Input to State Stabilizing Controller for Systems with Coarse Quantization

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Abstract

We consider the problem of achieving input-to-state stability (ISS) with respect to external disturbances for control systems with quantized measurements. Quantizers considered in this paper take finitely many values and have an adjustable "center" and "zoom" parameters. Both the full state feedback and the output feedback cases are considered. Similarly to previous techniques from the literature, our proposed controller switches repeatedly between "zooming out" and "zooming in". However, here we use two modes to implement the "zooming in" phases, which gives us the important benefit of using the minimal number of quantization regions. Our analysis is trajectory-based and utilizes a cascade structure of the closed-loop hybrid system. We further show that our method is robust to modeling errors in the plant dynamics using a specially adapted small-gain theorem. The main results are developed for linear systems, but we also discuss their extension to nonlinear systems under appropriate assumptions.

Index Terms

Quantized systems, Stability of hybrid systems, Input-to-state stability (ISS), Output feedback and Observers, Disturbances

I. Introduction

A *quantizer* is a device that converts a real-valued signal into a piecewise constant one taking a finite set of values. In the context of feedback control systems, the real-valued signal is either the measurable output of the system or the control input. Quantization is generally a constraint related to the implementation of the control system. Digital sensors, digital controllers and data

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links with limited date rate are typical in many implementations of control systems, and they all induce some degree of quantization.

The study of the influence of quantization on the behavior of feedback control systems can be traced back at least to [1]. In the literature on quantization, the quantized control system is typically regarded as a perturbation of the ideal (unquantized) one. Two principal phenomena account for changes in the system's behavior caused by quantization. The first one is saturation: if the quantized signal is outside the range of the quantizer, then the quantization error is large, and the system may significantly deviate from the nominal behavior (e.g., become unstable). The second one is deterioration of performance near the target point (e.g., the equilibrium to be stabilized): as this point is approached, higher precision is required, and so the presence of quantization errors again distorts the properties of the system. These effects can be precisely characterized using the tools of system theory, specifically, Lyapunov functions and perturbation analysis; see, e.g., [2], [3], [4] for results in this direction. We refer to this line of work as the "perturbation approach". The more recent work [5], also falling into this category, is particularly relevant because it reveals the importance of Input-to-State Stability—a concept we define below—for characterizing the robustness of the controller to quantization errors for general nonlinear systems.

An alternative point of view which this paper follows, pioneered by Delchamps [3], is to regard the quantizer as an information-processing device, i.e., to view the quantized signal as providing a limited amount of information about the real quantity of interest (system state, control input, etc.) which is encoded using a finite alphabet. This "information approach" seems especially suitable in modern applications such as networked and embedded control systems. The main question then becomes: how much information is really needed to achieve a given control objective? In the context of stabilization of linear systems, one can explicitly calculate the minimal information transmission rate that will dominate the expansiveness of the underlying system dynamics. Results in this direction are reported in [6], [4], [7], [8], [9], [10] and in the papers cited in the next paragraph; see also [11], [12], [13], [14] for extensions to nonlinear systems.

All the aforementioned works only addressed stability in the absence of external disturbances. The papers that did address the issue of external disturbances are cited below. They differ mainly in the stability property they aim to achieve, and in their assumptions on the external disturbance. Papers [15], [16] and [17] designed a controller which guarantees stability only for a disturbance whose magnitude is lower than some known value. In the paper [18] mean square stability in the stochastic setting is obtained by utilizing statistical information about the disturbance (a bound on its appropriate moment). The paper [19] designed a controller with which it is possible to bound the plant's state in probability. With the expense of one additional feedback bit, no further information about the disturbance is required. Note that these two latter papers use (and prove) stochastic stability notions. All of these papers followed the information approach. Deterministic stability for a completely unknown bounded disturbance was initially shown in [20]. By generalizing the perturbation approach of [4], [5], the deterministic stability property achieved in [20] is Input-to-State Stability (ISS) which, apart from ensuring a bounded state response to every bounded disturbance, also ensures asymptotic stability (convergence to the origin) when the disturbance converges to zero. The approach of [20] was also shown to produce ℓ_2 stability in [14] (also, [21]).

In this paper we also address the problem of achieving ISS for deterministic systems and completely unknown disturbance. In contrast to [20] which followed the perturbation approach, our first, and main contribution here is that we do this following the information approach. The main advantage of using the information approach is that it requires fewer, possibly much fewer, quantization regions which also translates to lower data rate. As a result, a better understanding is achieved of how much information is required for ISS disturbance attenuation. In fact, when all state variables are observed (quantized state feedback) we are able to achieve a data rate which can be arbitrarily close to the minimal data rate required for stabilization with no disturbance. We stress that following the information approach and not the perturbation approach necessitates significantly different design and analysis tools than what is described in [20].

Our second contribution is that we also consider the case where the state space is only partially measured, the situation commonly referred to as output feedback. This is a significant generalization of the approach described in [10], where only a specific observer was given and no disturbances were considered. The papers [18], [19] and [13] do formulate a system with output feedback, but it is assumed there that a state estimate is generated before the quantization is applied ([13] does not deal with disturbances). In our paper we generate the state estimate from the quantized measurements. We argue that this setting is much more reasonable when the quantization is due to physical or practical constraints on the sensor (as opposed to just a data

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rate constraint); refer to Remark 2 for more details. We emphasize that our results are novel even for the state feedback case.

Our third contribution, which was not discussed in any of the previous papers, is robustness to modeling errors where the system model is known only approximately, and may also vary over time. We show that under small enough modeling errors the system remains ISS in a local practical sense. We prove this robustness result using a specially adapted Small-Gain Theorem.

This paper is preceded by two conference papers, [22] and [23]. Not appearing in those papers but given here are the treatment of robustness to modeling errors and the proofs of all of the results which were omitted from the conference papers due to length limitations.

The paper is organized as follows: In §II-A we define the system and the specific quantizer we will use; in §II-B we define the desired stability property, an extension of the Input-to-State Stability (ISS) property; in §III we present the proposed controller; in §IV we state and prove our main results; in §V we show that we can arbitrarily approach the minimum data-rate for the unperturbed system; finally, in §VI we show how our results can be extended to nonlinear systems. We defer to part A of the appendix the proofs of our technical lemmas. In part B of the appendix we show that the small-gain theorem applies to our modified ISS notion.

II. Problem Statement

A. System Definition

The continuous-time dynamical system we are to stabilize is as follows ($t \in \mathbb{R}_{\geq 0}$):

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t) + D\boldsymbol{w}(t) \\ \boldsymbol{y}(t) &= C\boldsymbol{x}(t) \end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_x}$ is the state, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ is the control input, $\boldsymbol{w} \in \mathbb{R}^{n_w}$ is an unknown disturbance, and $\boldsymbol{y} \in \mathbb{R}^{n_y}$ is the measured output ($n_y \leq n_x$).

While $\boldsymbol{y}$ is what the sensors measure, we assume that the information available to the controller is $\boldsymbol{z} : \{kT_s \,|k \in \mathbb{Z}_{\geq 0}\} \to \mathbb{R}^{n_y}$, which is a sampled and quantized version of $\boldsymbol{y}$:

$$\boldsymbol{z}(kT_s) = Q\left(\boldsymbol{y}(kT_s); \boldsymbol{c}(kT_s), \mu(kT_s)\right) \tag{2}$$

where Q is a quantization function and $T_s > 0$ is the time-sampling interval. The quantization parameters, $\boldsymbol{c} : \{kT_s \,|k \in \mathbb{Z}_{\geq 0}\} \to \mathbb{R}^{n_y}$ and $\mu : \{kT_s \,|k \in \mathbb{Z}_{\geq 0}\} \to \mathbb{R}_{>0}$ are generated by the

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controller. For convenience we will use the notation $\boldsymbol{z}_k \doteq \boldsymbol{z}(kT_s)$, and similarly for other variables, so 2 becomes $\boldsymbol{z}_k = Q(\boldsymbol{y}_k; \boldsymbol{c}_k, \mu_k)$. We refer to the special case where $C = I$, the identity matrix, as the quantized state feedback problem. We refer to the general case where C is arbitrary as the quantized output feedback problem.

