$$

$$\alpha \int f_{x|y}(x | Y = y) dx = \int x \cdot f_{x|y}(x | Y = y) dx$$

$$\alpha = \int x \cdot f_{x|y}(x | Y = y) dx$$

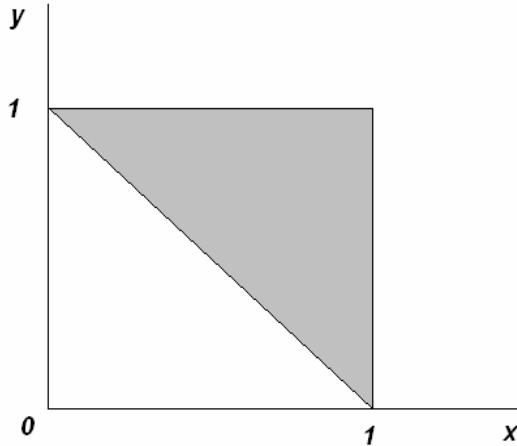
$$\alpha = E[X | Y = y]$$

$$\therefore \hat{x}(y) = E[X | Y = y]$$

This is the MMSE estimator for X given the observation that  $\mathbf{Y} = y$ .

**Example: Find the MMSE estimator for X given Y=y.**

Setup: X, Y are two Random Variables with joint density  $f_{x,y}(x,y)$ . Given  $Y = y$ , find  $\hat{x}_{MMSE}(y) = E[x | Y = y]$



$f_{x,y}(x,y) = C$  in the shaded region.

$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2} bh \\ &= \frac{1}{2} * 1 * 1 \\ &= \frac{1}{2} \end{aligned}$$

Since the total area of  $f_{x,y}(x,y) = 1$ , that must mean that  $\frac{1}{2} * C = 1$ .

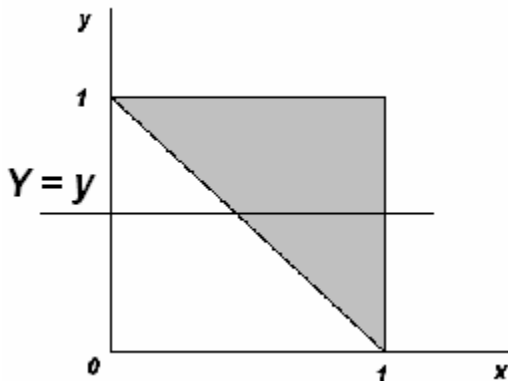
Therefore,  $C = 2$ .

$$f_{x|Y}(x | Y = y) = \frac{f_{x,y}(x,y)}{f_Y(y)}$$

$$f_{x,y}(x,y) = 2 \quad 0 < x < 1 \quad 1-x < y < 1$$

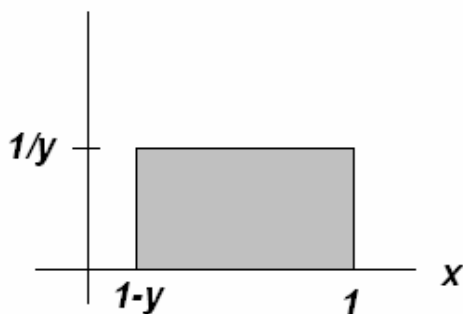
Since there is a constant distribution in the shaded region,  $f_{x|Y}(x | Y = y)$  is a uniform probability density from  $1-y < x < 1$ .

To illustrate, look at the following picture:



Take any slice of the 2-dimensional space along  $Y = y$ . Since there is an even probability distribution from the point of intersection of the diagonal line ( $x = 1 - y$ ) and the vertical line at  $x = 1$ ,  $f_{x|Y}(x | Y = y)$  will have a uniform distribution from  $1-y < x < 1$ .

$f_{x|y}(x | Y = y)$



From here it is easy to find the mean, and hence, the Minimum Mean Square Error (MMSE) estimator.

$$\text{Midpoint of } 1 \text{ and } 1-y = \frac{1 + (1-y)}{2} = \frac{y}{2}$$

$$\hat{x}_{MMSE}(y) = 1 - \frac{y}{2}$$

### Linear Minimum Mean Square Error (LMMSE) Estimators

LMMSE focuses on estimators that minimize the MSE but restricts itself to linear estimators. We do this because the distribution  $f_{x|y}(x | Y = y)$  is often hard to calculate or unavailable (whereas the qualities needed for LMMSE estimation are available). The LMMSE has the following form:



Our job is still to minimize the mean square error  $E[(\hat{x}(y) - x)^2]$  given by

$$E[(\hat{x}(y) - x)^2] = \iint (ay + b - x)^2 f_{x,y}(x, y) dx dy.$$

To minimize this expression we must find appropriate coefficients  $a$  and  $b$ . Therefore, we take two separate derivatives (one with respect to  $a$ , the other with respect to  $b$ ) and set them to 0.

$$E[(\hat{x}(y) - x)^2] = E[(ay + b - x)^2] = E[a^2 y^2 + 2ayb + b^2 - 2ayx - 2bx + x^2]$$

$$\frac{\partial}{\partial a} E[(\hat{x}(y) - x)^2] = E[2ay^2] + E[2yb] - E[2yx] = 0$$

$$aE[y^2] + bE[y] = E[yx]$$

$$\frac{\partial}{\partial b} E[(\hat{x}(y) - x)^2] = E[2ay] + E[2b] - E[2x] = 0$$

$$aE[y] + b = E[x]$$

By solving this system of equations, we obtain the following closed form solution for the LMMSE estimator:

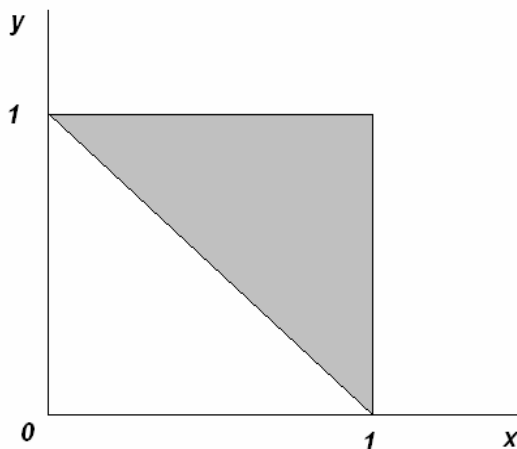
$$\hat{x}_{LMMSE}(y) = \mu_x + \rho(x, y) \frac{\sigma_x}{\sigma_y} (y - \mu_y)$$

Here  $\mu_x, \sigma_x$  ( $\mu_y, \sigma_y$ ) are the mean and standard deviation of  $\mathbf{X}$  ( $\mathbf{Y}$ ) and  $\rho(x, y)$  is the correlation coefficient between  $\mathbf{X}$  and  $\mathbf{Y}$ .

**Example: Revisiting the same problem but finding the LMMSE estimator for X given Y = y**

Setup:  $\mathbf{X}, \mathbf{Y}$  are two Random Variables with known  $f_{x,y}(x,y)$ . Find

$$\hat{x}_{LMMSE}(y) = \mu_x + \rho(x, y) \frac{\sigma_x}{\sigma_y} (y - \mu_y)$$



$$f_x(x) = \int_{1-x}^1 2dy = 2y \Big|_{1-x}^1 = 2 - 2 + 2x = 2x$$

$$E[X] = \int_0^1 2x^2 dx = \frac{2x^3}{3} \Big|_0^1 = \frac{2}{3}$$

$$\text{var}(X) = E[X^2] - (E[X])^2 = \left( \int_0^1 2x^3 dx = \frac{x^4}{2} \Big|_0^1 = \frac{1}{2} \right) - \left( \frac{2}{3} \right)^2 = \frac{1}{2} - \left( \frac{2}{3} \right)^2 = \frac{1}{18}$$

Due to symmetry,  $E[X] = E[Y]$  and  $\text{var}(X) = \text{var}(Y)$ .

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_x \cdot \sigma_y} = \frac{E[XY] - E[X]E[Y]}{\sigma_x \cdot \sigma_y}$$

$$E[XY] = \int_0^1 \left( \int_{1-x}^1 2xy dy \right) dx = xy^2 \Big|_{1-x}^1 = \int_0^1 (2x^2 - x^3) dx = \frac{2x^3}{3} - \frac{x^4}{4} \Big|_0^1 = \frac{5}{12}$$

$$\rho(X, Y) = \frac{E[XY] - E[X]E[Y]}{\sigma_X \cdot \sigma_Y} = \frac{\frac{5}{12} - \left(\frac{2}{3}\right)^2}{\left(\frac{1}{18}\right)} = -\frac{1}{2}$$

$$\hat{x}_{LMMSE}(y) = \frac{2}{3} - \frac{1}{2}\left(y - \frac{2}{3}\right) = 1 - \frac{y}{2}$$

Notice we get the same result as in the unrestricted MMSE case. In general, the MMSE and LMMSE estimators will not be the same; However, if the MMSE estimator is linear, then MMSE = LMMSE. The following property always holds:

$$E[(\hat{x}_{MMSE}(y) - x)^2] \leq E[(\hat{x}_{LMMSE}(y) - x)^2]$$