ECE 459 Midterm 2 Solution

Problem 1

Part A (i) True. X^2 & Y^2 are indep.

$$\begin{aligned} Cov(X^2, Y^2) &= E[(X^2 - E(X^2))(Y^2 - E(Y^2))] \\ &= E[X^2]E[Y^2] - 2E[X^2]E[Y^2] + E[Y^2]E[X^2] \\ &= 0 \end{aligned}$$

(ii) True.

$$E(Z) = 2\mu_X + 3\mu_Y = 0$$

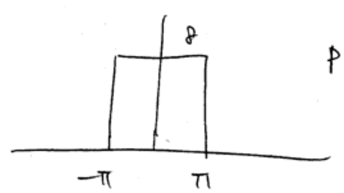
$$\begin{aligned} E[Z^2] &= 4E[X^2] + 9E[Y^2] + 12E[XY] \\ &= 4 + 36 + 12 \cdot 0.75 \cdot 1 \cdot 2 = 58 = \sigma_Z^2 \end{aligned}$$

(iii) False. Sinc fcn cannot correspond to PSD $\because$ it is NOT non-negative.

(iv) True. More information helps reducing the estimation error

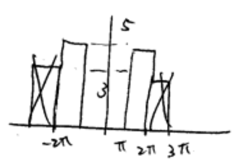
(v) False.

PSD for $N_I^{(1)}$



$$P_{ave} = \frac{1}{2\pi} \cdot 2\pi \cdot 8 = 8$$

PSD for $N_I^{(2)}$



$$P_{ave} = \frac{1}{2\pi} \cdot 5 \cdot 2\pi = 5$$

Part B

(i) power $= \frac{A_c^2}{2} = 32$ watt

(ii) $\beta = \frac{k_p \cdot A_m}{\omega_m} = 2$, $B = 2\omega_m(\beta+1) = 2\pi \times 6 \times 10^3$ rad/s

Problem 2

(a)

	$\frac{1}{2}$ $S_0 = 000$	$\frac{1}{4}$ $S_1 = 101$	$\frac{1}{8}$ $S_2 = 110$	$\frac{1}{8}$ $S_3 = 011$
"000"	$\frac{1}{2}(\frac{3}{4})^3$	$\frac{1}{4}(\frac{1}{4})^2(\frac{3}{4})$	$(\frac{1}{8})(\frac{1}{4})^2(\frac{3}{4})$	$(\frac{1}{8})(\frac{1}{4})^2(\frac{3}{4})$
"001"	$\frac{1}{2}(\frac{1}{4})(\frac{3}{4})^2$	$\frac{1}{4}(\frac{3}{4})^2(\frac{1}{4})$	$\frac{1}{8}(\frac{1}{4})^3$	$\frac{1}{8}(\frac{3}{4})^2(\frac{1}{4})$
"101"	$\frac{1}{2}(\frac{1}{4})^2(\frac{3}{4})$	$\frac{1}{4}(\frac{3}{4})^3$	$\frac{1}{8}(\frac{3}{4})(\frac{1}{4})^2$	$\frac{1}{8}(\frac{3}{4})(\frac{1}{4})^2$

"000" $\rightarrow$ 000

"001" $\rightarrow$ 000

"101" $\rightarrow$ 101

$Pr(\text{error} \mid 001)$

$$= \frac{9/4 + 1/8 + 9/8}{9/2 + 9/4 + 1/8 + 9/8} = \frac{7}{16} = 0.4375$$

(b) "000" $\rightarrow$ 000 , 101

"001" $\rightarrow$ 000 , 101

"101" $\rightarrow$ 101 , 000

Problem 3

part A . $\mu_X = E[X] = 1$

$\mu_Y = E[W] \cdot E[X] + E[N] = 3/4$

$E[XY] = E[W] \cdot E[X^2] + E[X] \cdot E[N]$

$= \frac{3}{4} \int_1^2 \frac{1}{2} x^2 dx + 0 = 1$

$\sigma_{XY} = E[XY] - E[X]E[Y] = 1 - 3/4 = 1/4$ $\quad - E[Y]^2$

$\sigma_Y^2 = E[Y^2] - E[Y]^2 = E[W^2]E[X^2] + E[N^2] = \frac{16}{9} - \left(\frac{3}{4}\right)^2 = \frac{175}{144}$

where $E[W^2] = \int_{1/2}^{1} 2 \cdot \omega^2 d\omega = 2/3 \left(1 - \frac{1}{8}\right) = \frac{7}{12}$