We will present our results using the following (square) quantizer. We assume N is an odd number, $N \geq 3$, which counts the number of quantization regions per observed dimension. The quantizer is denoted by $\left(Q_1, \ldots, Q_{n_y}\right)^T = Q(\boldsymbol{y}; \boldsymbol{c}, \mu)$ where each scalar component is defined as follows (see Figure 1 for an illustration):

$$Q_i(\boldsymbol{y}; \boldsymbol{c}, \mu) \doteq c_i + \begin{cases} (-N+1)\mu & y_i - c_i \leq (-N+2)\mu \\ (-N+3)\mu & (-N+2)\mu < y_i - c_i \leq (-N+4)\mu \\ \vdots & \vdots \\ 0 & -\mu < y_i - c_i \leq \mu \\ \vdots & \vdots \\ (N-3)\mu & (N-4)\mu < y_i - c_i \leq (N-2)\mu \\ (N-1)\mu & (N-2)\mu < y_i - c_i. \end{cases} \tag{3}$$

We will refer to $\boldsymbol{c}$ as the *center* of the quantizer, and to μ as the *zoom factor*. Note that what will actually be transferred from the quantizer to the controller will be an index to one of the quantization regions. The controller, which either generates the values $\boldsymbol{c}$ and μ or knows the rule by which they are generated[1], will use this information to convert the received index to the value of Q as given in (3).

Remark 1: All of our results, except for those in §V, actually apply to a more general family of quantizers. For an arbitrary quantizer, we denote by $\mathcal{Q}(\boldsymbol{c}, \mu)$ the (finite) set of possible values of $Q(\cdot; \boldsymbol{c}, \mu)$. A quantizer belongs to the family of quantizers to which our results apply if there exist real numbers $M > 1$ and $0 \leq H \leq N-1$ such that for all $\boldsymbol{y}$, $\boldsymbol{c}$ and μ there exists a set

[1]The quantization parameters $\boldsymbol{c}$ and μ can be available to the sensors (or the sensor side of the communication link) depending on the source of quantization. When the source of quantization is the communication, and there is sufficient computation capability on the sensor side of the communication link, the quantization parameters $\boldsymbol{c}$ and μ may be generated simultaneously on both sides of the communication link. When the source of quantization is the sensors, these quantities can be generated by the controller only and then sent to the sensors.

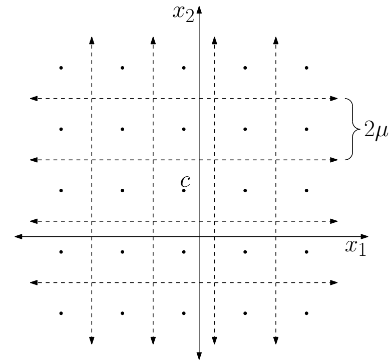

Fig. 1. Illustration of the quantizer for the two-dimensional output subspace, $N = 5$. The dashed lines define the boundaries of the quantization regions. The black dots define the quantization values.

$\mathcal{Q}_{INT}\left(\boldsymbol{c},\mu\right) \subsetneq \mathcal{Q}\left(\boldsymbol{c},\mu\right)$ with which the following implications hold:

$$|\boldsymbol{y}-\boldsymbol{c}| < M\mu \qquad \Rightarrow \qquad |Q\left(\boldsymbol{y};\boldsymbol{c},\mu\right)-\boldsymbol{y}| < \mu \tag{4}$$

$$|\boldsymbol{y}-\boldsymbol{c}| < \left(M-H\right)\mu \qquad \Rightarrow \qquad Q\left(\boldsymbol{y};\boldsymbol{c},\mu\right) \in \mathcal{Q}_{INT}\left(\boldsymbol{c},\mu\right) \tag{5}$$

$$Q\left(\boldsymbol{y};\boldsymbol{c},\mu\right) \in \mathcal{Q}_{INT}\left(\boldsymbol{c},\mu\right) \qquad \Rightarrow \qquad |Q\left(\boldsymbol{y};\boldsymbol{c},\mu\right)-\boldsymbol{y}| < \mu. \tag{6}$$

It is easy to see that the square quantizer above belongs to this family with $M = N$, $H = 2$ and $\mathcal{Q}_{INT}\left(\boldsymbol{y};\boldsymbol{c},\mu\right) = \left\{\left(c_1+q_1\mu,\ldots,c_{n_y}+q_{n_y}\mu\right)|q_i \in \left[-N+3,-N+5,\ldots N-3\right], \forall i\right\}$ when the ∞-norm is considered.

Remark 2: In the literature on quantization there appear to be two different methods of positioning the partial measurement constraint (output feedback) in the feedback loop. One approach, followed by [18], [19] and [13], assumes that while not all the state variables are observed, those that are observed are measured continuously. These continuous measurements are fed into an observer which generates a state estimate. This state estimate is sent through a communication link to the controller (and thus has to be quantized). The second approach, followed by [10] and this paper, assumes that the measurements of the observed state variables are quantized, and from these quantized measurements a state estimate needs to be generated. The reason for having two approaches is the different possible sources of quantization: Both approaches can handle the case when the communication is the source of quantization; however, only the second approach can handle the case when the sensors are the source of quantization.

In this paper we will use the ∞-norm unless otherwise specified. For vectors, $|\boldsymbol{x}| \doteq |\boldsymbol{x}|_\infty \doteq \max_i |x_i|$. For continuous-time signals, $\|\boldsymbol{w}\|_{[t_1,t_2]} \doteq \max_{t\in[t_1,t_2]} |\boldsymbol{w}(t)|_\infty$, $\|\boldsymbol{w}\| \doteq \|\boldsymbol{w}\|_{[0,\infty)}$. For discrete-time signals, $\|\boldsymbol{z}\|_{\{k_1\ldots k_2\}} \doteq \max_{k\in\{k_1\ldots k_2\}} |\boldsymbol{z}_k|_\infty$, $\|\boldsymbol{z}\| \doteq \|\boldsymbol{z}\|_{\{0\ldots\infty\}}$. For matrices we use the induced norm corresponding to the specified norm (∞-norm if none specified). For piecewise continuous signals we will use the superscripts $+$ and $-$ to denote the right and left continuous limits, respectively: $\boldsymbol{x}_k^+ \doteq \boldsymbol{x}^+ (kT_s) \doteq \lim_{t \searrow 0} \boldsymbol{x} (kT_s + t)$, $\boldsymbol{x}_k^- \doteq \boldsymbol{x}^- (kT_s) \doteq \lim_{t \nearrow 0} \boldsymbol{x} (kT_s + t)$.

B. Desired Stability Property

Ideally we would want our closed-loop system to be asymptotically stable. In the presence of a non-vanishing disturbance, even linear state feedback systems without quantization can not be driven asymptotically to the origin. Therefore, we aim for a weaker property: that the system be bounded and converge to a ball around the origin whose size depends on the magnitude of the disturbance. Furthermore, when the disturbance vanishes, we expect to recover asymptotic stability. This desired behavior is encapsulated by the (global) Input-to-State Stability property, originally defined in [24] as follows:

$$|\boldsymbol{x}(t)| \leq \beta\left(|\boldsymbol{x}(t_0)|, t - t_0\right) + \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right), \qquad \forall t \geq t_0 \geq 0 \tag{7}$$

where γ is a function of class $\mathcal{K}_\infty$ and β is a function of class $\mathcal{KL}$[2].

