University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 459: Communications I

Fall 2004

# Problem Set 8

## Signal Detection in Noise, Matched Filtering, Noise Analysis in AM Schemes

**Issued:** Thurs., November 11th. **Due:** Thurs., November 18th (beginning of lecture).

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**Reading from Lathi:** Chapter 13, Sections 13.1–13.2; Chapter 12, Sections 12.1–12.2.
**Reading from Haykin (3rd Edition):** Chapter 7, Sections 7.1–7.3; Chapter 5, Sections 5.1–5.4.

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### Problem 8.1 (Optional)

Consider a bandpass Gaussian noise process $N(t)$ with zero mean and power spectral density

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}\,, & \omega_c - 2\pi B \le \omega \le \omega_c + 2\pi B\,, \quad -\omega_c - 2\pi B \le \omega \le -\omega_c + 2\pi B\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Find the probability density function of a sample of the envelope of $n(t)$, i.e., find $f_E(e)$ of the random variable $E = c(t)$, where $c(t)$ is the envelope of the band-limited random process $N(t)$.

### Problem 8.2

Consider the signal

$$s(t) = \begin{cases} A/2\,, & 0 \le t \le T/2\,, \\ -A/2\,, & T/2 \le t \le T\,. \end{cases}$$

(a) Sketch the impulse response of a filter that is matched to this signal.

(b) Plot the output of the matched filter in part (a) when $s(t)$ is the input.

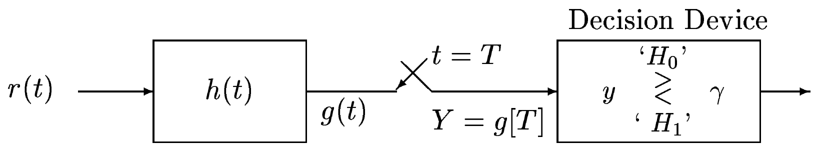
### Problem 8.3

In a binary on-off signaling scheme the transmitted signal $s(t)$ is either set to zero (under hypothesis $H_1$), or is given by

$$s(t) = \begin{cases} A, & 0 \le t \le T\,, \\ 0, & \text{otherwise} \end{cases}$$

(under hypothesis $H_0$). The a priory probabilities for the two hypotheses are given by $\Pr(H_0) = p_0$ and $\Pr(H_1) = p_1$. The signal $r(t)$ seen at the receiving end includes additive noise, i.e.,

$r(t) = s(t) + n(t)$, where $n(t)$ is a sample function of a white Gaussian random process $N(t)$ with autocorrelation function $R_{NN}(\tau) = \frac{N_0}{2}\delta(\tau)$. In order to make a decision as to whether hypothesis $H_0$ or $H_1$ took place we use the approach shown below.

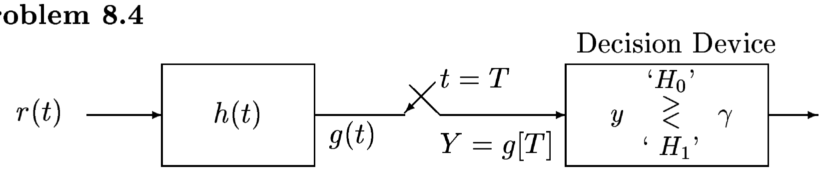



The impulse response $h(t)$ is given by $h(t) = 1$ for $0 \le t \le T$ (0 otherwise). At the decision device, the value $y$ observed for random variable $Y$ (i.e., the value of $g(t)$ at time $t = T$) is compared to a threshold $\gamma$. The decision is "$H_0$" if $y > \gamma$, and "$H_1$" otherwise.

(a) Find $f_{Y|H_0}(y|H_0)$, the conditional probability density of $Y$ given hypothesis $H_0$.

(b) Choose $\gamma$ so that the probability of error is minimized.

**Problem 8.4**




Consider the binary hypothesis testing situation shown above. Signal $r(t) = s(t) + n(t)$, where $s(t)$ is a deterministic signal and $n(t)$ is noise. Under hypothesis $H_0$, signal $s(t)$ is given by

$$s(t) = \begin{cases} \cos(\frac{\pi}{2}t)\,, & 0 \le t \le 2\,, \\ 0, & \text{otherwise.} \end{cases}$$

Under hypothesis $H_1$, signal $s(t)$ is zero. Assume that the a priory probabilities for the two hypotheses are equal and that the noise $n(t)$ is a sample function of a white Gaussian random process $N(t)$ with average power $N_0/2$. At the decision device, the value $y$ observed for random variable $Y$ is compared to a threshold $\gamma$. The decision is "$H_0$" if $y > \gamma$, and "$H_1$" otherwise.

