ECE 359: Communications I

Supplemental Notes on Hypothesis Testing

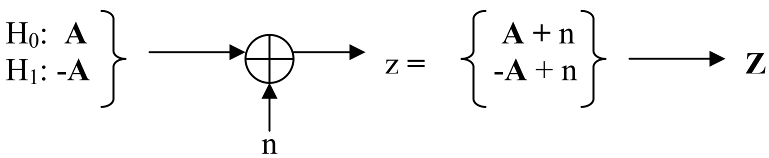
Prepared by: Suneil Hosmane
Based on Lectures by Prof. Hadjicostis
University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

Introduction

Hypothesis testing is a recurrent theme in many disciplines, especially that of Communications Engineering. In Digital Communications, the fundamental idea behind hypothesis testing is the following: We have two hypotheses H_0 and H_1 which could represent respectively a "0" or "1" being transmitted. If we are given the probability of receiving a "0" given that H_0 occurred (meaning that we actually sent a 0) and the probability of receiving a "1" given that H_1 occurred, how do we decide which signal was actually sent?

Example: Oversimplified Digital Communication System

We are given a Digital Communication System that transmits a single bit (0 or 1). During transmission, noise (represented by a R.V. **N**) is introduced into the system. On the receiving end, the receiver is presented with two hypotheses. Hypothesis H_0 states that a "0" was sent, and Hypothesis H_1 states that a "1" was sent. If we are also given the information below, how do we determine which bit was actually received so that we minimize the probability of error?

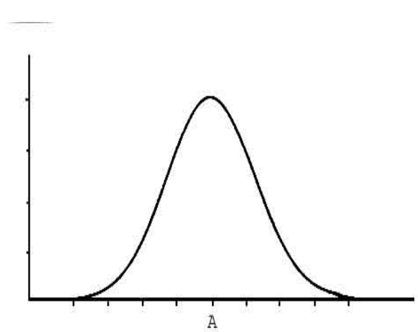


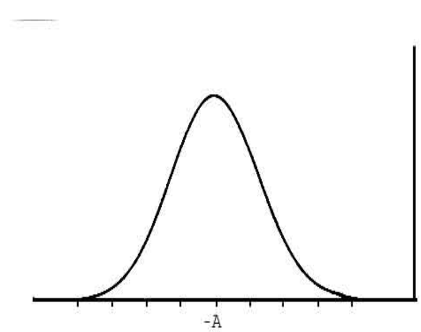
Assume that a priori, the two hypotheses have the following probabilities:
$Pr(H_0) = P_0,\ Pr(H_1) = P_1,\ P_0 + P_1 = 1$
And that the noise **N** is a Gaussian R.V.

$$f_N(n) = \mathbf{N}(0, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{n^2}{2\sigma^2})$$

Under H_0:
$F_{z|H0}(z|H_0)$ is $\mathbf{N}(A, \sigma^2)$

Under H_1:
$F_{z|H1}(z|H_1)$ is $\mathbf{N}(-A, \sigma^2)$





Note: These pictures are a little deceiving. The Gaussian distribution does not have a finite bound like depicted in the pictures above. It actually extends from – infinity to +

infinity. In other words, if you were to plot both functions above on the same graph, there would be overlap between the two graphs.

We now need to determine what was sent given the information above. One intuitive solution would be to devise a threshold test of the following form:

$$\mathbf{Z} \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} \gamma$$

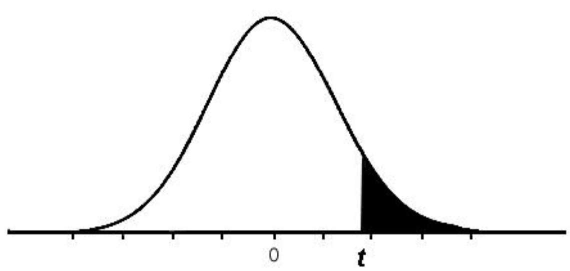
(If **Z** is larger than some set threshold γ, we decide that H_0 occurred otherwise we decide that H_1 occurred) The natural question is how to choose the threshold (γ)? One reasonable criterion is to choose γ so that we minimize the error probability, i.e., the probability of choosing “H_0” when H_1 was true and choosing “H_1” when H_0 was true.

$$\begin{aligned}\Pr(\text{Error}) &= \Pr(z \le \gamma \cap H_0) + \Pr(z \ge \gamma \cap H_1) \\ &= P_0 \cdot \Pr(z \le \gamma \mid H_0) + P_1 \cdot \Pr(z \ge \gamma \mid H_1) \\ &= P_0 \cdot \int_{-\infty}^{\gamma} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z-A)^2}{2\sigma^2}\right) dz \;+\; P_1 \cdot \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z+A)^2}{2\sigma^2}\right) dz\end{aligned}$$

Let Q(t), $t \ge 0$, be defined as follows:

$$Q(t) = \int_{t}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$

i.e., Q(t) is the area to the right of t under a Normal (or Gaussian distribution) with zero mean and variance one. This is indicated by the shaded area in the graph below.