$\hat{X}_{LMMSE} = 1 + \frac{1/4}{175/144} \left(y - \frac{3}{4}\right)$

part B: If ω is known

$y = \omega X + N$

$\mu_X = 1$

$\mu_Y = w \cdot E[X] + 0 = \omega$

$\sigma_{XY} = \omega \cdot E[X^2] + E[X] \cdot E[N] - 1 \cdot \omega = \omega/3$

$\sigma_Y^2 = \omega^2 E[X^2] + E[N^2] - E[Y]^2 = \frac{1}{3}\omega^2 + 1$

$\hat{X}'_{LMMSE}(y) = 1 + \frac{\frac{1}{3}\omega}{\frac{1}{3}\omega^2 + 1} (y - \omega)$

Problem 4

part A

$$\frac{f_{R|H_0}(r|H_0)}{f_{R|H_1}(r|H_1)} \underset{\text{"}H_1\text{"}}{\overset{\text{"}H_0\text{"}}{\gtrless}} \frac{P_1}{P_0}$$

$$\rightarrow \frac{r\cdot e^{-\frac{r^2}{2}}}{r/2 \cdot e^{-r^2/4}} \gtrless 2 \rightarrow e^{-\frac{r^2}{4}} \gtrless 1$$

$$\rightarrow -\frac{r^2}{4} \underset{\text{"}H_1\text{"}}{\overset{\text{"}H_0\text{"}}{\gtrless}} \ln 1 = 0 \Rightarrow \text{Always say "}H_1\text{"}$$

$$P_e = P_0 = 1/3$$

part B

(a) $E[x(t+\tau)x(t)] = E[w(t+\tau)w(t)]$

$+ E[w(t+\tau)v(t)]$

$+ E[v(t+\tau)w(t)]$

$+ E[v(t+\tau)v(t)]$

$= R_{ww}(\tau) + R_{vv}(\tau) + 0 + 0$

$\Rightarrow$ w.s.s.

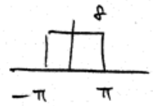
(b) $S_{xx}(\omega) = S_{vv}(\omega) + S_{ww}(\omega)$

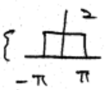
$$= 1 + \frac{144}{25+\omega^2} = \frac{169+\omega^2}{25+\omega^2} = \frac{(13+j\omega)(13-j\omega)}{(5+j\omega)(5-j\omega)}$$

$$\Rightarrow H_w(\omega) = \frac{5+j\omega}{13+j\omega} = 1 - \frac{8}{13+j\omega}$$

$$\rightarrow h_w(t) = \delta(t) - 8\cdot e^{-13t}u(t)$$

Problem 5

(a) $R_{XX}(\tau) = R_{YY}(\tau) = \frac{1}{4}\, \mathcal{F}^{-1}\{$ 

$\} = \mathcal{F}^{-1}\{$ 

$\}$

$$= 2\,\frac{\pi}{\pi}\,\mathrm{sinc}(\pi t) = 2\,\mathrm{sinc}(\pi t)$$

(b)

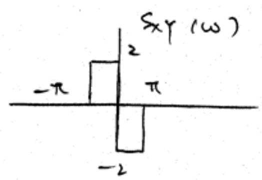
$Z \sim N(0, 2)$

(c)

$Z \sim N(0, 2)$
$W \sim N(0, 2)$ } independent

$\Rightarrow f_{Z,W}(z, w) = f_Z(z)\, f_W(w)$

$\because S_{XY} = j\,[\,S_{NN}(\omega + \omega_c) - S_{NN}(\omega - \omega_c)\,] \quad |\omega| < \omega_c$



$\Rightarrow R_{XY}(0) = \int_{-\infty}^{\infty} S_{XY}(\omega)\, d\omega = 0$