In addition to the original state variables, $\boldsymbol{x}$, the closed-loop system will contain other variables. Of these additional variable, the zoom factor in particular will not exhibit an ISS relation with respect to the disturbance. We refer the reader to the discussion in [20, §III.B] which explains why it is hard and probably impossible to have both the original state and the zoom factor exhibit an ISS relation with respect to the disturbance. Nevertheless, the value of the zoom factor at an arbitrary initial time will affect the ISS relation between the disturbance and the state. Therefore, the property that we will achieve, which we refer to as *parameterized Input-to-State Stability*, is defined as:

$$\begin{aligned} &|\boldsymbol{x}(t)| \leq \beta\left(|\boldsymbol{x}(t_0)|, t - t_0; \mu(t_0)\right) + \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}; \mu(t_0)\right), \qquad \forall t \geq t_0 \geq 0 \\ &\mu(t) \leq \delta\left(\|\boldsymbol{x}\|_{[t_0,t]}; \mu(t_0)\right) \end{aligned} \tag{8}$$

where the functions $\beta(\cdot,\cdot;\mu)$ and $\gamma(\cdot;\mu)$ are of class $\mathcal{KL}$ and class $\mathcal{K}_\infty$, respectively, for every fixed $\mu \in \mathbb{R}_{\geq 0}$; the function $\delta(\cdot,\mu)$ is continuous, non-decreasing and non-negative for every fixed $\mu \in \mathbb{R}_{\geq 0}$, but not necessarily zero at zero; all the functions are continuous in μ when their other arguments are fixed.

[2]A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and also unbounded. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geq 0$.

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In the case of modeling errors, even this cannot in general be achieved. Namely, we can not achieve a global result, only local; furthermore, even with no external disturbance, the system will only be practically stable, not asymptotically stable. The weaker result we do achieve in the case of modeling error is *local practical parameterized Input-to-State Stability*: There exist x_{max}, w_{max} such that

$$\begin{aligned} &|\boldsymbol{x}(t)| \leq \beta\left(|\boldsymbol{x}(t_0)|, t-t_0\right) + \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right) + d \quad \forall t \geq t_0 \geq 0, \\ &\forall\,|\boldsymbol{x}(0)| < x_{max} \quad \forall\,\|\boldsymbol{w}\|_{[0,t]} < w_{max} \end{aligned} \tag{9}$$

where d is some positive constant. This property is along the lines of the Input-to-State practical Stability (ISpS) [25].

III. Controller Design

A. Overview of the Controller Design

Our controller switches between three different modes of operation. The motivation for each of these modes is given in this subsection.

Our quantizer consists of quantization regions of finite size, for which the quantization error, $\boldsymbol{e}_k = \boldsymbol{z}_k - \boldsymbol{y}_k$, can be bounded, and regions of infinite size, where the quantization error is unbounded. We will refer to these regions as bounded and unbounded quantization regions, respectively. Due to the fact that there are only a finite number of quantization regions to cover the infinite-size output subspace $\mathbb{R}^{n_y}$, only a subset of finite size of this subspace can be covered by the bounded quantization regions. The size of this subset, however, can be adjusted dynamically by changing the parameters of the quantizer. We refer to this subset which is covered by only bounded quantization regions as the unsaturated region. Our controller follows the general framework which was introduced in [4], [5] to stabilize the system from an unknown initial condition using dynamic quantization. In [20] this approach was developed further to achieve disturbance attenuation. This framework consists of two main modes of operation, generally referred to as the "*zoom-in*" and the "*zoom-out*" mode. During the *zoom-out* mode, the unsaturated region is enlarged until the measured output is captured in this region and a state estimate with a bounded estimation error can be established. This is followed by a switch to the *zoom-in* mode. During the *zoom-in* mode, the size of the quantization regions is reduced in order to have the state estimate converge to the true state. The reduction of the size of the

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quantization regions inevitably reduces the size of the unsaturated region. As the size of this region is reduced, eventually the unknown disturbance may drive the measured output outside the unsaturated region. To regain a new state estimate with a bounded estimation error, the controller will switch back to the *zoom-out* mode. By switching repeatedly between these two modes, an ISS relation can be established. In this paper we use the name "*capture*" mode for the *zoom-out* mode.

In our quantizer there are $2n_y$ unbounded quantization regions. To achieve the minimum data-rate, however, we are required to use the unbounded regions not only to detect saturation, but also to reduce the estimation error. This dual use is accomplished by dividing the *zoom-in* mode into two modes: a "*measurement-update*" mode and an "*escape-detection*" mode. After receiving r successive measurements in bounded quantization regions, where r is the observability index of the pair (A, C), we are able to define a region in the state space which must contain the state if there were no disturbance. We enlarge this region proportionally to its current size to accommodate some disturbance. In the *measurement-update* mode we cover this containment region using both the bounded and the unbounded regions of the quantizer. This way we are able to use the smallest quantization regions, which leads to the fastest reduction in the estimation error. The drawback with this mode is that if a strong disturbance comes in, we will not be able to detect it. Therefore, in the *escape-detection* mode we use larger quantization regions to cover the containment region using only the bounded regions. Thus, if a strong disturbance does come in, we will be able to detect it as the quantized output measurement will correspond to one of the unbounded regions. Note that having these two *zoom-in* modes is especially critical when there are only 3 quantization regions per dimension. If we would have used only the *escape-detection* mode, which is necessary to detect escape, then the unsaturated region would contain only one quantization region. Having only one quantization in the unsaturated region does not provide any additional information, beside the distinction of whether an escape occurred, that can be used to reduce the estimation error.

The precise details on how to design the controller are given in the remainder of this section.

B. Preliminaries

In this section we assume that $A \equiv A_0$ is fixed and known. Extension to varying, unknown A will be discussed in §IV-C.

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We define the sampled-time versions of A, $\boldsymbol{u}$ and $\boldsymbol{w}$ as ($k \in \mathbb{Z}_{\geq 0}$):

$$A_d \doteq \exp\left(T_s A_0\right), \quad \boldsymbol{x}_k \doteq \boldsymbol{x}\left(kT_s\right), \quad \boldsymbol{u}_k^d \doteq \int_0^{T_s} \exp\left(A_0\left(T_s - t\right)\right) B\boldsymbol{u}\left(kT_s + t\right) dt,$$

$$\boldsymbol{w}_k^d \doteq \int_0^{T_s} \exp\left(A_0\left(T_s - t\right)\right) D\boldsymbol{w}\left(kT_s + t\right) dt.$$

With these definitions we can write

$$\boldsymbol{x}_{k+1}^d = A_d \boldsymbol{x}_k^d + \boldsymbol{u}_k^d + \boldsymbol{w}_k^d. \tag{10}$$

We assume that (A, B) is a controllable pair, so there exists a matrix K such that $A + BK$ is Hurwitz. By construction A_d is full rank, and in general (unless T_s belongs to some set of measure zero) the observability of the pair (A, C) implies that (A_d, C) is an observable pair (see [26, Proposition 6.2.11]). Thus there exists a positive integer r, the observability index, such that:

$$\widetilde{C} \doteq \begin{pmatrix} CA_d^{-r+1} \\ \vdots \\ CA_d^{-1} \\ C \end{pmatrix} = \begin{pmatrix} C \\ CA_d \\ \vdots \\ CA_d^{r-1} \end{pmatrix} A_d^{-r+1} \tag{11}$$

has full column rank. For state feedback systems $r = 1$ and $\tilde{C}$ is the identity matrix.

C. Implementation

Our controller consists of three elements: an observer which generates a state estimate $\hat{x}\left(t\right)$ (with the notation $\hat{x}_k \doteq \hat{x}\left(kT_s\right)$); a switching logic which updates the parameters for the quantizer and sends update commands to the observer; and a stabilizing control law which computes the control input based on the state estimate. For simplicity of presentation, we assume the stabilizing control law consists of a static nominal state feedback:

$$\boldsymbol{u}\left(t\right) = K\hat{\boldsymbol{x}}\left(t\right). \tag{12}$$

However, any control law that will render the closed-loop system ISS with respect to the disturbance and the state estimation error will work with our controller.

Given an update command from the switching logic, the observer generates an estimate of the state based on current and previous quantized measurements. We require the state estimate to be exact in the absence of measurement error and disturbance, and to be a linear function of the measurements. For concreteness, we use the following state estimate from [10] which is based on the pseudo-inverse:

$$\hat{\boldsymbol{x}}_k = G\left(\boldsymbol{z};\boldsymbol{u}^d;k\right) \doteq \widetilde{C}^\dagger \begin{bmatrix} \boldsymbol{z}_{k-r+1} + C\sum_{i=1}^{r-1} A_d^{-i}\boldsymbol{u}^d_{k-r+i} \\ \vdots \\ \boldsymbol{z}_{k-1} + CA_d^{-1}\boldsymbol{u}^d_{k-1} \\ \boldsymbol{z}_k \end{bmatrix}, \qquad \widetilde{C}^\dagger \doteq \left(\widetilde{C}^T\widetilde{C}\right)^{-1}\widetilde{C}^T \tag{13}$$

In [23] we presented additional approaches to generate a state estimate that satisfy the above requirements, and compared their properties. Note that we must have at least r successive measurements to generate a state estimate. Therefore, (13) is defined only for $k \geq r-1$. In the special case of state feedback, on which we will comment further as we present our results, the state estimate will then be generated simply as $\hat{\boldsymbol{x}}_k = \boldsymbol{z}_k$.

