University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: Communications I

Fall 2002

**Problem Set 6**

**Review of Random Variables, Decision Rules, Estimators**

**Issued:** Thursday, Oct. 17th. **Due:** Thursday, October 24th (beginning of lecture).

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**Reading from Proakis (2nd Edition):** Chapter 4, Section 4.1.

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### Problem 6.1 (Optional)

A Gaussian random variable $X$ of zero mean and variance $\sigma_X^2$ is transformed to random variable $Y$ via the transformation

$$Y = X^2 \; .$$

Show that the probability density function of $Y$ is given by

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}\sigma_X} exp\left(-\frac{y}{2\sigma_X^2}\right) \; , & y \geq 0 \\ 0 \; , & y < 0 \; . \end{cases}$$

### Problem 6.2

The joint pdf of two random variables $X$ and $Y$ is given by

$$f_{X,Y}(x,y) = \begin{cases} A(1-|x-y|), & 0<x<1, \; 0<y<1 \\ 0, & \text{otherwise.} \end{cases}$$

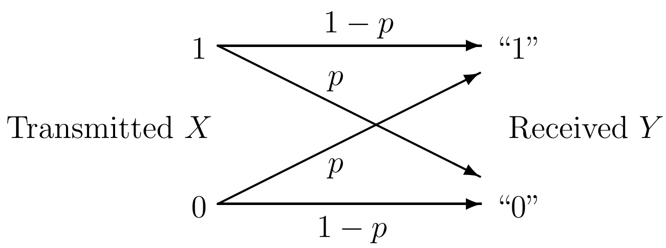
(a) Find $A$.

(b) **(optional)** Find the marginal pdf of $X$ and $Y$.

(c) Find $\Pr(X+Y<1 \mid X>\frac{1}{2})$.

### Problem 6.3 (Optional)

The discrete, binary, symmetric and memoryless communication channel shown below can transmit one bit (i.e., at each time step, $X$ could be 0 or 1). The number next to an arrow denotes the conditional probability of receiving the bit to the right of the arrow given transmission of the bit to the left of the arrow (transmissions between different time steps are independent).




The above channel with $p = \frac{1}{4}$ is used to transmit two symbols $s_0$ and $s_1$. Symbol $s_0$ is transmitted with probability $\Pr(s_0) = 1/3$ and symbol $s_1$ is transmitted with probability $\Pr(s_1) = 2/3$.

(i) Suppose that $s_0$ is encoded into the two-bit sequence 00 and that $s_1$ is encoded into the two-bit sequence 11. Determine the decoding rule that the receiver should use to minimize the probability of error. In particular, you need to specify a rule that minimizes the probability of error and determines what symbol was sent (given the reception of a "00," "01," "10" or "11"). What is the corresponding (minimal) probability of error?

(ii) In order to increase the communication rate of the system, while keeping the probability of error low, two symbols are encoded into a three-bit sequence as follows: $s_0 s_0$ is encoded into 011, $s_1 s_1$ is encoded into 111, $s_1 s_0$ is encoded into 101 and $s_0 s_1$ is encoded into 000. Suppose that the receiver receives "011." Given that your objective is to minimize the probability of error, determine what the transmitted pair of symbols was.

**Problem 6.4**

(a) Show that if two random variables $X$ and $Y$ are related by

$$y = \alpha x + \beta$$

for arbitrary constants $\alpha$ and $\beta$, then their correlation coefficient $\rho_{yx}$ equals 1 if $\alpha > 0$ and equals $-1$ if $\alpha < 0$.

(b) Given that $X = \cos\Phi$ and $Y = \sin\Phi$, where $\Phi$ is a uniform random variable in the interval $[0, 2\pi]$, show that $X$ and $Y$ are uncorrelated but not independent.

(c) Determine whether the following statements are TRUE or FALSE.