Continuing with our calculation of Pr(Error):

$$\Pr(\text{Error}) = P_0 \cdot \int_{-\infty}^{\frac{\gamma-A}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z')^2}{2}\right) dz' \;+ P_1 \cdot \int_{\frac{\gamma+A}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z'')^2}{2}\right) dz'' \qquad z' = \frac{z-A}{\sigma}$$

$$z'' = \frac{z+A}{\sigma}$$

$$= \mathbf{P_0} \cdot \left(1 - Q\left(\frac{\gamma - A}{\sigma}\right)\right) + \mathbf{P_1} \cdot Q\left(\frac{\gamma + A}{\sigma}\right)$$

We can then solve for γ and obtain our optimal solution for the threshold test:

$$\mathbf{Z} \underset{\text{"}H_1\text{"}}{\overset{\text{"}H_0\text{"}}{\gtrless}} \gamma$$

Notice that even if we choose γ in an optimal fashion, it is still unclear at this point that the above rule will be the best we can do in terms of minimizing the probability of error. This is so because we arbitrarily set the rule to be of the form $\mathbf{Z} \gtrless \gamma$.

Likelihood Ratio Test

It turns out that to minimize the probability of error we need to use a MAP rule.

Choose H_0 if:

$$\Pr(H_0 \mid \mathbf{Z} = z) > \Pr(H_1 \mid \mathbf{Z} = z)$$

Choose H_1 if:

$$\Pr(H_0 \mid \mathbf{Z} = z) < \Pr(H_1 \mid \mathbf{Z} = z)$$

If we arrange terms, we can arrive at the following rule:

$$P_0 \cdot f_{z|H0}(z \mid H_0) \underset{\text{"}H_0\text{"}}{\overset{\text{"}H_1\text{"}}{\gtrless}} P_1 \cdot f_{z|H1}(z \mid H_1)$$

$$\frac{f_{z \mid H_0}(z \mid H_0)}{f_{z \mid H_1}(z \mid H_1)} \underset{\text{"}H_1\text{"}}{\overset{\text{"}H_0\text{"}}{\gtrless}} \frac{P_1}{P_0}$$

This is know as the Likelihood Ratio Test

By looking at the weighted probabilities of $P_0 \cdot f_{z|H0}(z \mid H_0)$ and $P_1 \cdot f_{z|H1}(z \mid H_1)$ we can choose which hypothesis occurred.

Let us use the Likelihood Ratio Test to find the optimal decision rule in the previous example.

$$\frac{\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(z-A)^2}{2\sigma^2})}{\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(z+A)^2}{2\sigma^2})} \underset{\text{"H}_1\text{"}}{\overset{\text{"H}_0\text{"}}{\frac{>}{<}}} \frac{P_1}{P_0}$$

$$\frac{\exp(-\frac{(z-A)^2}{2\sigma^2})}{\exp(-\frac{(z+A)^2}{2\sigma^2})} \underset{\text{"H}_1\text{"}}{\overset{\text{"H}_0\text{"}}{\frac{>}{<}}} \frac{P_1}{P_0}$$

$$\exp(-\frac{(z-A)^2}{2\sigma^2}+\frac{(z+A)^2}{2\sigma^2}) \underset{\text{"H}_1\text{"}}{\overset{\text{"H}_0\text{"}}{\frac{>}{<}}} \frac{P_1}{P_0}$$

$$\exp(\frac{-z^2+2zA-A^2}{2\sigma^2}+\frac{z^2+2zA+A^2}{2\sigma^2}) \underset{\text{"H}_1\text{"}}{\overset{\text{"H}_0\text{"}}{\frac{>}{<}}} \frac{P_1}{P_0}$$

$$\exp(\frac{4zA}{2\sigma^2}) \underset{\text{"H}_1\text{"}}{\overset{\text{"H}_0\text{"}}{\frac{>}{<}}} \frac{P_1}{P_0}$$

$$z \underset{\text{"H}_1\text{"}}{\overset{\text{"H}_0\text{"}}{\frac{>}{<}}} \frac{\sigma^2}{2A}\ln(\frac{P_1}{P_0})$$

so we have now solved for our γ

$$\boxed{\gamma = \frac{\sigma^2}{2A}\ln(\frac{P_1}{P_0})}$$

Notice that if $P_1 = P_0 = ½$, then $\gamma = 0$, which makes intuitive sense. Also notice that the threshold rule we arbitrarily chose to study earlier turns out to be optimal (assuming we choose the right γ).

Example: Optical Communications Setup

In an optical communication system, the receiver operates by counting the number of photons (**E**) incident on the photocell during the interval (0, T). A bit 0 or 1, represented by Hypotheses H_0 and H_1, is transmitted by sending or not sending light.

Under H_0, the probability that k photons are counted is given by:

$\Pr(\mathbf{E} = k \mid H_0) = A_0 \cdot u_0^k$ where k = 1, 2, ….