The observer continuously updates the state estimate based on the nominal system dynamics:

$$\dot{\hat{\boldsymbol{x}}}(kT_s+t) = A_0\hat{\boldsymbol{x}}(kT_s+t) + B\boldsymbol{u}\left(kT_s+t\right), \qquad t\in[0,T_s). \tag{14}$$

The control input is integrated continuously to generate $\boldsymbol{u}^d_k$ (initialized to zero at every $t = kT_s$):

$$\dot{\boldsymbol{u}}^d_k = \exp\left(A\left(\left(k+1\right)T_s - t\right)\right)B\boldsymbol{u}\left(t\right) \qquad \forall t\in\left[kT_s,\left(k+1\right)T_s\right]$$

D. Switching Logic

The switching logic will keep and update a discrete time step variable, $k\in\mathbb{N}$, whose value will correspond to the current sampling time of the continuous system – at each sampling time, the switching logic will update $\hat{x}_k \doteq \hat{x}\left(kT_s\right)$ where k is the discrete time step. At each discrete time step, the switching logic will operate in one of three modes: *capture*, *measurement update* or *escape detection*. The current mode will be stored in the variable $mode(k) \in \{capture, update, detect\}$. The switching logic will also use $p_k \in \mathbb{Z}$ and $saturated(k) \in \{\textbf{true}, \textbf{false}\}$ as auxiliary variables.

We assume the control system is activated at $k=0$ ($t=0$). We initialize $\hat{\boldsymbol{x}}_0 = \boldsymbol{0}$, $mode(0) = capture$, $p_0 = 0$, and $\mu_{-1} = s$, where s is a positive constant which will be regarded as a design parameter. We also have following design parameters: $\alpha\in\mathbb{R}_{>0}$, $\Omega_{\text{out}}\in\mathbb{R}$ such that $\Omega_{\text{out}} > \|A\|$, and $P\in\mathbb{Z}$ such that $P\geq r+1$. We refer the reader to [22, §V] for a detailed qualitative discussion on how each design parameter affects the system performance. We also define

$$F\left(\mu;k\right) \doteq \left\|CA_d\widetilde{C}^\dagger\right\| \|\mu\|_{\{k-r\ldots k-1\}} \tag{15}$$

which in the case of state feedback reduces to $F(\mu;k) \doteq \|A_d\|\,\mu_{k-1}$.

At each discrete time step, k, the switching logic is implemented by sequentially executing the following algorithms:

Algorithm 1 Preliminaries

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if $mode(k) = capture$ **then**
  set $\mu_k = \Omega_{out}\mu_{k-1}$
else if $mode(k) = update$ **then**
  set
$$\mu_k = \frac{F(\mu;k) + \alpha\|\mu\|_{\{k-r-p_{k-1}\ldots k-1-p_{k-1}\}}}{N} \tag{16}$$
else if $mode(k) = detect$ **then**
  set
$$\mu_k = \frac{F(\mu;k) + \alpha\|\mu\|_{\{k-r-p_{k-1}\ldots k-1-p_{k-1}\}}}{N-2} \tag{17}$$
end if
have the observer record $\boldsymbol{z}_k = Q(\boldsymbol{y}(kT_s); C\hat{\boldsymbol{x}}_k, \mu_k)$
if $\exists i$ such that $(\boldsymbol{z}_k)_i = (C\hat{\boldsymbol{x}}_k)_i \pm (N-1)\mu_k$ **then**
  set $saturated(k) =$ **true**
else
  set $saturated(k) =$ **false**
end if
initialize the next mode to be the same as the current mode: $mode(k+1) = mode(k)$

Algorithm 2 *capture* mode

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if $mode(k) = capture$ **then**
  if $saturated(k)$ **then**
    set $p_k = 0$
  else
    set $p_k = p_{k-1} + 1$
    if $p_k = r$ **then**
      set $p_k = 0$
      have the observer update the state estimate: $\hat{\boldsymbol{x}}_k = G(\boldsymbol{z}; \boldsymbol{u}_d; k)$
      switch to the *measurement update* mode: set $mode(k+1) = update$
    end if
  end if
end if

Algorithm 3 *measurement update* mode

```
if mode(k) = update then
  set p_k = p_{k-1} + 1
  have the observer update the state estimate: x̂_k = G(z; u_d; k)
  if p_k = P − r then
    switch to the escape detection mode: set mode(k + 1) = detect
  end if
end if
```

Algorithm 4 *escape detection* mode

```
if mode(k) = detect then
  if not saturated(k) then
    set p_k = p_{k-1} + 1
    have the observer update the state estimate: x̂_k = G(z; u_d; k)
    if p_k = P then
      set p_k = 0
      switch to the measurement update mode: set mode(k + 1) = update
    end if
  else
    set p_k = 0 and μ_k = s
    switch to capture mode: set mode(k + 1) = capture
  end if
end if
```

IV. Main Results

A. The Convergence Property

We define the following convergence property which implies that in an infinite sequence in which the switching logic is never in the *capture* mode (a result of having no disturbance), $\lim_{k\to\infty} \mu_k = 0$.

Set μ' as:

$$\begin{aligned} \mu'_k &= 1, && k \in \{0 \dots r-1\} \\ \mu'_k &= \frac{F(\mu'; k) + \alpha}{N}, && k \in \{r \dots P-1\} \\ \mu'_k &= \frac{F(\mu'; k) + \alpha}{N-2}, && k \in \{P \dots P+r-1\}, \end{aligned} \tag{18}$$

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If there exists $\sigma < 1$ for which the following holds

$$\|\mu'\|_{\{P \ldots P+r-1\}} \leq \sigma \tag{19}$$

then we say that the observer has the *convergence property*.

Whether the observer has the convergence property depends on the choice of the design parameters α and P. The following Lemma (proved in the Appendix) gives a sufficient and easy to verify condition for the existence of design parameters with which the observer will have the convergence property.

Lemma 1: If the following condition holds:

$$\sigma_{pi} \doteq \frac{1}{N} \left\| C A_d \widetilde{C}^{\dagger} \right\| < 1 \tag{20}$$

then it is always possible to choose P and α such that the observer will possess the convergence property.

In the state feedback case we do not need an observer as the updates of the state estimate become simply $\hat{\boldsymbol{x}}_k = G\left(\boldsymbol{z}, \boldsymbol{u}_d, k\right) = \boldsymbol{z}_k$. In this case we just say that the control system has or does not have the convergence property. Note also that in this case (20) becomes $\|A_d\| / N < 1$.

B. Results for When the System Model Is Known

The state estimation error is defined as:

$$\tilde{\boldsymbol{x}}\left(t\right) = \hat{\boldsymbol{x}}\left(t\right) - \boldsymbol{x}\left(t\right). \tag{21}$$

In the simpler case where $A \equiv A_0$, the evolution of the state estimation error is independent of the state. This property is critical in proving the following Proposition, which is the main technical step for deriving the desired stability results.

Proposition 2: If we implement the controller with the above algorithm and the observer has the convergence property then the state estimation error of the closed-loop system will satisfy the parameterized-ISS property with respect to the disturbance:

$$\begin{aligned} &\left|\tilde{\boldsymbol{x}}\left(t\right)\right| \leq \beta_e\left(\left|\tilde{\boldsymbol{x}}\left(t_0\right)\right|, t - t_0; \mu\left(t_0\right)\right) + \gamma_e\left(\|\boldsymbol{w}\|_{[t_0,t]}; \mu\left(t_0\right)\right), \qquad \forall t \geq t_0 \geq 0 \\ &\mu\left(t\right) \leq \delta_e\left(\|\tilde{\boldsymbol{x}}\|_{[t_0,t]}, \mu\left(t_0\right)\right) \end{aligned} \tag{22}$$

where β_e, γ_e and δ_e have the same properties as β, γ and δ in (8), respectively.

