Input/output-to-state stability of switched nonlinear systems

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Abstract— **In this paper, we study the property of input/output-to-state stability (IOSS) for switched nonlinear systems under average dwell-time switching signals, both when each of the constituent systems is IOSS as well as when only some of the constituent systems are IOSS and others are not. This extends available results on input-to-state stability for switched nonlinear systems whose constituent systems are all ISS. We show that if the average dwell-time is big enough and if the fraction of time where one of the non-IOSS systems is active is not too big, then IOSS of the switched system can be established.**

I. Introduction

Switched systems arise in a situation where several dynamical subsystems are present together with a switching signal, i.e. a rule specifying the active subsystem at each point of time. In recent years, different properties of switched systems, especially stability issues, were extensively studied in literature (see e.g. [1] and the references therein). In general, a switched system does not necessarily inherit the properties of its subsystems; for example, it is well known that a switched system consisting of linear exponentially stable subsystems might become unstable [1]. In [2] it was shown that such a switched system is exponentially stable if the switching signal satisfies a certain dwell-time condition. This approach was generalized to the average dwell-time concept in [3]. In [4], the authors considered the situation of a linear switched system consisting of both stable and unstable subsystems, by imposing a condition on the fraction of time where those unstable subsystems are active. For nonlinear randomly switched systems including unstable subsystems, stochastic stability was established in [5] under the condition that the probability of activating the unstable modes is not too high.

When external inputs are present, the concept of input-to-state stability (ISS), introduced in [6], has proved very useful when investigating stability properties of continuous-time nonlinear systems. The dual of this concept, output-to-state stability (OSS), and the combination of the two, input/output-to-state stability (IOSS), were established in [7] and [8]. Loosely speaking, the IOSS property means that no matter what the initial state is, if the inputs and the observed outputs are small, then eventually also the state of the system will become small. Studied a lot for continuous-time nonlinear systems, the ISS property has also been investigated for other classes of systems like switched systems (see e.g. [9], [10], [11], [12]) and recently also for impulsive systems [13]. In [9], the authors examine the ISS property for a deterministic switched system under a dwell-time switching signal where all of the constituent subsystems are ISS. This was extended to average dwell time switching signals in [11]. ISS properties of randomly switched systems were studied in [12].

In this paper, we give sufficient conditions under which a deterministic switched nonlinear system with an average dwell-time switching signal is IOSS, also examining the case where some of the constituent subsystems are not IOSS. In fact, the results of the latter case are also new for systems with no outputs, i.e. if we consider ISS of switched nonlinear systems where not all of the constituent subsystems are ISS, which is an extension of the results in [11]. Even for systems with no inputs we give some novel results, namely on asymptotic stability for switched nonlinear systems where not all of the constituent subsystems are asymptotically stable; this is an extension of the results in [4], where these issues were considered for switched linear systems.

The paper is structured as follows. In section II, the notation and definitions we use are introduced. In section III the main results are stated and proven. Section IV gives a short summary and an outlook on future work.

II. Preliminaries

Consider a family of systems

$$\begin{aligned} \dot{x} &= f_p(x,u) \\ y &= h_p(x) \end{aligned} \qquad p \in \mathcal{P} \tag{1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, the output $y \in \mathbb{R}^l$ and $\mathcal{P}$ is an index set. For every $p \in \mathcal{P}$, $f_p(\cdot,\cdot)$ and $h_p(\cdot)$ are locally Lipschitz and $f_p(0,0) = h_p(0) = 0$. A *switched system*

$$\begin{aligned} \dot{x} &= f_\sigma(x,u) \\ y &= h_\sigma(x) \end{aligned} \tag{2}$$

is generated by the family of systems (1) and a switching signal $\sigma(\cdot)$, where $\sigma : [0,\infty) \to \mathcal{P}$ is a piecewise constant, right continuous function which specifies at each time t the index of the active system.

