University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 359: COMMUNICATIONS I

Fall 2000

Final Exam

Tuesday, December 12, 1:30–4:30pm, 161 Everitt Laboratory

READ THESE COMMENTS BEFORE STARTING THE EXAM!!!

- This is a **closed-book** exam, but **three** sheets of notes (both sides) are allowed. Calculators should not be necessary but feel free to use one.
- **Write your name on the answer booklet.**
- There are **five** problems. The weighting is **not** equal and is indicated within each problem.
- A correct answer does not guarantee credit; an incorrect answer does not guarantee loss of credit. **Provide clear explanations, show all relevant work and justify your answers!** If we cannot make sense of your writing or reasoning, you may lose points.
- Read each problem carefully and *think* before performing detailed calculations.
- Only the supplied answer booklet is to be handed in. **No additional pages will be considered in the grading.** You may want to work things through in the blank areas of the exam and then neatly transfer to the answer sheet the work you would like us to look at.

Useful Formulas:

- $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
- $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
- $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$
- $\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$
- $\sin a \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\sin^2\theta + \cos^2\theta = 1$
- $e^{j\theta} = \cos\theta + j\sin\theta$
- $\mathcal{FT}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + j\omega}, \quad \alpha > 0, \qquad \mathcal{FT}\{e^{\alpha t}u(-t)\} = \frac{1}{\alpha - j\omega}, \quad \alpha > 0$
- $\mathcal{FT}\{\frac{W}{\pi}\text{sinc}(Wt)\} = \text{rect}(\frac{\omega}{2W})$
- $\mathcal{HT}\{\cos(\omega_0 t)\} = \sin(\omega_0 t)$

Problem 1 (10/100, equally weighted parts)

Consider a wide-sense stationary (WSS) Gaussian random process $N(t)$ with zero mean and autocorrelation function

$$R_{NN}(\tau) = \begin{cases} 3 - |\tau|\,, & -3 \leq \tau \leq 3 \\ 0\,, & \text{otherwise.} \end{cases}$$

(a) What is the average power of the random process $N(t)$? What is the pdf $f_Y(y)$ of the random variable $Y = N(2)$, the sample of the random process $N(t)$ at time $t = 2$?

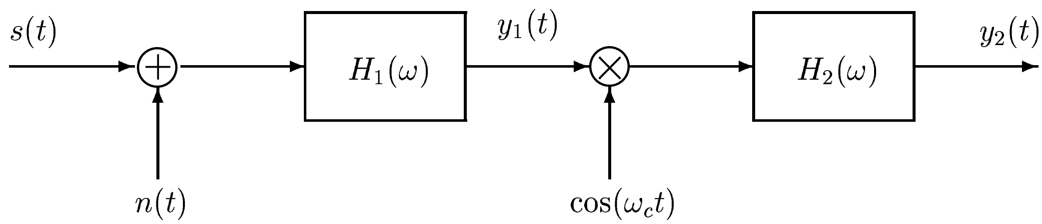
(b) Let $X = N(1)$ and $Y = N(2)$; find $\hat{Y}_{LMMSE}(x)$, the linear minimum mean square error (LMMSE) estimator of Y given observation $X = x$.

Problem 2 (16/100, equally weighted parts)

The (deterministic) message signal

$$m(t) = 40\cos(2\pi \times 20t) + 20\sin(2\pi \times 10t)$$

is modulated using a DSB-SC modulation scheme with center frequency $\omega_c = 2\pi \times 10^6$rad/s. The transmitted signal $s(t) = m(t)\cos(\omega_c t)$ gets corrupted by additive white noise $n(t)$ with power spectral density $S_{NN}(\omega) = 10^{-1}$W/rad. The resulting signal $r(t) = s(t) + n(t)$ is processed using the following arrangement.



The frequency responses of $H_1(\omega)$ and $H_2(\omega)$ are given by

$$H_1(\omega) = \begin{cases} 1, & 2\pi \times (10^6 - 25) < |\omega| < 2\pi \times (10^6 + 25) \\ 0, & \text{otherwise,} \end{cases}$$

$$H_2(\omega) = \begin{cases} 1, & |\omega| < 2\pi \times 25 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the signal-to-noise ratio at the output $y_1(t)$ of filter $H_1(\omega)$.

(b) Find the signal-to-noise ratio at the output $y_2(t)$ of filter $H_2(\omega)$.