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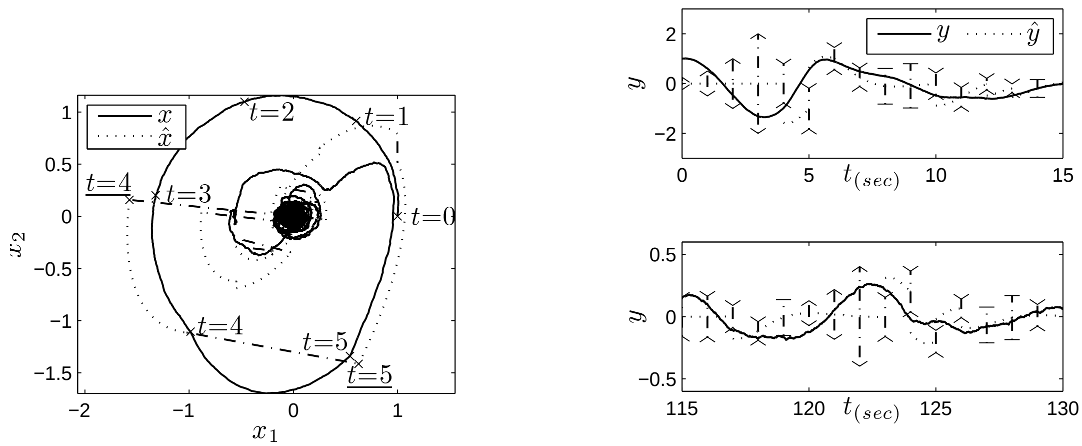

Fig. 2. Simulation of the proposed controller. Simulated here is a two dimensional dynamical system: $\dot{\boldsymbol{x}}(t) = [0.1, -1; 1, 0.1]\,\boldsymbol{x}(t) + [0; 1]\,\boldsymbol{u}(t) + [1, 0; 0, 1]\,\boldsymbol{w}(t)$, where only the first dimension is observed, $\boldsymbol{y}(t) = [1, 0]\,\boldsymbol{x}\,(t)$, through a quantizer with $N = 3$. The solid line in the left plot is the trajectory of the system (starting at $\boldsymbol{x}\,(0) = [1; 0]$). The dotted line in that plot is the state estimate. The dash-dotted lines represent the jumps in the state estimate after a new measurement is received. The locations of the trajectory and the state estimate at the first few sampling times are marked by $\times$. The underlined time indications corresponds to the state estimate. The two plots on the right show time segments of the measured output ($T_s = 1s$). The solid line is the unquantized output ($\boldsymbol{y}$) of the system and the dotted line is its estimate. The vertical dash-dotted lines depict the single bounded quantization region. The controller is in the *capture* mode where these vertical lines are bounded by arrows facing outward; in the *update* mode where the arrows are facing inward; and in the *detect* mode where the vertical lines are bounded by small horizontal lines. Looking both at the left plot and the top right plot, one can observe the initial transient of the system: At $t = 3$ a sufficient number (two) of unsaturated measurements were collected and the controller switches to the *update* mode; this causes the state estimate to jump at $t = 4$ from the origin to $\approx [-1.6; 0.2]$; and at $t = 5$ the state estimate jumps even closer to the true state. Looking at the bottom right plot, one can observe the steady-state behavior of the simulation, where an escape of the trajectory due to disturbance is detected at $t = 119s$, and then the trajectory is recaptured at $t = 122s$. The design parameters were: $P = 6$, $\mu\,(0) = 0.25$, $\Omega_{out} = 2$, $\alpha = 0.02$, $s = 0.05$, $K = [0.6, -1.5]$. The disturbance followed the zero-mean normal distribution with standard deviation of 0.2.

Our first stability result is the following:

Theorem 1: With the assumptions stated in Proposition 2, the closed-loop system will be parameterized-ISS with respect to the disturbance, in the sense of (8).

An illustrative simulation of the proposed controller is given in Figure 2. The proofs of Proposition 2 and Theorem 1 will follow the statements of the technical lemmas below. The proofs of the technical lemmas are deferred to appendix A.

Lemma 3: Assume that for some time step k' we have $mode(k'+1) = update$ and $p(k') = 0$ (i.e. an *update measurement* sequence starts at $k'+1$). If $\forall k \in \{k'+1 \ldots k'+P+1\}$,

$mode(k) \neq capture$ (i.e. by time step $k' + P$ the controller has not switched to the *capture* mode) then $\|\mu\|_{\{k'-r+1+P\ldots k'+P\}} \leq \sigma \|\mu\|_{\{k'-r+1\ldots k'\}}$.

Lemma 4: There exist constants $\zeta_D > 0$ and $\zeta_\mu > 0$ with the following properties: If for some time step k' we have $mode(k'+1) = update$ and $p(k') = 0$, and the input is such that

$$\|\mu\|_{\{k'-r+1\ldots k'\}} > \frac{1}{\alpha}\zeta_D \left\|\boldsymbol{w}^d\right\|_{\{k'-r+1,k'+P-2\}}, \tag{23}$$

then $mode(m) = update\ \forall m \in \{k'+2\ldots k'+P-r\}$, $mode(m) = detect\ \forall m \in \{k'+P-r+1\ldots k'+P\}$, $mode(k'+P+1) = update$, and

$$\|\tilde{\boldsymbol{x}}\|_{\{k'\ldots k'+P-1\}} \leq \zeta_\mu \|\mu\|_{\{k'-r+1\ldots k'\}}. \tag{24}$$

Lemma 5: Assume that for some time step k' we have $mode(k'+1) = update$ and $p(k') = 0$. Let $k_2 = \min\{k'+P, \min\{k\,|mode(k+1) = capture, k > k'\}\}$. There exists a constant $\zeta_w > 0$ such that if the disturbance does not satisfy (23), then

$$\|\tilde{\boldsymbol{x}}\|_{\{k'\ldots k_2-1\}} \leq \zeta_w \left\|\boldsymbol{w}^d\right\|_{\{k'-r+1\ldots k'+P-2\}}.$$

Lemma 6: Let k_2 be any time step such that $mode(k_2+1) = capture$. There exist constants $\zeta_C > 0$ and $\zeta_b > 0$, and functions $\tilde{\delta}_1(\nu;\rho) : \mathbb{R}^2_{>0} \to \mathbb{R}_{>0}$ and $T_1^*(\nu;\rho) : \mathbb{R}^2_{>0} \to \mathbb{R}_{>0}$, each nondecreasing in ν when ρ is fixed, with the following properties: There exists $k_3 > k_2$ such that $k_3 < k_2 + T_1^*\left(|\tilde{x}_{k_2}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_2}\right)$, $mode(k_3+1) = update$, $p(k_3) = 0$, $\|\tilde{\boldsymbol{x}}\|_{\{k_2\ldots k_3\}} \leq \tilde{\delta}_1\left(|\tilde{x}_{k_2}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_2}\right)$ and $\|\mu\|_{k_3-r+1\ldots k_3} \leq \mu_{k_2}\Omega_{out}^{T_1^*(\nu;\rho)}$; the functions $\tilde{\delta}_1$ and T_1^* satisfy $\tilde{\delta}_1(\nu;\rho) \leq \rho\zeta_b\Omega_{out}^{T_1^*(\nu;\rho)}\ \forall \nu, \rho$.