According to [3] we say that a switching signal has *average dwell-time* τ_a if there exist numbers $N_0, \tau_a > 0$

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such that

$$\forall T \geq t \geq 0: \qquad N_\sigma(T,t) \leq N_0 + \frac{T-t}{\tau_a}, \tag{3}$$

where $N_\sigma(T,t)$ is the number of switches occurring in the interval $(t,T]$.

Denote the switching times in the interval $(0,t]$ by $\tau_1, \tau_2, ..., \tau_{N_\sigma(t,0)}$ (by convention, $\tau_0 := 0$) and the index of the system that is active in the interval $[\tau_i, \tau_{i+1})$ by p_i.

The switched system (2) is *input/output-to-state stable (IOSS)* [7] if there exist functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ [1] and $\beta \in \mathcal{KL}$ [2] such that for all $x_0 \in \mathbb{R}^n$ and each input $u(\cdot)$, the corresponding solution satisfies

$$|x(t)| \leq \beta(|x_0|,t) + \gamma_1(\|u\|_{[0,t]}) + \gamma_2(\|y\|_{[0,t]}) \tag{4}$$

for all $t \geq 0$, where $\|\cdot\|_J$ denotes the supremum norm on an interval J.

III. Input/Output-to-state properties of switched systems

In this section, input/output-to-state stability of switched nonlinear systems will be examined more closely, both for the case where all of the constituent subsystems are IOSS and where only some are IOSS and some are not. In the following, a Lyapunov-like characterization of the IOSS property will be given.

A. All subsystems IOSS

Theorem 1: Consider the family of systems (1). Suppose there exist functions $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \mathcal{K}_\infty$, continuously differentiable functions $V_p : \mathcal{R}^n \to \mathcal{R}$ and constants $\lambda_s > 0$, $\mu \geq 1$ such that for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$ we have

$$\alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|) \tag{5}$$

$$|x| \geq \varphi_1(|u|) + \varphi_2(|h(x)|) \Rightarrow \frac{\partial V_p}{\partial x} f_p(x,u) \leq -\lambda_s V_p(x) \tag{6}$$

$$V_p(x) \leq \mu V_q(x). \tag{7}$$

If σ is a switching signal with average dwell-time

$$\tau_a > \frac{\ln \mu}{\lambda_s}, \tag{8}$$

then the switched system (2) is IOSS.

In the following, the assumptions of Theorem 1 will be discussed shortly. Conditions (5) and (6) imply that all subsystems are IOSS [8]. The function V_p, which is positive definite (ensured by (5)), is called an exponential decay IOSS-Lyapunov function for the p-th subsystem [8]. The set of possible IOSS-Lyapunov functions for the subsystems is constrained by (7). For example, this condition doesn't hold if V_p is quadratic and V_q is quartic for some $p, q \in \mathcal{P}$.

[1] A function α: $[0,\infty) \to [0,\infty)$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_\infty$.

[2] A function β: $[0,\infty) \times [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$, and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

Proof of Theorem 1: Let $\nu(t) := \varphi_1(\|u\|_{[0,t]}) + \varphi_2(\|y\|_{[0,t]})$ and $\xi(t) := \alpha_1^{-1}(\mu^{N_0}\alpha_2(\nu(t)))$, where N_0 comes from (3). Furthermore, define the balls around the origin $B_\nu(t) := \{x \mid |x| \leq \nu(t)\}$ as well as $B_\xi(t) := \{x \mid |x| \leq \xi(t)\}$. Note that ν, and thus also ξ, are non-decreasing functions of time, and therefore the balls B_ν and B_ξ are dynamic sets with non-decreasing volume.

