University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 459: Communications I

Fall 2004

Mid-Semester Exam II

Tuesday, November 9, 5:00–7:00pm, 165 Everitt Laboratory

READ THESE COMMENTS BEFORE STARTING THE EXAM!!

- This is a **closed-book** exam, but **two** sheets of notes (both sides) are allowed. Calculators should not be necessary, but feel free to use one.
- **Write your name on the answer booklet.**
- There are **five equally weighted** problems for a total of **100 points**. Problems are *not* necessarily in order of difficulty.
- A correct answer does not guarantee credit; an incorrect answer does not guarantee loss of credit. **Provide clear explanations, show all relevant work and justify your answers!** If we cannot make sense of your writing or reasoning, you may loose points.
- Read each problem carefully and *think* before performing detailed calculations.
- Only the supplied answer booklet is to be handed in. **No additional pages will be considered in the grading.** You may want to work things through in the blank areas of the exam and then neatly transfer to the answer sheet the work you would like us to look at.

Useful Formulas:

- $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
- $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
- $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$
- $\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$
- $\sin a \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\sin^2\theta + \cos^2\theta = 1$
- $e^{j\theta} = \cos\theta + j\sin\theta$
- $\mathcal{FT}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + j\omega}, \quad \alpha > 0, \qquad \mathcal{FT}\{e^{\alpha t}u(-t)\} = \frac{1}{\alpha - j\omega}, \quad \alpha > 0$
- $\mathcal{FT}\{\frac{W}{\pi}\mathrm{sinc}(Wt)\} = \mathrm{rect}(\frac{\omega}{2W})$
- $\mathcal{HT}\{\cos(\omega_0 t)\} = \sin(\omega_0 t)$

Problem 1 (20/100, equally weighted parts)

This problem has two **independent** parts.

Part A. Determine whether the following **five** statements are TRUE or FALSE.

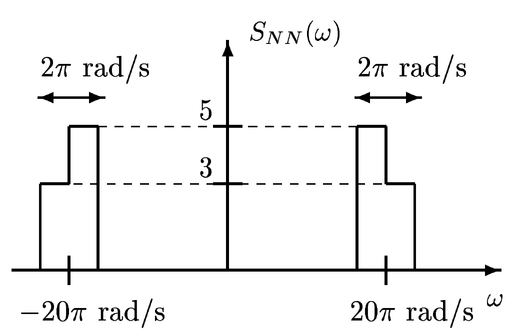
(i) **(2 points)** If X and Y are independent random variables then $\text{cov}(X^2, Y^2) = 0$.

(ii) **(2 points)** Let X and Y be two (not necessarily jointly Gaussian) random variables with means $\mu_x = \mu_y = 0$, variances $\sigma_x^2 = 1$ and $\sigma_y^2 = 4$, and correlation coefficient $\rho_{XY} = 0.75$. Then, the random variable $Z = 2X + 3Y$ has mean $\mu_Z = 0$ and variance $\sigma_Z^2 = 58$.

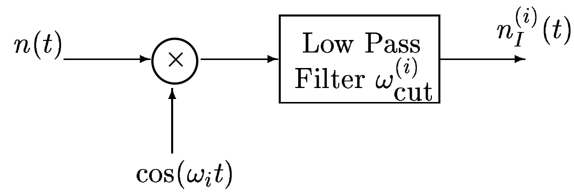
(iii) **(2 points)** The function $R_{XX}(\tau) = \text{rect}(1)$ could be the autocorrelation function of a wide sense stationary random process $X(t)$.

(iv) **(2 points)** Let X_1 and X_2 be two random variables that are jointly distributed with another random variable Y; let $\widehat{Y}(x_1) \equiv \alpha x_1 + \beta$ be the linear mean square error estimator (LMMSE) of Y based on the measurement $X_1 = x_1$ (where α and β are appropriate constants); let $\widehat{Y}'(x_1, x_2) \equiv \alpha' x_1 + \beta' x_2 + \gamma'$ be the linear mean square error estimator (LMMSE) of Y based on the measurements $X_1 = x_1$ and $X_2 = x_2$ (where α' and β' and γ' are appropriate constants). Then, the mean square error $E[(\widehat{Y}(x_1) - Y)^2]$ resulting from the use of estimator $\widehat{Y}(x_1)$ is always larger than or equal to the mean square error $E[(\widehat{Y}'(x_1, x_2) - Y)^2]$ resulting from the use of estimator $\widehat{Y}'(x_1, x_2)$.

(iv) **(2 points)** Consider the narrowband noise process $N(t)$ whose power spectral density $S_{NN}(\omega)$ is as shown below.