Lemma 7: Let k_2 be an arbitrary time step. There exist a constant $\zeta_s > 0$, a class $\mathcal{K}$ function ε, and functions $\tilde{\delta}_2(\nu;\rho) : \mathbb{R}^2_{>0} \to \mathbb{R}_{>0}$ and $T_2^*(\nu;\rho) : \mathbb{R}^2_{>0} \to \mathbb{R}_{>0}$ with the following properties: If $|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C \left\|\boldsymbol{w}^d\right\| \leq \varepsilon(\mu_{k_2})$ then $k_3 \doteq k_2 + T_2^*\left(|\tilde{x}_{k_2}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_2}\right)$ satisfies $\|\tilde{\boldsymbol{x}}\|_{\{k_2\ldots k_3\}} \leq \tilde{\delta}_2\left(|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_2}\right)$,$\|\mu\|_{k_3-r+1\ldots k_3} \leq \mu_{k_2}\zeta_s\sigma^{T_2^*/P}$, $mode(k_3+1) = update$ and $p(k_3) = 0$; when ρ is fixed the function $\tilde{\delta}_2(\cdot;\rho)$ is of class $\mathcal{K}_\infty$; the functions $\tilde{\delta}_2$ and T_2^* satisfy $\tilde{\delta}_2(\nu;\rho) \leq \rho\zeta_s\sigma^{T_2^*(\nu;\rho)/P}/\|C\|\ \forall \nu, \rho$.

Proof of Proposition 2: Assume first t_0 is at a sampling time and let k_0 be such that $k_0 T_s = t_0$. Let k_1 be some time step to be defined below with the following *SS properties*: $mode(k_1+1) = update$, $p(k_1) = 0$ and (23) does not hold with $k' = k_1$. The proof will proceed in four steps: in the first step we will derive a bound on the trajectory from k_0 to k_1; in the second step we will derive a bound on the trajectory from k_1 to infinity; in the third step we

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will combine these two bounds and derive the ISS bound on the estimation error; in the fourth step we will derive the bound on the zoom factor.

1) Assume first that $mode\,(k_0) = capture$. Define k_1 to be the first time step after k_0 with the SS properties. If such a time step does not exist, define $k_1 \doteq \infty$. By Lemma 6 we know there exists $k^* \leq k_0 + T_1^*\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$ such that $\|\tilde{\boldsymbol{x}}\|_{k_0 \dots k^*} \leq \tilde{\delta}_1\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$. With Lemma 6, together with Lemmas 3 and 4, we also have that if $k_1 > k^*$ then $|\tilde{\boldsymbol{x}}_k| \leq \zeta_\mu \mu_{k_0} \Omega_{out}^{T_1^*} \sigma^{\left\lfloor \frac{k-k^*}{P} \right\rfloor} \leq \zeta_\mu \mu_{k_0} \Omega_{out}^{T_1^*} \sigma^{\left\lfloor \frac{k-T_1^*}{P} \right\rfloor}$ $\forall k \in \{k^* \dots k_1\}$. As Lemma 6 also states $\tilde{\delta}_1\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right) \leq \mu_{k_0} \zeta_b \Omega_{out}^{T_1^*}$, we can finally derive $|\tilde{\boldsymbol{x}}_k| \leq \tilde{\beta}_c\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|, k - k_0; \mu_{k_0}\right)$ $\forall k \in \{k_0 \dots k_1\}$ where

$$\tilde{\beta}_c(\nu, k; \rho) \doteq \min\left\{\tilde{\delta}_1(\nu; \rho), \rho\left(\frac{\Omega_{out}}{\sigma^{1/P}}\right)^{T_1^*(\nu;\rho)} \sigma^{\frac{k}{P}-1} \max\{\zeta_\mu, \zeta_b\}\right\}. \tag{25}$$

If $mode\,(k_0) \neq capture$ then there is a time step k_1', $k_0 - P < k_1' \leq k_0$, such that $mode(k_1'+1) = update$ and $p(k_1') = 0$. If in addition (23) does not hold with $k' = k_1'$, then we define $k_1 = k_1'$, and thus we have, vacuously, $|\tilde{\boldsymbol{x}}_k| \leq 0$ $\forall k \in \{k_0 \dots k_1\}$. If (23) does hold with $k' = k_1'$, then with k_1 defined as the first time step after k_0 with the SS properties, we can write: $|\tilde{\boldsymbol{x}}_k| \leq \zeta_\mu \mu_{k_0} \sigma \left\lceil \frac{k-k_0}{P} \right\rceil$ $\forall k \in \{k_0 \dots k_1\}$. Taking into consideration that $mode\,(k_0) = capture$ only if $\mu_{k_0} \geq s$, we can now write $|\tilde{\boldsymbol{x}}_k| \leq \tilde{\beta}_1\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|, k - k_0; \mu_{k_0}\right)$ $\forall k \in \{k_0 \dots k_1\}$ where

$$\tilde{\beta}_1(\nu, k; \rho) \doteq \begin{cases} \max\left\{\tilde{\beta}_c(\nu, k; \rho), \zeta_\mu \rho \sigma \left\lceil \frac{k-k_0}{P} \right\rceil\right\} & \rho \geq s \\ \zeta_\mu \rho \sigma \left\lceil \frac{k-k_0}{P} \right\rceil & \rho < s. \end{cases} \tag{26}$$

Assume now that $|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\| \leq \varepsilon(\mu_{k_0})$ where $\varepsilon(\cdot)$ comes from Lemma 7. Then by Lemma 7 we know there exists $T_2^* = T_2^*\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$ such that $\|\tilde{\boldsymbol{x}}\|_{k_0 \dots T_2^*} \leq \tilde{\delta}_2\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$. With Lemma 7, together with Lemmas 3 and 4, we also have that if $k_1 > k^*$ then $|\tilde{\boldsymbol{x}}_k| \leq \zeta_\mu \mu_{k_0} \zeta_s \sigma^{T_2^*/P} \sigma^{\left\lfloor \frac{k-T_2^*}{P} \right\rfloor}$ $\forall k \in \{T_2^* \dots k_1\}$. As we're also given from Lemma 7 that $\tilde{\delta}_2\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right) \leq \mu_{k_0} \zeta_s \sigma^{T_2^*/P} / \|C\|$, we can finally derive $\forall k \in \{k_0 \dots k_1\}$: $|\tilde{\boldsymbol{x}}_k| \leq \tilde{\beta}_2\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|, k - k_0; \mu_{k_0}\right)$ where

$$\tilde{\beta}_2(\nu, k; \rho) \doteq \min\left\{\tilde{\delta}_2(\nu; \rho), \rho \zeta_s \sigma^{\lfloor k/P \rfloor} \max\{\zeta_\mu, 1/\|C\|\}\right\}. \tag{27}$$

For fixed ν and ρ, both $\lim_{k\to\infty} \tilde{\beta}_1(\nu, k; \rho) = 0$ and $\lim_{k\to\infty} \tilde{\beta}_2(\nu, k; \rho) = 0$. Also, for fixed k and ρ, both $\tilde{\beta}_1(\nu, k; \rho)$ and $\tilde{\beta}_2(\nu, k; \rho)$ are continuous and nondecreasing with respect to ν. However, only $\tilde{\beta}_2$ satisfies $\tilde{\beta}_2(0,k;\rho)=0\ \forall k\forall\rho$, and $\tilde{\beta}_2$ is a valid bound on the trajectory only when $|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|\leq\varepsilon(\mu_{k_0})$. Nevertheless, it is possible to construct a class $\mathcal{KL}$ function, $\hat{\beta}(\nu,k;\rho)$, such that $\hat{\beta}(\nu,k;\rho)\geq\tilde{\beta}_2(\nu,k;\rho)$ when $\nu\leq\varepsilon(\rho)$ and $\hat{\beta}(\nu,k;\rho)\geq\tilde{\beta}_1(\nu,k;\rho)$ otherwise. With this $\hat{\beta}(\nu,k;\rho)$ we can write $|\tilde{\boldsymbol{x}}_k|\leq\hat{\beta}\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|,k-k_0;\mu_{k_0}\right)\ \forall k\in\{k_0\ldots k_1\}$.

Note that all the functions mentioned above are continuous in ν and ρ, $\forall\nu\in\mathbb{R}_{\geq 0}$ and $\forall\rho\in\mathbb{R}_{>0}$. They are not, however, all continuous at $\rho=0$ (or even defined) since $\lim_{\rho\searrow 0}T_1^*(\nu;\rho)=\infty$ for every $\nu>0$. Nevertheless, $\hat{\beta}(\nu,k;\rho)$ is continuous at $\rho=0$. This is due to ε being a of class $\mathcal{K}$, which implies that for sufficiently small ρ, $\hat{\beta}(\nu,k;\rho)=\tilde{\beta}_1(\nu,k;\rho)=\zeta_\mu\rho\sigma\lceil(k-k_0)/P\rceil$.