If $|x(t)| \geq \nu(t) \geq \varphi_1(|u(t)|) + \varphi_2(|y(t)|)$ during some time interval $t \in [t', t'']$, we have according to (6) that

$$\frac{\partial V_p}{\partial x} f_p(x(t), u(t)) \leq -\lambda_s V_p(x(t)) \tag{9}$$

for all $p \in \mathcal{P}$. Thus, for all $t \in [t', t'']$, $|x(t)|$ is bounded above [3] by

$$\begin{aligned} |x(t)| &\leq \alpha_1^{-1}(\mu^{N_0} e^{-\lambda(t-t')}\alpha_2(|x(t')|)) \\ &:= \beta(|x(t')|, t-t') \end{aligned} \tag{10}$$

for some $\lambda \in (0, \lambda_s)$. To see why this is true, consider the function $W(t) := e^{\lambda_s t} V_{\sigma(t)}(x(t))$. On any interval $[\tau_i, \tau_{i+1}) \subseteq [t', t'']$, we have according to (9) $\dot{W}(t) \leq 0$. Using (7), we arrive at $W(\tau_{i+1}) \leq \mu W(\tau_{i+1}^-) \leq \mu W(\tau_i)$ and thus, for any $t \in [t', t'']$, we obtain $W(t) \leq \mu^{N_\sigma(t,t')} W(t')$ and therefore

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\leq e^{N_\sigma(t,t')\ln\mu - \lambda_s(t-t')} V_{\sigma(t')}(x(t')) \\ &\leq e^{N_0 \ln \mu} e^{(\frac{\ln \mu}{\tau_a} - \lambda_s)(t-t')} V_{\sigma(t')}(x(t')) \end{aligned} \tag{11}$$

If τ_a satisfies the condition (8), then $V_{\sigma(t)}(x(t))$ decays exponentially in the time interval $[t', t'']$, namely for every $t \in [t', t'']$, it is upper bounded by

$$V_{\sigma(t)}(x(t)) \leq e^{N_0 \ln \mu} e^{-\lambda(t-t')} V_{\sigma(t')}(x(t'))$$

for some $\lambda \in (0, \lambda_s)$. Using (5), we arrive at (10).
Denote the first time when $x(t) \in B_\nu(t)$ by $\check{t}_1$, i.e. $\check{t}_1 := \min\{t \geq 0 : |x(t)| \leq \nu(t)\}$. As $\nu(t) \geq \nu(0) = \varphi_1(|u(0)|) + \varphi_2(|y(0)|)$ for all $t \geq 0$, we get according to (10) that

$$0 \leq \check{t}_1 \leq \max\{0, -\frac{1}{\lambda}\ln(\frac{\alpha_1(\varphi_1(|u(0)|) + \varphi_2(|y_0|))}{\mu^{N_0}\alpha_2(|x_0|)})\}.$$

Thus, for $0 \leq t \leq \check{t}_1$ we get

$$|x(t)| \leq \beta(|x_0|, t). \tag{12}$$

For $t > \check{t}_1$, $|x(t)|$ can be bounded above in terms of $\nu(t)$. Namely, let $\hat{t}_1 := \inf\{t > \check{t}_1 : |x(t)| > \nu(t)\}$. If this is an empty set, let $\hat{t}_1 := \infty$. Clearly, for all $t \in [\check{t}_1, \hat{t}_1)$, it holds that $|x(t)| \leq \nu(t)$. Furthermore, according to (10),

$$\begin{aligned} |x(t)| &\leq \beta(\nu(\hat{t}_1), t - \hat{t}_1) \\ &= \alpha_1^{-1}(\mu^{N_0} e^{-\lambda(t-\hat{t}_1)}\alpha_2(\nu(\hat{t}_1))) \qquad (13) \\ &\leq \alpha_1^{-1}(\mu^{N_0}\alpha_2(\nu(\hat{t}_1))) \\ &= \xi(\hat{t}_1) \qquad (14) \end{aligned}$$

for all $t \in [\hat{t}_1, \check{t}_2)$, where $\check{t}_2 := \min\{t \geq \hat{t}_1 : |x(t)| \leq \nu(t)\}$. Note that $\check{t}_2 < \infty$ either due to the decrease of $x(t)$ according