The noise process $N(t)$ is processed via the system shown below where the pair $(\omega_i, \omega_{\text{cut}}^{(i)})$ is either $(\omega_1 = 20\pi, \omega_{\text{cut}}^{(1)} = \pi)$ or $(\omega_2 = 18\pi, \omega_{\text{cut}}^{(2)} = 2\pi)$. Note that ω_i is the frequency of the cosine and $\omega_{\text{cut}}^{(i)}$ is the cutoff frequency of the lowpass filter.



Then, the average power of process $N_I^{(2)}(t)$ is larger than the average power of the process $N_I^{(1)}(t)$.

Part B. Consider the FM signal

$$s_{FM}(t) = A_c \cos\left[\omega_c t + k_f \int_{-\infty}^{t} m(\tau) d\tau\right] ,$$

where $m(t) = A_m \cos(\omega_m t)$. The following parameters are given:

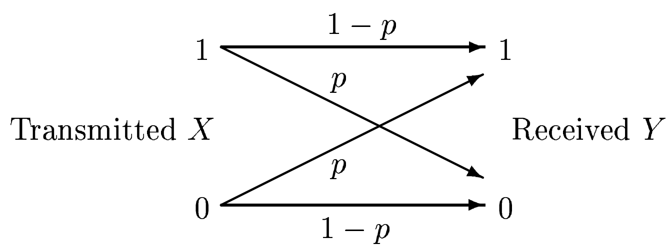
$A_c = 8V, \quad \omega_c = 2\pi \times 10^6 \text{rad/s}, \quad k_f = 2\pi \times 10^2 \text{rad/sVolt}, \quad A_m = 20V, \quad \omega_m = 2\pi \times 10^3 \text{rad/s}$.

(i) What is the power of $s_{FM}(t)$?

(ii) What is the bandwidth of $s_{FM}(t)$? Justify your answer.

Problem 2 (20/100, equally weighted parts)

The discrete, binary, symmetric and memoryless communication channel shown below can transmit one bit (i.e., at each time step, X could be 0 or 1). The number next to an arrow denotes the conditional probability of receiving the bit to the right of the arrow given transmission of the bit to the left of the arrow (transmissions between different time steps are independent).



The above channel with $p = \frac{1}{4}$ is used to transmit four symbols s_0, s_1, s_2 and s_3 with rate 2/3. The prior probability of each symbol and the encoding used to transmit each one are as shown in the table below.

Symbol	Prior Probability	Encoding
s_0	1/2	000
s_1	1/4	101
s_2	1/8	110
s_3	1/8	011

(a) Determine the decoding rule that the receiver should use to minimize the probability of error for the following 3-bit received sequences: "000," "001" and "101." What is the probability of error given that the 3-bit received sequence is 001?

(b) For *list decoding* the receiver is allowed to choose multiple possible symbols for each 3-bit received sequence. The probability of a list error is then the probability that the transmitted symbol is not *listed* by the receiver. Repeat Part (a) allowing the receiver to list *two* possible symbols for the 3-bit received sequences "000," "001" and "101" in a way that minimizes the probability of a list error.

Problem 3 (20/100, equally weighted parts)

This problem has two **relatively independent** parts.

In a certain wireless communication system, the transmitted value X is attenuated by a random attenuation and is corrupted by channel noise so that the measurement Y at the receiving end is related to X as

$$Y = WX + N\ .$$

The transmitted value X is a uniform random variable in the interval $[0,\ 2]$, the attenuation W is a uniform random variable in the interval $[\frac{1}{2},\ 1]$, and the additive noise N is a Gaussian random variable with zero mean and unit variance. Furthermore, X, W, N are mutually independent.

Part A. Given that you observe the value $Y = y$ at the receiving end, find the linear minimum mean square error (LMMSE) estimate for the transmitted value, i.e., find α and β so that

$$\hat{X}_{LMMSE}(y) = \alpha y + \beta$$

and $E[(\hat{X}_{LMMSE}(y) - X)^2]$ is minimized.

Part B. A different (more expensive) receiving structure is able to correctly estimate the attenuation W for each reception $Y = y$. Assuming that the attenuation is known ($W = w$) find the linear minimum mean square error (LMMSE) estimate for the transmitted value, i.e., find $\alpha'(\omega)$ and $\beta'(\omega)$ so that

$$\hat{X'}_{LMMSE}(y, \omega) = \alpha'(\omega) y + \beta'(\omega)$$

and, for a given ω, $E_\omega[(\hat{X'}_{LMMSE}(y, \omega) - X)^2]$ is minimized.