(a) Let $T = 1$ and $h(t)$ be given by

$$h(t) = \begin{cases} 1, & 0 \le t \le T\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Find $f_{Y|H_0}(y|H_0)$ and $f_{Y|H_1}(y|H_1)$, the conditional probability densities of $Y$ given each of the two hypotheses. Choose $\gamma$ to minimize the probability of error and find the corresponding probability of error.

(b) Repeat part (a) for $T = 2$.

(c) Let $T = 2$. Choose $h(t)$ and $\gamma$ so that you minimize the probability of error. What is the corresponding probability of error?

**Problem 8.5**

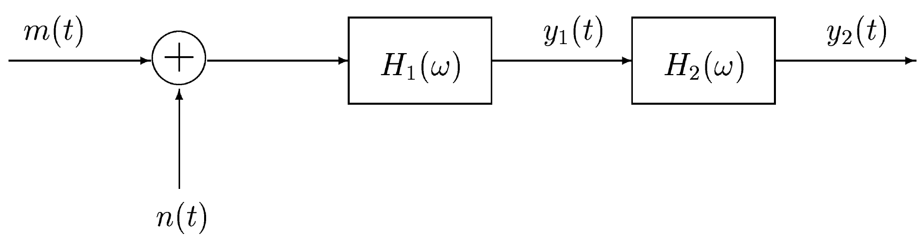
The signal $m(t) = A_c \cos(\omega_c t)$ (where $A_c$ and $\omega_c$ are constants) is corrupted by additive white Gaussian noise. More specifically, the corrupted signal $x(t)$ is given by

$$x(t) = m(t) + n(t) \ ,$$

where $n(t)$ is a sample path of a white Gaussian random process $N(t)$ with zero mean and power spectral density $S_{NN}(\omega) = \frac{N_0}{2}$. Find an expression for the output signal-to-noise ratio after the signal $x(t) = m(t) + n(t)$ is applied to an LTI filter with impulse response $h(t) = e^{-t}u(t)$.

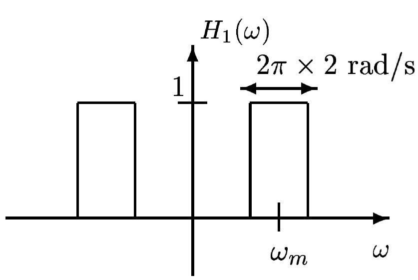
**Problem 8.6**

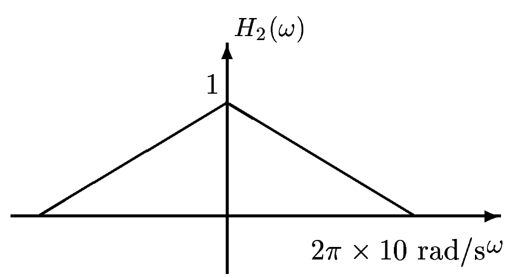
The message signal $m(t) = \cos(\omega_m t)$ gets corrupted by additive white Gaussian noise $N(t)$ with zero mean and power spectral density $S_{NN}(\omega) = 10^{-1}$ W/rad/s. The resulting signal gets filtered by the cascade of the two filters shown below.




The frequency response of the filters is given by

$$H_1(\omega) = \begin{cases} 1, & \omega_m - 2\pi \text{ rad/s} < |\omega| < \omega_m + 2\pi \text{ rad/s} \ , \\ 0, & \text{otherwise.} \end{cases} \qquad H_2(\omega) = \begin{cases} 1 - \frac{|\omega|}{2\pi \times 10}, & |\omega| < 2\pi \times 10 \text{ rad/s} \ , \\ 0, & \text{otherwise.} \end{cases}$$




(a) For this part, assume that $\omega_m = 2\pi \times 5$ rad/s.

(i) Find the signal to noise ratio at the output $y_1(t)$ of filter $H_1(\omega)$.

(ii) Find the signal to noise ratio at the output $y_2(t)$ of filter $H_2(\omega)$.

(b) Repeat part (a) for $\omega_m = 2\pi \times 9$ rad/s.

**Note:** The signal to noise ratio is defined as

$$\text{SNR} = \frac{\text{Power of component of } y(t) \text{ that is due to } m(t)}{\text{Average power of component of } y(t) \text{ that is due to } n(t)}\,.$$