to (13) or due to the increase of $\nu(t)$.
Continuing in this way, we define

$$\begin{aligned} \check{t}_i &:= \min\{t \geq \hat{t}_{i-1} : |x(t)| \leq \nu(t)\}, \qquad i = 2, 3, ... \\ \hat{t}_i &:= \inf\{t > \check{t}_i : |x(t)| > \nu(t)\}, \qquad i = 2, 3, ... \end{aligned}$$

and we obtain the result that for any i, $|x(t)| \leq \nu(t) \leq \xi(t)$ if $t \in [\check{t}_i, \hat{t}_i)$ and $|x(t)| \leq \xi(\hat{t}_i)$ if $t \in [\hat{t}_i, \check{t}_{i+1})$. Note that this sequence of intervals is either infinite and all subintervals are finite, or the sequence is finite and the last subinterval is infinite.

Thus, as $\xi(t)$ is non-decreasing, $|x(t)|$ cannot leave the ball $B_\xi(t)$ anymore for all $t \geq \check{t}_1$, i.e. it can be bounded above by

$$\begin{aligned} |x(t)| &\leq \xi(t) \\ &= \alpha_1^{-1}\Big(\mu^{N_0}\alpha_2\big(\varphi_1(\|u\|_{[0,t]}) + \varphi_2(\|y\|_{[0,t]})\big)\Big) \\ &\leq \alpha_1^{-1}\Big(\mu^{N_0}\alpha_2\big(2\varphi_1(\|u\|_{[0,t]})\big)\Big) + \\ &\qquad + \alpha_1^{-1}\big(\mu^{N_0}\alpha_2(2\varphi_2(\|y\|_{[0,t]}))\big)\Big) \\ &=: \gamma_1(\|u\|_{[0,t]}) + \gamma_2(\|y\|_{[0,t]}). \end{aligned} \tag{15}$$

Combining (12) and (15) we arrive at

$$|x(t)| \leq \beta(|x_0|, t) + \gamma_1(\|u\|_{[0,t]}) + \gamma_2(\|y\|_{[0,t]})$$

for all $t \geq 0$, which means according to (4) that the switched system (2) is IOSS. □

Remark 1: If (7) holds for $\mu = 1$, then $V_p(x) = V_q(x) =: V(x)$ for all $p, q \in \mathcal{P}$. In this case, the function $V(x)$ is a common IOSS-Lyapunov function for the switched system (2), and thus the system is IOSS for arbitrary switching (see also [10]). This can also be seen by noting that in the case of $\mu = 1$, the condition (8) which the average dwell-time has to satisfy in order that the system is IOSS reduces to $\tau_a > 0$, which means that the system is IOSS for arbitrary switching.

Remark 2: If no outputs are present in the system, Theorem 1 includes as a special case the first statement of Theorem 3.1 in [11], where ISS of switched nonlinear systems with an average dwell-time switching signal (all subsystems ISS) is considered. However, the proof of Theorem 1 is quite different than the proof of Theorem 3.1 in [11].

Remark 3: In the proof of Theorem 1, one major difference compared to the non-switched case is the proceeding after the time $\check{t}_1$. Namely, if we denote the index of the subsystem active at this time by p_1^* and if we define the level set $\Omega_p(t) := \{x \mid V_p(x) \leq \alpha_2(\nu(t))\}$, then the solution $x(t)$ couldn't leave the level set $\Omega_{p_1^*}(t)$ again if no switching occurred for $t > \check{t}_1$, because $\dot{V}_{p_1^*}$ is negative on its boundary. Thus in this case, we could conclude the proof by simply noting that $|x(t)| \leq \alpha_1^{-1}(\alpha_2(\nu(t)))$ for all $t > \check{t}_1$. Due to switching, however, $x(t)$ can leave the level set $\Omega_{p_1^*,\eta}(t)$ again and thus we have to proceed with the proof as shown above.

B. Some subsystems not IOSS

In the following, the result of Theorem 1 will be extended to the case where not all systems of the family (1) are IOSS, i.e. (6) doesn't hold for all $p \in \mathcal{P}$, but only for a subset $\mathcal{P}_s$ of $\mathcal{P}$.