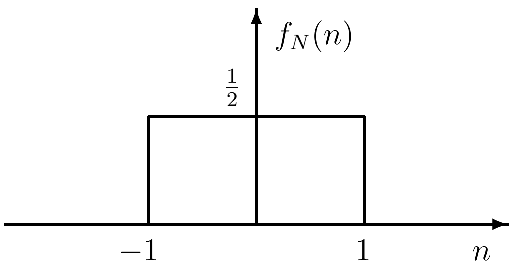
(i) $E[X + Y] = E[X] + E[Y]$ regardless of whether random variables $X$ and $Y$ are independent.

(ii) Let $X$ and $Y$ be two independent random variables with zero mean and unit variance and let $Z = X + 2Y$. Then, random variable $Z$ has variance 3.

(iii) If $X$ is a zero-mean random variable, the linear estimator of $X$ based on some random variable $Y$ which also has zero mean will necessarily be of the form $\widehat{X}(y) = \alpha y$ (where $\alpha$ is a constant and $y$ is the observed value of $Y$).

**Problem 6.5**

Consider the following communication scenario: under hypothesis $H_0$ no signal is transmitted and the receiver observes a (constant) signal $R = N$, whereas under hypothesis $H_1$ a constant signal is transmitted and the receiver observes $R = 1 + N$. The random variable $N$ models noise in the channel and is assumed to have the pdf shown below.




The prior probabilities for hypotheses $H_0$ and $H_1$ are given by

$$\Pr(H_0) = 1/4\ , \qquad \Pr(H_1) = 3/4\ .$$

As we discussed in class, given a received signal $R = r$, we can use the MAP rule at the receiving end in order to minimize the probability of error. Furthermore, we know that the MAP rule reduces to the likelihood ratio test

$$\frac{f_{R|H_1}(r|H_1)}{f_{R|H_0}(r|H_0)} \underset{`H_0\text{'}}{\overset{`H_1\text{'}}{\gtrless}} \frac{\Pr(H_0)}{\Pr(H_1)}\ .$$

(a) Find the range or ranges of values of the observation $r$ for which you would decide that the signal was transmitted ("$H_1$" has taken place).

(b) Find the corresponding (minimal) probability of error.

**Problem 6.6**

Consider a system for determining whether a certain communication channel is being used or not. Let $H_1$ denote the hypothesis that the channel is being used, and let $H_0$ denote the hypothesis that the channel is not being used. The decision between hypotheses $H_1$ and $H_0$ is based on a single scalar measurement $R$ (e.g., the output of our antenna at a particular time).

This measurement $R$ is a zero-mean Gaussian random variable that has larger variance if the radio channel is indeed being used. More specifically,

$$H_0 \quad : \quad f_{R|H_0}(r|H_0) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2}$$

$$H_1 \quad : \quad f_{R|H_1}(r|H_1) = \frac{1}{\sqrt{4\pi}} e^{-r^2/4} \ .$$

Assume that the a priory probabilities for these two hypotheses are $\Pr(H_0) = p_0$ and $\Pr(H_1) = p_1$.

(a) Find the decision rule that minimizes the probability of error.

(b) Find the probability of error for the decision rule in part (a). Express your answer as

$$\Pr(\text{error}) = \alpha_1 Q(\gamma_1) + \alpha_2 Q(\gamma_2) + \alpha_3 \ ,$$

where $\alpha_1$, $\alpha_2$, $\alpha_3$, $\gamma_1$ and $\gamma_2$ are appropriate constants, and $Q(\gamma)$ is defined as

$$Q(\gamma) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\tau^2/2} d\tau \ .$$

(c) Is there a choice of $p_0$ and $p_1$ such that the rule of part (b) always decides in favor of one of the hypothesis regardless of the measurement $R$?

**Problem 6.7**

A signal $X$ which has Gaussian distribution with mean $\mu_X = 2$ and variance $\sigma_X^2 = 0.5$ is sent via a cable. The receiver needs to form an estimate $\widehat{X}$ for the transmitted signal $X$ based on the received signal $Y = X + N$, where $N$ models the (*additive*) noise. Assume that $N$ also has Gaussian distribution with mean $\mu_N = 0$ and variance $\sigma_N^2 = 2$. Find $\widehat{X}_{\text{MMSE}}$, the minimum mean square error estimator for $X$ given $Y$.