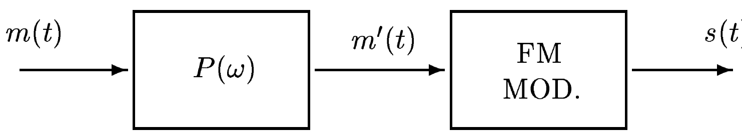
Note: The signal-to-noise ratio is defined as

$$SNR_i = \frac{\text{Power of component of } y_i(t) \text{ that is due to } m(t)}{\text{Expected power of component of } y_i(t) \text{ that is due to } n(t)}.$$

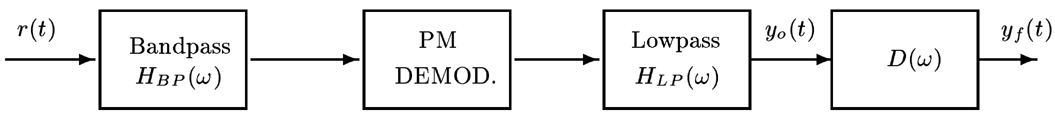
Problem 3 (25/100, equally weighted parts)

A message signal $m(t)$, with known average power P and with bandwidth $2\pi B$ (i.e., $M(\omega) = 0$ outside the frequency interval $[-2\pi B, 2\pi B]$), is transmitted using the angle modulation scheme shown below: $m(t)$ is first filtered using $P(\omega)$ and the resulting signal $m'(t)$ is FM-modulated so that

$$s(t) = A\cos\left[\omega_c t + k_f \int_{-\infty}^{t} m'(\alpha)d\alpha\right] .$$



The transmitted signal $s(t)$ goes through a distortionless channel and gets corrupted by additive white Gaussian noise $n(t)$ with power spectral density $S_{NN}(\omega) = \mathcal{N}/2$. The received signal $r(t) = s(t) + n(t)$ is demodulated using the following approach.



The bandpass and lowpass filters have the following frequency responses (where $2(\Delta\omega + 2\pi B)$ is the bandwidth of $s(t)$ as given by Carson's rule):

$$H_{BP}(\omega) = \begin{cases} 1, & \omega_c - (\Delta\omega + 2\pi B) < |\omega| < \omega_c + (\Delta\omega + 2\pi B) \\ 0, & \text{otherwise} \end{cases}$$

$$H_{LP}(\omega) = \begin{cases} 1, & |\omega| < 2\pi B \\ 0, & \text{otherwise.} \end{cases}$$

Assume that the conditions we discussed in class hold so that the output $y_o(t)$ of the lowpass filter can be written as

$$y_o(t) = \psi(t) + x_s(t) ,$$

where $\psi(t)$ depends exclusively on the message signal $m(t)$ and $x_s(t)$ is a sample path of a wide-sense stationary (WSS) random process $X_s(t)$ that depends exclusively on the noise $n(t)$.

Part A. For Part A, assume that $P(\omega) = 1$.

(a) State what $\psi(t)$ is in terms of $m(t)$.

(b) Sketch the power spectral density of random process $X_s(t)$. What is the average power of $X_s(t)$?

(c) Find an appropriate choice of $D(\omega)$ so that the final output $y_f(t)$ contains a scaled version of $m(t)$ and a noise component $w(t)$, i.e.

$$y_f(t) = km(t) + w(t)$$

for some constant k. What is the signal-to-noise ratio at the output $y_f(t)$?

Part B. For Part B, assume that $P(\omega) = \frac{1}{j\omega}$.

(d) Find an appropriate choice for $D(\omega)$ so that the output $y_f(t)$ contains a scaled version of $m(t)$ and a noise component $w'(t)$, i.e.

$$y_f(t) = k'm(t) + w'(t) \ ,$$

where k' is a constant and $w'(t)$ is a sample path from a wide-sense stationary (WSS) random process $W'(t)$. Also determine the power spectral density of the random process $W'(t)$.

(e) Find the signal-to-noise ratio at the output $y_f(t)$. For what values of B will the scheme in **Part B** have a higher signal-to-noise ratio than the scheme in **Part A**?