Let $\mathcal{P} = \mathcal{P}_s \cup \mathcal{P}_u$ such that $\mathcal{P}_s \cap \mathcal{P}_u = \emptyset$. Denote by $T^u(t, \tau)$ the total activation time of the systems in $\mathcal{P}_u$ and by $T^s(t, \tau)$ the total activation time of the systems in $\mathcal{P}_s$ during the time interval $[\tau, t)$, where $0 \leq \tau \leq t$. Clearly, $T^s(t, \tau) = t - \tau - T^u(t, \tau)$.

For the later examinations of the IOSS property for these kind of systems, the following lemma, where we consider asymptotic stability of a switched system without inputs (i.e. a switched system (2) with $u \equiv 0$), will be helpful.

Lemma 1: Suppose there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, continuously differentiable functions $V_p : \mathcal{R}^n \to \mathcal{R}$ and constants $\lambda_s, \lambda_u > 0$, $\mu \geq 1$ such that (5) and (7) hold for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$ and furthermore the following holds for all $x \in \mathbb{R}^n$:

$$\frac{\partial V_p}{\partial x} f_p(x, 0) \leq -\lambda_s V_p(x) \quad \forall p \in \mathcal{P}_s \tag{16}$$

$$\frac{\partial V_p}{\partial x} f_p(x, 0) \leq \lambda_u V_p(x) \quad \forall p \in \mathcal{P}_u \tag{17}$$

If there exist constants $\tau_0, \rho \geq 0$ such that

$$\rho < \frac{\lambda_s}{\lambda_s + \lambda_u} \tag{18}$$

$$\forall t \geq 0 : \quad T^u(t, 0) \leq \tau_0 + \rho t \tag{19}$$

and if $\sigma(\cdot)$ is a switching signal with average dwell-time

$$\tau_a > \frac{\ln \mu}{\lambda_s(1-\rho) - \lambda_u \rho}, \tag{20}$$

then the switched system (2) is globally asymptotically stable.

Remark 4: The constant τ_0 in (19) can be interpreted as an initial offset on the activation time T^u of the systems in $\mathcal{P}_u$, which allows us to start with a system in $\mathcal{P}_u$ (if $\tau_0 = 0$, we have to start with a system in $\mathcal{P}_s$ in order to be able to satisfy (19) because $\rho < 1$).

Remark 5: The idea to restrict the fraction of time during which the unstable systems are active was also used in [4], where asymptotic stability for switched linear systems, including some unstable systems, was considered. In [4], an upper bound for this fraction of time is gained in terms of the maximum eigenvalues of the unstable and the stable system matrices. In Lemma 1, where switched nonlinear systems are considered, the maximal instability margin and minimal stability margin are characterized by the constants λ_u and λ_s, respectively, which give a bound for the (exponential) growth and decay, respectively, of the Lyapunov functions V_p. The upper bound for the fraction of time during which the unstable systems are active (18) - (19) is given in terms of these constants.

Proof of Lemma 1: Consider the function $W(t) := e^{\lambda_s t} V_{\sigma(t)}(x(t))$. On any interval $[\tau_i, \tau_{i+1})$ we have according to (16) and (17)

$$\begin{array}{rcll} \dot{W}(t) & \leq & 0 & \text{if } p_i \in \mathcal{P}_s \\ \dot{W}(t) & \leq & (\lambda_s + \lambda_u) W(t) & \text{if } p_i \in \mathcal{P}_u. \end{array}$$

