University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 459: COMMUNICATIONS I

Fall 2005

Problem Set 7

Random Processes, Autocorrelation, Stationarity, Random Processes through LTI Systems, Power Spectral Density

Issued: Thursday, Oct. 27th. **Due:** Thursday, Nov. 3rd (beginning of lecture).

Reading from Lathi: Chapter 11, Sections 11.1-11.5.

Announcement: The second Mid-Semester Exam will be held on Thursday, November 10th, from 7:00pm to 9:00pm in 165 Everitt. The exam will cover all material from the beginning of the term *up to and including* the lecture on Thursday, November 3rd. The corresponding material includes Problem Sets 1 through 7 and Chapters 1, 2, 3, 4, 5, 10 and 11 (excluding Section 11.6) from Lathi. Emphasis will be placed on the material not covered in the first Mid-Semester Exam.

For the exam, you can bring *two* 8.5×11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 7.1 (Optional)

Let $X(t)$ be a random process defined by $X(t) = At + B$.

(a) If B is constant and A is a random variable that is uniformly distributed in $(-1, 1)$, sketch a sample function of this process and find $E[X(t)]$.

(b) If A is constant and B is a random variable that is uniformly distributed in $(0, 2)$, sketch a sample function of this process and find $E[X(t)]$.

(c) If A, B are independent, identically distributed (i.i.d.) Gaussian random variables with mean 0 and variance σ^2, find the joint pdf $f_{X(t_1),X(t_2)}(x_1, x_2)$.

Problem 7.2

Consider a random process $X(t) = A\cos(\omega_0 t)$ where ω_0 is a constant and A is a random variable that is uniformly distributed in $[0, 1]$.

(a) Find the autocorrelation $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$ of random process $X(t)$.

(b) Find the autocovariance $C_{XX}(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))]$ of random process $X(t)$.

(c) Is $X(t)$ wide-sense stationary?

Problem 7.3

Show that the random process $X(t)$ defined as

$$X(t) = \sin(\Omega t) ,$$

where Ω is a random variable uniformly distributed in the interval $[0, 2\pi W]$ is non-stationary.

Problem 7.4

Problem 11.1-7 from Lathi, p. 526.

Problem 7.5

Let X and Y be independent, identically distributed (i.i.d.) Gaussian random variables with zero mean and unit variance. Define the Gaussian random process $Z(t)$ as

$$Z(t) = X \cos(2\pi t) + Y \sin(2\pi t) .$$

Determine the joined probability density function $f_{Z(t_1),Z(t_2)}(z_1, z_2)$ of random variables $Z(t_1)$ and $Z(t_2)$ (obtained by observing the random process $Z(t)$ at times t_1 and t_2). Is $Z(t)$ stationary?

Problem 7.6

A communication channel has an input signal $S(t)$ which can be modeled as a modulated sine wave with random phase and random amplitude at any given time, i.e.,

$$S(t) = X(t) \sin(\omega_0 t + \Theta) ,$$

where ω_0 is a constant, Θ is a random variable that is uniformly distributed in $[0, 2\pi]$ and is independent of the amplitude, and amplitude $X(t)$ is a wide-sense stationary random process with

$$\mu_X(t) = 0 , \qquad -\infty < t < +\infty ,$$

$$R_{XX}(t + \tau, t) \equiv R_{XX}(\tau) = Ae^{-|\tau|} , \qquad -\infty < \tau < +\infty .$$

Find the autocorrelation function $R_{SS}(t_1, t_2)$ for the signal $S(t)$. Is $S(t)$ a wide-sense stationary random process?

Problem 7.7

Let $X(t)$ be a wide-sense stationary random process with

$$R_{XX}(\tau) = 2e^{-|\tau|} , \quad -\infty < \tau < +\infty .$$

(a) What is the average power in the random process $X(t)$?

(b) Find the value of $E[(X(t+1) - X(t-1))^2]$.

(c) Let $Y(t)$ be a random process defined by

$$Y(t) = 5X(2t) - X(t-1) \; , \quad -\infty < t < +\infty \; .$$

Find $R_{YY}(t_1, t_2)$. Is $Y(t)$ a wide-sense stationary random process?

Problem 7.8 (Optional)

Problem 11.3-1 from Lathi, p. 528.

Problem 7.9

A random process $X(t)$ is defined as $X(t) = A\cos(\omega_c t)$, where ω_c is a constant and A is a Gaussian random variable with zero mean and variance σ_A^2. This random process is applied to an ideal integrator producing the output $Y(t) = \int_0^t X(\tau)d\tau$.

(a) Determine the first order pdf of the output $Y(t)$ (i.e., find $f_{Y(t_1)}(y)$ for any time instant $t_1 > 0$.

(b) Is $Y(t)$ wide-sense stationary (WSS)? Is $Y(t)$ strict-sense stationary (SSS)?

Problem 7.10

Problem 11.2-1 from Lathi, p. 526.

Problem 7.11

A zero-mean Gaussian random process $X(t)$ has power spectral density

$$S_{XX}(\omega) = \frac{4}{1+\omega^2} \; , \qquad -\infty < \omega < +\infty \; .$$

(a) Determine $R_{XX}(\tau)$, the autocorrelation function of the random process $X(t)$.