Problem 4 (25/100, equally weighted parts)

In this problem we analyze a signal detection scheme in a communication channel that distorts the transmitted signal. The transmitter sends one bit ("0" or "1") by setting the transmitted signal $s(t)$ either to zero or to a pulse $p(t)$ given by:

$$p(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

The signal $s(t)$ goes through a communication channel that can be modeled as linear time-invariant (LTI) system with impulse response $c(t)$ as given below:

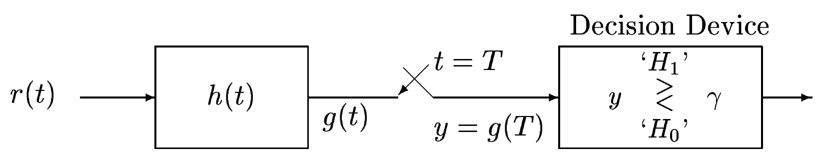
$$c(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

The signal $r(t)$ received at the receiver consists of the output $p'(t) = p(t) * c(t)$ of this communication channel corrupted by additive white Gaussian noise $n(t)$ with power spectral density $S_{NN}(\omega) = \mathcal{N}/2$.

The receiver is faced with the following binary hypothesis testing problem:

Hypothesis H_0: "0" being transmitted (no pulse is transmitted),
Hypothesis H_1: "1" being transmitted ($p(t)$ is transmitted).

The a priory probabilities for the two hypotheses are equal. In order to make a decision as to whether hypothesis H_0 or H_1 took place the receiver uses the system shown below.



Part A. For **Part A** assume that $T = 1$ and that $h(t) = p(1-t)$ (i.e., $h(t)$ is "matched" to pulse $p(t)$).

(a) Find $f_{Y|H_0}(y|H_0)$ and $f_{Y|H_1}(y|H_1)$, the conditional probability densities of the random variable $Y = g(1)$ under hypotheses H_0 and H_1.

(b) Choose γ so that you minimize the probability of error. Express the corresponding probability of error in terms of the Q-function.

(c) Given the restriction that $T = 1$, is there a better choice for the $h(t)$? If so, find $h(t)$, γ and the corresponding probability of error. If not, explain why.

Part B. In **Part B**, the sampling time T the filter $h(t)$ and the threshold γ will be determined so that the probability of error is minimized.

(d) Choose T, $h(t)$ and γ so that the probability of error is minimized.

(e) What is the corresponding probability of error and how does it compare to the probability of error in part (b)?

Problem 5 (24/100, equally weighted parts)

This problem has **six** independent **TRUE/FALSE** questions. Make sure you provide a sufficient **explanation** for your answer in each part.

A. If $h(t)$ denotes the impulse response of a BIBO stable system, then the system with impulse response $g(t) = h(t) * h(t)$ is also BIBO stable.

B. Let Y be a uniform random variable in the interval $[-1, 1]$ and let U be a discrete random variable that takes values 1 or -1 with equal probability, independent of Y. If random variable $X = YU$, then the linear minimum mean square error (LMMSE) estimator of Y given observation $X = x$ is given by

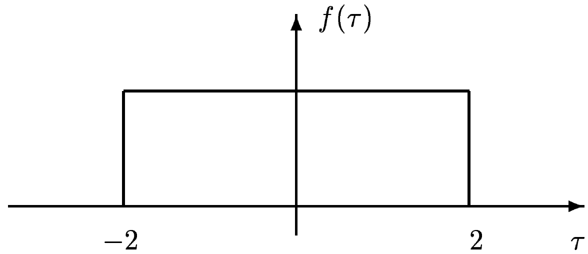
$$\hat{Y}_{LMMSE}(x) = x \; .$$

C. Let Y and X be two random variables. If $\hat{Y}_{LMMSE}(x) = \hat{Y}_{MMSE}(x)$, then Y and X are jointly Gaussian (j.G.) random variables.

[Note: $\hat{Y}_{LMMSE}(x)$ denotes the linear minimum mean square error (LMMSE) estimator of Y given observation $X = x$ and $\hat{Y}_{MMSE}(x)$ denotes the minimum mean square error (MMSE) estimator of Y given observation $X = x$.]

D. If random variables $X_1 \sim N(0,1)$ and $X_2 \sim N(0,1)$ are jointly Gaussian (j.G.), then $Y = X_1 + X_2$ must be a Gaussian random variable with mean zero and variance 2.

E. The following function $f(\tau)$ cannot be the autocorrelation function $R_{XX}(\tau)$ of a zero mean wide-sense stationary (WSS) random process $X(t)$:



F. If $X(t)$ is a Gaussian random process with zero mean and autocorrelation function $R_{XX}(\tau) = \dfrac{\sin(\pi\tau)}{\pi\tau}$, then $X(1)$ and $X(3)$ are independent random variables.