Using (7), we thus arrive at

$$W(\tau_{i+1}) \leq \mu W(\tau_{i+1}^-) \leq \mu W(\tau_i)$$

if $p_i \in \mathcal{P}_s$ and

$$W(\tau_{i+1}) \leq \mu W(\tau_{i+1}^-) \leq \mu W(\tau_i) e^{(\lambda_s + \lambda_u)(\tau_{i+1} - \tau_i)}$$

if $p_i \in \mathcal{P}_u$. Thus, for any $t \geq 0$ we obtain

$$W(t) \leq \mu^{N_\sigma(t,0)} W(0) e^{(\lambda_s + \lambda_u) T^u(t,0)}$$

and therefore, using (19),

$$\begin{array}{rcl} V_{\sigma(t)}(x(t)) & \leq & e^{N_\sigma(t,0) \ln \mu + (\lambda_s + \lambda_u) T^u(t,0) - \lambda_s t} V_{\sigma(0)}(x_0) \\ & \leq & e^{N_0 \ln \mu + (\lambda_s + \lambda_u)\tau_0} \times \\ & \times & e^{(\frac{\ln \mu}{\tau_a} + (\lambda_s + \lambda_u)\rho - \lambda_s) t} V_{\sigma(0)}(x_0). \end{array} \quad (21)$$

We conclude that if ρ and τ_a satisfy the conditions (18) and (20) respectively, then $V_{\sigma(t)}(x(t))$ decays exponentially, namely it is upper bounded by

$$V_{\sigma(t)}(x(t)) \leq e^{N_0 \ln \mu + (\lambda_s + \lambda_u)\tau_0} e^{-\lambda t} V_{\sigma(0)}(x_0)$$

for some $\lambda \in (0, \lambda_s - (\lambda_s + \lambda_u)\rho)$.
Finally, using (5), we obtain

$$|x(t)| \leq \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s + \lambda_u)\tau_0} e^{-\lambda t} \alpha_2(|x_0|)) \quad (22)$$

which proves global asymptotic stability. □

Combining Theorem 1 and Lemma 1, we obtain the following result concerning input/output-to-state stability for switched systems including unstable systems:

Theorem 2: Consider the switched system (2). Suppose there exist functions $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \mathcal{K}_\infty$, positive definite functions $V_p : \mathcal{R}^n \to \mathcal{R}$ and constants $\lambda_s, \lambda_u > 0$, $\mu \geq 1$ such that (5) and (7) hold for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$ and furthermore, the following holds:

$$\begin{array}{rcl} |x| & \geq & \varphi_1(|u|) + \varphi_2(|h(x)|) \\ & & \Rightarrow \left\{ \begin{array}{lll} \frac{\partial V_p}{\partial x} f_p(x,u) & \leq -\lambda_s V_p(x) & \forall p \in \mathcal{P}_s \\ \frac{\partial V_p}{\partial x} f_p(x,u) & \leq \lambda_u V_p(x) & \forall p \in \mathcal{P}_u. \end{array} \right. \end{array} \quad (23)$$

If there exist constants $\rho \geq 0$ satisfying (18) and $\tau_0 \geq 0$ such that

$$\forall\, t \geq \tau \geq 0: \quad T^u(t,\tau) \leq \tau_0 + \rho(t - \tau) \quad (24)$$

and if σ is a switching signal with average dwell-time τ_a satisfying (20), then the switched system (2) is IOSS.

Proof: The proof of Theorem 2 follows the lines of the proof of Theorem 1. Define $\nu(t)$, $B_\nu(t)$ and the points $\check{t}_i$ and $\hat{t}_i$ where the solution enters (respectively, leaves) the ball $B_\nu(t)$ as in the proof of Theorem 1. Furthermore, define $\xi(t) := \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s + \lambda_u)\tau_0} \alpha_2(\nu(t)))$.

According to Lemma 1, for $0 \leq t \leq \check{t}_1$ we get

$$|x(t)| \leq \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s + \lambda_u)\tau_0} e^{-\lambda t} \alpha_2(|x_0|)) := \beta(|x_0|, t) \quad (25)$$

for some $\lambda \in (0, \lambda_s - (\lambda_s + \lambda_u)\rho)$.

Similarly to Theorem 1, we obtain that for any $i \geq 1$, $|x(t)| \leq \nu(t) \leq \xi(t)$ if $t \in [\check{t}_i, \hat{t}_i)$ and $|x(t)| \leq \xi(\hat{t}_i)$ if $t \in [\hat{t}_i, \check{t}_{i+1})$.