(b) The random process $X(t)$ is passed through a stable LTI system with frequency response

$$H(\omega) = \begin{cases} 1 \; , & |\omega| < 2 \; , \\ 0 \; , & \text{otherwise} \; . \end{cases}$$

Determine the average power $E[Y^2(t)]$ of the output random process $Y(t)$.

Problem 7.12

(a) Let random processes $X(t)$ and $Y(t)$ be the input and output respectively of a stable LTI system with frequency response $H(\omega)$. Assume $X(t)$ is wide-sense stationary and define random process $Z(t)$ to be

$$Z(t) = Y(t) - X(t) \ .$$

Find $S_{ZZ}(\omega)$, the power spectral density of $Z(t)$, in terms of $H(\omega)$ and $S_{XX}(\omega)$.

(b) The input voltage $X(t)$ to a stable LTI filter with frequency response

$$H(\omega) = \frac{2}{2 + j\omega}$$

can be modeled as a wide-sense stationary random process with zero mean and autocorrelation function $R_{XX}(\tau) = 2e^{-|\tau|}$. Find $S_{ZZ}(\omega)$, the power spectral density of random process $Z(t) = Y(t) - X(t)$, where $Y(t)$ is the output of the filter.

Problem 7.13

A wide-sense stationary random process $X(t)$ with autocorrelation function $R_{XX}(\tau) = e^{-|\tau|}$ is processed by a stable LTI system with real-valued impulse response $h(t)$.

(a) For this part, assume that the output of the filter $Y(t)$ is a wide-sense stationary random process with autocorrelation function $R_{YY}(\tau) = 3e^{-3|\tau|}$.

(i) Find $|H(\omega)|$, the magnitude of the frequency response of the filter.

(ii) Suppose that $h(t)$ is causal. Find a possible impulse response $h(t)$ for the LTI system. Is your answer unique?

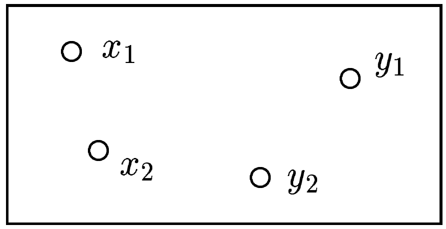
(b) Suppose that the LTI system is known to be stable and that

$$R_{YX}(\tau) = e^{-\tau}u(\tau) - 2e^{-2\tau}u(\tau) + e^{-3\tau}u(\tau) \ .$$

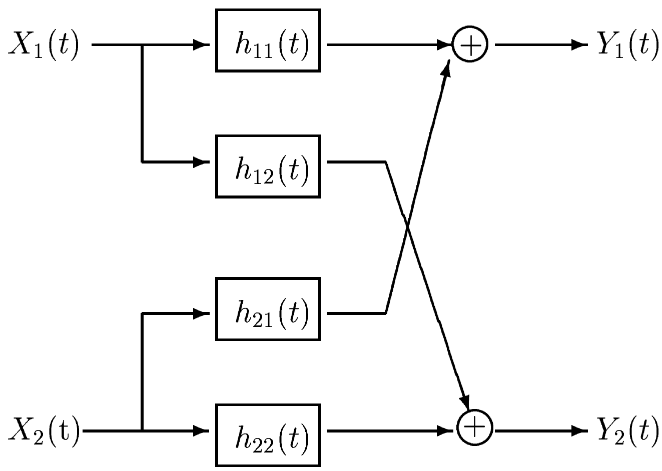
Find a possible impulse response $h(t)$. Is your answer unique?

Problem 7.14 (Optional)

In a variety of real communication situations two (or more) uncorrelated sources are received through channels or systems with crosstalk. Since signals interfere with each other, it is essential that they are separated at the receiving end. One such scenario is shown below, where we have two sources, denoted by x_1 and x_2, and two receivers, denoted by y_1 and y_2.



The above scenario is modeled in terms of LTI systems as shown below. The impulse response from source i to receiver j is denoted by $h_{ij}(t)$.



Approaches for recovering the signals from sources x_1 and x_2 usually involve estimating the impulse responses $h_{ij}(t)$. In this problem we investigate such techniques; to simplify the problem we assume that $h_{11}(t) = h_{22}(t) = \delta(t)$. We also assume that $h_{12}(t)$ and $h_{21}(t)$ are stable and causal and that the sources can be modeled by uncorrelated wide-sense stationary random processes $X_1(t)$ and $X_2(t)$ (with zero mean and known autocorrelation functions $R_{X_1X_1}(\tau)$ and $R_{X_2X_2}(\tau)$).

(a) Show that $Y_1(t)$ and $Y_2(t)$ are jointly wide-sense stationary and determine $R_{Y_1Y_1}(\tau)$, $R_{Y_2Y_2}(\tau)$ and $R_{Y_1Y_2}(\tau)$ in terms of $R_{X_1X_1}(\tau)$, $R_{X_2X_2}(\tau)$, $h_{12}(t)$ and $h_{21}(t)$.

(b) Suppose that $h_{12}(t) = h_{21}(t) = h(t)$ and that you can measure *only one* of $R_{Y_1Y_1}(\tau)$, $R_{Y_2Y_2}(\tau)$ or $R_{Y_1Y_2}(\tau)$. Which one would be most helpful in determining $h_{12}(t)$?