Problem 4 (20/100, equally weighted parts)

This problem has two **independent** parts.

Part A. Consider a simple binary communication system in which the random variable $R = r$ can be received under two different hypothesis, H_0 and H_1. The received value R under each of the hypothesis is Rayleigh distributed as

$$H_0 \quad : \quad f_{R|H_0}(r|H_0) = \begin{cases} re^{-\frac{r^2}{2}} & \text{for } r \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$H_1 \quad : \quad f_{R|H_1}(r|H_1) = \begin{cases} \frac{r}{2}e^{-\frac{r^2}{4}} & \text{for } r \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that the a priory probabilities for these two hypotheses are $\Pr(H_0) = 1/3$ and $\Pr(H_1) = 2/3$. Given the observation $R = r$, find the decision rule that minimizes the probability of error. Calculate this minimal probability of error.

Part B. Suppose that random process $X(t)$ is given by

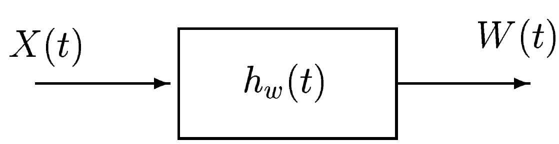
$$X(t) = W(t) + V(t) \ ,$$

where $W(t)$ and $V(t)$ are zero-mean jointly wide-sense stationary random processes such that

$$S_{VV}(\omega) = 1 \ , \quad S_{WW}(\omega) = \frac{144}{25 + \omega^2} \ , \quad \text{and} \quad R_{VW}(\tau) = 0 \text{ for all } \tau \ .$$

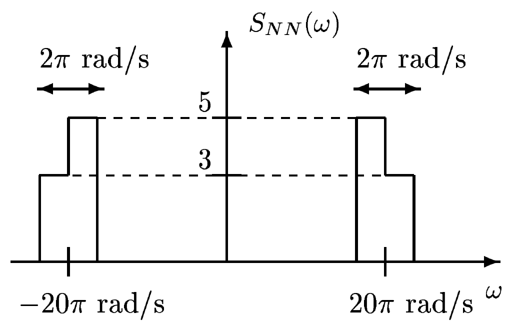
(a) Explain why $X(t)$ is a wide-sense stationary random process and find its power spectral density $S_{XX}(\omega)$.

(b) Find a whitening filter for the random process $X(t)$, i.e., find the impulse response $h_w(t)$ of a stable, linear time-invariant system such that, when its input is $X(t)$, the output of the filter is a process $W(t)$ whose autocorrelation function $R_{WW}(\tau) = \delta(\tau)$.

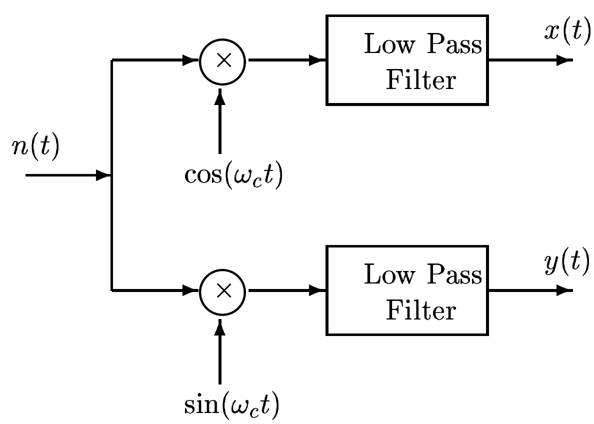


Problem 5 (20/100, unequally weighted parts)

A zero mean narrowband Gaussian noise process $N(t)$ has power spectral density $S_{NN}(\omega)$ as shown on the figure below.



Process $N(t)$ is processed as shown below to obtain $X(t)$ and $Y(t)$, where $\omega_c = 20\pi$ rad/s and the filter is an ideal low pass filter with cutoff frequency π.



(a) **(5 points)** Find the autocorrelation functions $R_{XX}(\tau)$ and $R_{YY}(\tau)$ of the processes $X(t)$ and $Y(t)$ respectively.

(c) **(5 points)** Describe the probability density function $f_Z(z)$ of the random variable $Z = X(5)$ (the sample of $X(t)$ at time $t = 5$s). Justify your answer.

(d) **(10 points)** Describe the joint probability density function $f_{Z,W}(z, w)$ of the random variables $Z = X(5)$ (the sample of $X(t)$ at time $t = 5$s) and $W = Y(5)$ (the sample of $Y(t)$ at time $t = 5$s). Justify your answer.