Thus, as $\xi(t)$ is non-decreasing, we conclude that for all $t \geq 0$

$$\begin{array}{rcl} |x(t)| & \leq & \beta(|x_0|, t) + \xi(t) \\ & \leq & \beta(|x_0|, t) + \gamma_1(\|u\|_{[0,t]}) + \gamma_2(\|y\|_{[0,t]}), \end{array}$$

where

$$\begin{array}{rcl} \gamma_1(r) & := & \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s + \lambda_u)\tau_0} \alpha_2(2\varphi_1(r))) \\ \gamma_2(r) & := & \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s + \lambda_u)\tau_0} \alpha_2(2\varphi_2(r))), \end{array}$$

which means according to (4) that the switched system (2) is IOSS. □

Remark 6: Condition (24) is stricter than condition (19), i.e. we have to impose a stricter condition on the activation time for the unstable systems in the case where IOSS is considered than if we just consider asymptotic stability. Namely, in (19), we just require that for any $t \geq 0$ the amount of time during the interval $[0, t)$ where systems in $\mathcal{P}_u$ are active doesn't exceed a certain fraction of this interval (plus an offset τ_0), whereas in (24) we require this upper bound to hold uniformly over any interval $[\tau, t)$ with arbitrary starting point $\tau \leq t$. This means that in contrast to (19), we cannot "save up" activation time for systems in $\mathcal{P}_u$ for a later point in time. This is the case because in the proof of Theorem 2, in order to be able to apply Lemma 1 in each time interval $t \in [\hat{t}_i, t_{i+1}^*)$, i.e. to ensure the decaying of $|x(t)|$ outside the ball $B_\nu(t)$, we need that in each of these intervals, $T^u(t, \hat{t}_i) \leq \tau_0 + \rho(t - \hat{t}_i)$. As the points $\hat{t}_i$ may be different for each $u(\cdot)$ and $\sigma(\cdot)$, this results in the condition (24).

In the case where we consider asymptotic stability, "saving up" activation time for systems in $\mathcal{P}_u$ is no problem as this means that systems in $\mathcal{P}_s$ have been active for a longer time before and thus $|x|$ is already small during the longer activation time of systems in $\mathcal{P}_u$. Hence also the growth of $|x|$ during this period of time is small and the switched system is still asymptotically stable. However, if inputs and outputs are present in the switched system, they might increase $|x|$ again before respectively while the systems in $\mathcal{P}_u$ are active, and thus the growth of $|x|$ might be large

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and $|x(t)|$ cannot be bounded in terms of $\sup_{0\le\tau\le t}|y(\tau)|$ anymore.

Similar considerations apply to the average dwell-time property. Namely, if we consider systems with no inputs, it is enough to require that the average dwell time property (3) is satisfied for an interval starting at the initial time, whereas in the case where inputs are present the average dwell-time property (3) has to hold uniformly over any interval $[\tau, t)$ with arbitrary starting point $\tau \le t$ (see e.g. [1] p.61).

IV. Conclusions and future research

In this paper, we studied the IOSS property of switched nonlinear systems. We showed that if the average dwell time and the activation time of the non-IOSS subsystems satisfy appropriate sufficient conditions, IOSS can be established for the switched system.

An interesting topic in the context of IOSS is the existence of a state-norm estimator, introduced in [7]. For continuous-time nonlinear systems, and even more for switched nonlinear systems, reconstructing the state of the system from the observations of the input and the output is a challenging task far from being solved completely. For control purposes, it is often sufficient to gain an estimate of the magnitude, i.e. the norm, of the state ([7], [8]). It was shown in [8], that for continuous-time nonlinear systems, the existence of a state-norm estimator is equivalent to the system being IOSS. Introducing the concept of state-norm estimators to switched systems and examining its relation to the IOSS property is a topic of our current research.

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