Under H_1, the probability that k photons are counted is given by:

$\Pr(\mathbf{E} = k \mid H_1) = A_1 \cdot u_1^k$ where k = 1, 2, ….

We are also given that:

$0 < u_1 < u_0 < 1$ $\Pr(H_0) = P_0$, $\Pr(H_1) = P_1$, $P_0 + P_1 = 1$

Likelihood Decision via MAP Rule

We need to find a decision rule so that we decide whether 0 or 1 was transmitted based on the number of photons that are counted at the receiving end. First, let us find A_0 and A_1.

Under H_0: $\Pr(\mathbf{E} = k \mid H_0) = A_0 \cdot u_0^k$

$$\sum_{k=0}^{\infty} \Pr(E = k \mid H_0) = 1$$

$$\sum_{k=0}^{\infty} A_0 \cdot u_0^k = 1$$

$$\frac{A_0}{1 - u_0} = 1$$

$$\boxed{A_0 = 1 - u_0}$$

Similarly, $\boxed{A_1 = 1 - u_1}$

Given **E** = k, we can use the MAP rule to decide whether 0 or 1 was transmitted. Remember that the MAP rule is the one that minimizes the probability of error.

$$\frac{A_0 \cdot u_0^{k}}{A_1 \cdot u_1^{k}} \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} \frac{P_1}{P_0}$$

$$\frac{A_0}{A_1} \cdot \left(\frac{u_0}{u_1}\right)^{k} \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} \frac{P_1}{P_0}$$

$$\left(\frac{u_0}{u_1}\right)^{k} \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} \frac{P_1}{P_0} \cdot \left(\frac{A_1}{A_0}\right)$$

$$\ln\left(\left(\frac{u_0}{u_1}\right)^{k}\right) \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} \ln\left(\frac{P_1}{P_0} \cdot \left(\frac{A_1}{A_0}\right)\right)$$

$$k \cdot \ln\left(\frac{u_0}{u_1}\right) \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} \ln\left(\frac{P_1}{P_0} \cdot \left(\frac{A_1}{A_0}\right)\right)$$

$$k \underset{\text{“}H_1\text{”}}{\overset{\text{“}H_0\text{”}}{\gtrless}} \frac{\ln\left(\frac{P_1}{P_0} \cdot \left(\frac{A_1}{A_0}\right)\right)}{\ln\left(\frac{u_0}{u_1}\right)}$$

$$k \underset{\text{"H}_1\text{"}}{\overset{\text{"H}_0\text{"}}{\underset{<}{\overset{>}{-}}}} \frac{\ln\left(\frac{P_1}{P_0} \cdot \left(\frac{1-u_1}{1-u_0}\right)\right)}{\ln\left(\frac{u_0}{u_1}\right)}$$

$$\boxed{k \underset{\text{"H}_1\text{"}}{\overset{\text{"H}_0\text{"}}{\underset{<}{\overset{>}{-}}}} \frac{\ln(P_1) + \ln(1-u_1) - \ln(P_0) - \ln(1-u_0)}{\ln(u_0) - \ln(u_1)}}$$

$$k \underset{\text{"H}_1\text{"}}{\overset{\text{"H}_0\text{"}}{\underset{<}{\overset{>}{-}}}} c \qquad \text{where } c = \frac{\ln(P_1) + \ln(1-u_1) - \ln(P_0) - \ln(1-u_0)}{\ln(u_0) - \ln(u_1)}$$

Since c can be any real number but k is an integer:

$$k \underset{\text{"H}_1\text{"}}{\overset{\text{"H}_0\text{"}}{\underset{<}{\overset{>}{-}}}} \lceil c \rceil$$

(if $k \geq \lceil c \rceil$ then "H_0" is selected, otherwise if $k < \lceil c \rceil$ then "H_1" is selected.)

So what is the error probability?

Error Probability

$$\begin{aligned}
\Pr(\text{Error}) &= \Pr(\text{Error} \cap H_0) + \Pr(\text{Error} \cap H_1) \\
&= \Pr(H_0) \cdot \Pr(\text{Error} \mid H_0) + \Pr(H_1) \cdot \Pr(\text{Error} \mid H_1) \\
&= P_0 \cdot \Pr(\text{Error} \mid H_0) + P_1 \cdot \Pr(\text{Error} \mid H_1)
\end{aligned}$$

$$= P_0 \cdot \sum_{k=0}^{\lceil c \rceil - 1} (1-u_0) \cdot u_0^{\,k} + P_1 \cdot \sum_{k=\lceil c \rceil}^{\infty} (1-u_1) \cdot u_1^{\,k}$$

$$= \boxed{(1-u_0) \cdot \frac{1-u_0^{\lceil c \rceil}}{1-u_0} \cdot P_0 + (1-u_1) \cdot \frac{u_1^{\lceil c \rceil}}{1-u_1} \cdot P_1}$$