University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 459: Communications I

Fall 2004

Problem Set 5

Probability Review, Hypothesis Testing, Detection and Estimation

Issued: Thursday, Oct. 14th. **Due:** Thursday, Oct. 21st (beginning of lecture).

Reading from Lathi: Chapter 10.
Reading from Haykin (3rd Edition): Chapter 4, Sections 4.1–4.5.

Problem 5.1 (Optional)

Problems 10.5-2 and 10.5-3 from Lathi, p. 486.

Problem 5.2

A Gaussian random variable X with zero mean and variance σ_X^2 is transformed to random variable Y via the transformation

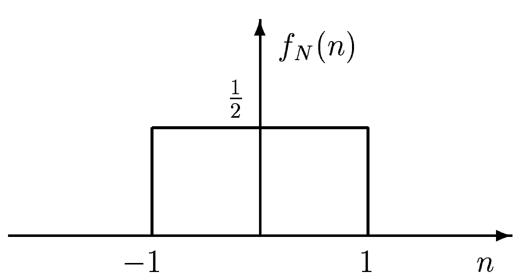
$$Y = X^2 .$$

Show that the probability density function of Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}\sigma_X} \exp\left(-\frac{y}{2\sigma_X^2}\right) , & y \geq 0 \\ 0 , & y < 0 . \end{cases}$$

Problem 5.3

Consider the following communication scenario: under hypothesis H_0 no signal is transmitted and the receiver observes a (constant) signal $R = N$, whereas under hypothesis H_1 a constant signal is transmitted and the receiver observes $R = 1 + N$. The random variable N models noise in the channel and is assumed to have the pdf shown below.



The prior probabilities for hypotheses H_0 and H_1 are given by

$$\Pr(H_0) = 1/4 \; , \qquad \Pr(H_1) = 3/4 \; .$$

As we discussed in class, given a received signal $R = r$, we can use the MAP rule at the receiving end in order to minimize the probability of error. Furthermore, we know that the MAP rule reduces to the likelihood ratio test

$$\frac{f_{R|H_1}(r|H_1)}{f_{R|H_0}(r|H_0)} \underset{\text{`}H_0\text{'}}{\overset{\text{`}H_1\text{'}}{\gtrless}} \frac{\Pr(H_0)}{\Pr(H_1)} \; .$$

(a) Find the range or ranges of values of the observation r for which you would decide that the signal was transmitted ("H_1" has taken place).

(b) Find the corresponding (minimal) probability of error.

Problem 5.4

Consider a system for determining whether a certain communication channel is being used or not. Let H_1 denote the hypothesis that the channel is being used, and let H_0 denote the hypothesis that the channel is not being used. The decision between hypotheses H_1 and H_0 is based on a single scalar measurement R (e.g., the output of our antenna at a particular time). This measurement R is a zero-mean Gaussian random variable that has larger variance if the radio channel is indeed being used. More specifically,

$$H_0 \; : \; f_{R|H_0}(r|H_0) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2}$$

$$H_1 \; : \; f_{R|H_1}(r|H_1) = \frac{1}{\sqrt{4\pi}} e^{-r^2/4} \; .$$

Assume that the a priory probabilities for these two hypotheses are $\Pr(H_0) = p_0$ and $\Pr(H_1) = p_1$.

(a) Find the decision rule that minimizes the probability of error.

(b) Find the probability of error for the decision rule in part (a). Express your answer as

$$\Pr(\text{error}) = \alpha_1 Q(\gamma_1) + \alpha_2 Q(\gamma_2) + \alpha_3 \; ,$$

where α_1, α_2, α_3, γ_1 and γ_2 are appropriate constants, and $Q(\gamma)$ is defined as

$$Q(\gamma) = \int_\gamma^\infty \frac{1}{\sqrt{2\pi}} e^{-\tau^2/2} d\tau \; .$$

(c) Is there a choice of p_0 and p_1 such that the rule of part (b) always decides in favor of one of the hypothesis regardless of the measurement R?

Problem 5.5

The joint pdf of two random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} A(1-|x-y|), & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find A.

(b) **(optional)** Find the marginal pdf of X and Y.

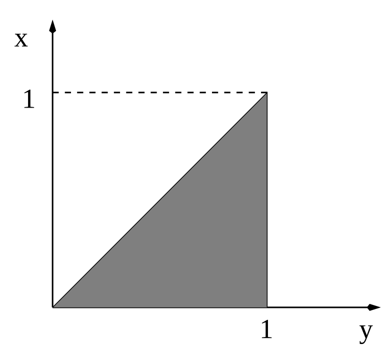
(c) Find $\Pr(X+Y<1 \mid X > \frac{1}{2})$.

Problem 5.6

A signal X which has Gaussian distribution with mean $\mu_X = 2$ and variance $\sigma_X^2 = 0.5$ is sent via a cable. The receiver needs to form an estimate $\widehat{X}$ for the transmitted signal X based on the received signal $Y = X + N$, where N models the (*additive*) noise. Assume that N also has Gaussian distribution with mean $\mu_N = 0$ and variance $\sigma_N^2 = 2$. Find $\widehat{X}_{\text{MMSE}}$, the minimum mean square error estimator for X given Y.

Problem 5.7

Random variables X and Y have joint pdf $f_{X,Y}(x,y)$ that is constant in the shaded region (and zero elsewhere).



(a) Make fully labeled sketches of the densities $f_X(x)$ and $f_Y(y)$.

(b) Are X and Y statistically independent? Explain.

(c) Determine $\widehat{X}_{MMSE}(y)$, the minimum mean square error estimator for X, given the observation $Y = y$.