2) Let k_2 be the first time step after k_1 such that $mode(k_2)=detect$ and $mode(k_2+1)=capture$. If such a time step does not exist, we set $k_2=\infty$. From Lemma 5 we have that $\|\tilde{\boldsymbol{x}}\|_{\{k_1\ldots k_2\}}\leq\zeta_w\left\|\boldsymbol{w}^d\right\|$. Let k_4 be the first time step after k_2 such that $mode(k_4+1)=update$, $p(k_4)=0$ and (23) does not hold with $k'=k_4$. Replacing k_0 with k_2 in the previous step, we can write

$$\|\tilde{\boldsymbol{x}}\|_{\{k_2\ldots k_4\}}\leq\hat{\beta}\left(|x_{k_2}|+\zeta_C\left\|\boldsymbol{w}^d\right\|,k-k_2;\mu_{k_2}\right)\leq\hat{\beta}\left((\zeta_w+\zeta_C)\left\|\boldsymbol{w}^d\right\|,0;s\right)\doteq\tilde{\gamma}\left(\left\|\boldsymbol{w}^d\right\|\right).$$

Since k_4 also satisfies the SS properties as does k_1, we can repeat these arguments for future time steps and arrive at $\|\tilde{\boldsymbol{x}}\|_{\{k_1\ldots\infty\}}\leq\hat{\gamma}\left(\left\|\boldsymbol{w}^d\right\|\right)$, where $\hat{\gamma}(\nu)\doteq\max\{\zeta_w\nu,\tilde{\gamma}(\nu)\}$. Note that $\hat{\delta}(\cdot)$ is of class $\mathcal{K}_\infty$.

3) Combining the last two steps, we can derive the first condition for the parametrized ISS property at the discrete times: for all $k\in\{0\ldots\infty\}$,

$$|\tilde{\boldsymbol{x}}_k|\leq\beta_e(|\tilde{\boldsymbol{x}}_{k_0}|,k;\mu_{k_0})+\gamma_e\left(\left\|\boldsymbol{w}^d\right\|;\mu_{k_0}\right)$$

where $\beta_e(\nu,k;\mu)\doteq\hat{\beta}(2\nu,k;\mu)$ and $\gamma_e(\nu;\mu)\doteq\hat{\beta}(2\zeta_C\nu,0;\mu)+\hat{\gamma}(\nu)$. Note that indeed β_e and γ_e of class $\mathcal{KL}$ and $\mathcal{K}_\infty$, respectively.

The extension from the discrete analysis to continuous time, with the estimation error defined as $\tilde{\boldsymbol{x}}(t)\doteq\hat{\boldsymbol{x}}(t)-\boldsymbol{x}(t)$ for every $t\geq t_0$, can be proved along the lines of [27, Theorem 6]. This proves the first line of (22).

4) To construct the bound on μ we consider the three phases of the trajectory: initial *capture* sequence, *zoom-in* sequences and subsequent *capture* sequences. If $mode(k_0)=capture$

we start with μ_{k_0} and we grow the zoom factor until for r successive time steps we have $(N-2)\,\mu_k > |\tilde{\boldsymbol{y}}_k|$. Thus at the initial *capture* sequence we have

$$\|\mu\| \leq \Omega_{out}^r \max\left\{\mu_{k_0}, \|C\|\,\|\tilde{\boldsymbol{x}}\| \,/\, (N-2)\right\}. \tag{28}$$

At a *zoom-in* sequence we may initially enlarge μ by a factor of $\|\mu'\|$ with μ' defined according to (18). However, after this possible initial enlargement, μ is decreased by a factor of σ every P steps. At subsequent *capture* sequences we start with $\mu_k = s$ and enlarge it again until for r successive time steps we have $(N-2)\,\mu_k > |\tilde{\boldsymbol{y}}_k|$. Combining all these observations together we can set γ_μ from (22) as:

$$\gamma_\mu\left(\nu, \mu_0\right) \doteq \|\mu'\|\,\Omega_{out}^r \max\left\{\mu_0, s, \|C\|\,\|\tilde{\boldsymbol{x}}\| \,/\, (N-2)\right\}.$$

■

Proof of Theorem 1: With $A+BK$ being Hurwitz, the stabilizing control law, $u = K\hat{x}$, renders the closed-loop system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + D\boldsymbol{w} = (A+BK)\,\boldsymbol{x} + BK\tilde{\boldsymbol{x}} + D\boldsymbol{w} \tag{29}$$

ISS with respect to the disturbance and the estimation error. Combining this ISS property with the ISS property proved in Proposition 2, and applying a cascade argument similar to what was used to prove [24, Proposition 7.2], we can conclude that the closed-loop system is ISS with respect to the disturbance. ■

C. Robustness to Modeling Errors

We represent modeling errors as $A(t) = A_0 + \Delta A(t)$ with only A_0 known and $\Delta A(t) \not\equiv 0$. It is assumed though that $\|\Delta A(t)\| \leq \delta_A$ where δ_A is given. To deal with such modeling errors the only change needed in the design is in the stabilizing control law, where K will be chosen such there exist two positive definite symmetric matrices, P and Q, for which the following holds:

$$P\left(A_0 + \Delta A + BK\right) + \left(A_0 + \Delta A + BK\right)^T P + Q < 0, \qquad \|\Delta A(t)\| < \delta_A. \tag{30}$$

It is well-known and easy to show using a Lyapunov argument that if (30) holds then the system (29) has the ISS stability property with respect to the estimation error and disturbance:

$$|\boldsymbol{x}(t)| \leq \beta_x\left(\boldsymbol{x}(t_0), t-t_0\right) + \gamma_{x,e}\left(\|\tilde{\boldsymbol{x}}\|_{[t_0,t]}\right) + \gamma_{x,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right), \quad \forall t > t_0 > 0 \tag{31}$$

where β_x is of class $\mathcal{KL}$ and $\gamma_{x,w}$ and $\gamma_{w,x}$ are of class $\mathcal{K}_\infty$. Such a stabilizing gain matrix K can be found by using Linear Matrix Inequality (LMI) techniques.

With this stabilizing control law, we derive our second stability result:

Theorem 2: Assume the observer has the convergence property and the stabilizing control law is chosen so that (30) holds. There exists $\delta_A > 0$ such that if $\|\Delta A\| < \delta_A$ then the closed-loop system has the local practical parameterized ISS property (9) for some $d \geq 0$, $x_{max} > 0$ and $w_{max} > 0$.

Proof: The estimation error now follows the following dynamics between sampling times:

$$\dot{\tilde{\boldsymbol{x}}} = A\tilde{\boldsymbol{x}} - \Delta A\boldsymbol{x} - D\boldsymbol{w}. \tag{32}$$

Therefore its evolution is no longer independent from the state of the system. The proposed controller in this case will render the estimation error parameterized-ISS with respect to both the disturbance and the system's state:

$$\begin{aligned} &|\tilde{\boldsymbol{x}}(t)| \leq \beta_e\left(\tilde{\boldsymbol{x}}(t_0), t - t_0; \mu(t_0)\right) + \gamma_{e,x}\left(\delta_A \|\boldsymbol{x}\|_{[t_0,t]}; \mu(t_0)\right) + \gamma_{e,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}; \mu(t_0)\right), \\ &\mu(t) \leq \gamma_\mu\left(\|\tilde{\boldsymbol{x}}\|_{[t_0,t]}, \mu(t_0)\right), \qquad \forall t_0 > 0 \quad \forall t > t_0. \end{aligned}$$

Due to the interleaved dependency of $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ on each other we can no longer apply the Cascade Theorem, but we can apply the Small-Gain Theorem. In Appendix B we state and prove the Small-Gain Theorem for the special case of parameterized-ISS systems. To apply the Small-Gain Theorem we need the small-gain condition

$$\gamma_{x,e}\left(\gamma_{e,x}\left(\delta_A r; \mu\right)\right) < r \text{ and } \gamma_{e,x}\left(\delta_A \gamma_{x,e}(r); \mu\right) < r, \quad \forall r \in [r_0, r_1] \subset \mathbb{R}_{\geq 0}, \forall \mu \in [0, \mu_1], \tag{33}$$

to hold. For every fixed μ, $\gamma_{e,x}(\cdot;\mu)$ and $\gamma_{x,e}(\cdot;\mu)$ are of class $\mathcal{K}_\infty$. Thus for every $r \in [r_0, r_1]$ and every $\mu \in [0, \mu_1]$ there exists small enough but strictly positive δ_A for which the small-gain condition holds. Since $[r_0, r_1] \times [0, \mu_1]$ is a compact set and all the functions in (33) are continuous, the minimum of δ_A satisfying the small-gain condition over this whole set must also be strictly positive. ■

Note that because for every fixed μ, $\gamma_{e,x}(r, \mu)$ grows faster than any linear function of r both at $r = 0$ and at $r = \infty$, we can not choose $r_0 = 0$ or $r_1 = \infty$ in the proof of Theorem 2. These super-linear gains are not an artifact of our design. Recently Nuno Martins showed using techniques from information theory that it is impossible to achieve ISS with linear gain for any linear system with finite data rate feedback [28].

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V. Approaching the Minimal Data Rate

Several papers ([8],[15],[16],[18],[19]) present the same lower bound on the data rate necessary to stabilize a given system. This bound, in terms of the bit-rate (R) to be transmitted, is:

$$R > R_{min} \doteq \frac{\sum_{|\eta_j|\geq 1} \log_2 |\eta_j|}{T_s} \tag{34}$$

where $\eta_j' s$ are the eigenvalues of the discrete open-loop matrix $\Phi \doteq \exp(AT_s)$. Note that the bound (34) is independent of the disturbance characteristics and is applicable to systems with no disturbances.

Lemma 8: To achieve ISS in the state feedback case, it is necessary and sufficient for the data rate to satisfy (34).

Proof (sketch): Since (34) is necessary for asymptotic stability in the disturbance-free case, it is also necessary to achieve disturbance rejection in the ISS sense (which reduces to asymptotic stability when the disturbance is zero). Now for the sufficiency. Consider the scalar unstable case, $n_x = 1$, $A \equiv a > 0$. In this case the minimum data rate is $R_{min} = \log_2 \exp(T_s a)/T_s$. The data rate of our scheme is $\log_2(N)/T_s$ where we require N to be an odd integer which, together with P and α, satisfies the convergence property. Note that for this simple case, in the limit as $\alpha \searrow 0$ the convergence property becomes

$$\exp(T_s a)^P / \left(N^{P-r}(N-2)^r\right) < 1. \tag{35}$$

As the above is equivalent to $(\exp(T_s a)/N)^P < (N-2)^r/N^r$, we can see that for any $N > \exp(T_s a)$ we can find P large enough to satisfy (35). Because of the continuous dependence of the convergence property on α, if (35) is satisfied then there exists $\alpha > 0$ which satisfies the convergence property. Thus to be able to find design parameters that satisfy the convergence property, we only need $N > \exp(T_s a)$. To deal with the constraint that N is an integer, we can use a different number of quantization regions at each sampling time. This makes our data rate $\log_2\left(\tilde{N}\right)/T_s$ with $\tilde{N}$ being the average number of quantization regions per sampling time. As $\tilde{N}$ does not have to be integer, we can have $\tilde{N}$ approach $\exp(T_s a)$ and make our data rate arbitrarily close to the minimum data rate.

Extension to the multidimensional case with distinct real eigenvalues is trivial if we allocate a different number of quantization regions for each unstable mode of the system. Extension to systems with imaginary eigenvalues and to non diagonalizable systems is explained in [16] (see also [22, §VI]). ∎

VI. Extension to Nonlinear Systems

The crucial properties of linear systems which are used in the proof of Theorem 1 are (a) that the continuous, unquantized, closed-loop system is ISS with respect to the estimation error and the disturbance, and (b) that the update law for the estimated state between the sampling times (14) is such that the estimation error grows between these sampling times according to:

$$\lim_{t \nearrow T_s} \left\|\tilde{\boldsymbol{x}}\left(kT_s+t\right)\right\| \leq \lambda_e \left\|\tilde{\boldsymbol{x}}\left(kT_s\right)\right\| + \lambda_w \left\|\boldsymbol{w}\right\|_{[kT_s,(k+1)T_s]} + \lambda_x \left\|\boldsymbol{x}\right\|_{[kT_s,(k+1)T_s]} \tag{36}$$

where λ_e, λ_w and λ_x are known constants. For linear systems these constants are $\lambda_e = \|A_d\|$, $\lambda_w = \left\|\int_0^{T_s} \exp\left(A_0\left(T_s-t\right)\right) D dt\right\|$ and $\lambda_x = \left\|\int_0^{T_s} \exp\left(A_0\left(T_s-t\right)\right) dt\right\| \delta_A$, which follows easily from (10). If (36) holds globally, $\lambda_x = 0$ (as in the case where the exact system model is known), and the number of quantization regions allows the observer to satisfy the convergence property, then the quantized closed-loop system will be parameterized ISS with respect to the disturbance. If $\lambda_x \neq 0$, as in the case where modeling errors exists, then an additional small-gain condition must be satisfied in order to achieve the local practical parameterized ISS property.

Both properties are not unique to linear systems and they can also be formulated for nonlinear systems. This leads to a better conceptualization of our results. Consider a nonlinear system

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)). \tag{37}$$

with $\boldsymbol{y}(t) = \boldsymbol{x}(t)$ (state feedback). State feedback control laws that render unquantized systems ISS with respect to either external disturbances or measurement errors have been proposed for certain nonlinear systems, see for example the discussions in [5], [14] and the references therein. Designing state feedback control laws that render unquantized systems ISS with respect to *both* external disturbances and measurement errors is still considered an open problem. The two closest results, for systems in strict feedback from, appear in [29, §6.2.2] and [30].

Assume that (37) satisfies the Lipschitz property: There exist $l_x > 0$, $l_w > 0$, $L_x > 0$ and $L_w > 0$ such that

$$|f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) - f(\hat{\boldsymbol{x}}, \boldsymbol{u}, 0)| \leq L_x |\boldsymbol{x} - \hat{\boldsymbol{x}}| + L_w |\boldsymbol{w}|, \qquad \forall |\boldsymbol{x}| < l_x,\ \forall |\hat{\boldsymbol{x}}| < l_x,\ \forall |\boldsymbol{w}| < l_w \tag{38}$$

holds. When the Lipschitz property holds globally, which is the case with linear systems, $l_x = l_w = \infty$. Assuming the exact system model is known, if we update our state estimate between sampling times according to $\dot{\hat{\boldsymbol{x}}} = f\left(\hat{\boldsymbol{x}}, \boldsymbol{u}, 0\right)$ then (36) holds with

$$\lambda_e \doteq e^{T_s L_x}, \qquad \lambda_w \doteq \int_0^{T_s} e^{(T_s-\tau)L_x} d\tau L_w. \tag{39}$$

To make the convergence property applicable to state feedback nonlinear systems, the only change needed is to redefine:

$$F(\mu;k) \doteq \lambda_e \|\mu\|_{\{k-r\ldots k-1\}}. \tag{40}$$

A sufficient condition for the control system to have the convergence property remains $\lambda_e/N < 1$.

The above discussion leads to our third stability result:

Theorem 3: Consider a state feedback nonlinear system:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)), \qquad \boldsymbol{z}_k = Q(\boldsymbol{x}_k; c_k, \mu_k) \tag{41}$$

where f has the Lipschitz property (38), and for which there exists a static feedback $\boldsymbol{u} = k(\boldsymbol{x})$ which renders the dynamics $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}, k(\boldsymbol{x}+\boldsymbol{e}), \boldsymbol{w})$ ISS with respect to $\boldsymbol{e}$ and $\boldsymbol{w}$. If $e^{T_s L_x}/N < 1$ then there exists a choice of α and P with which the control system has the convergence property with $F(\mu;k)$ defined in (40). With this choice of α and P and a choice of $\Omega_{out} > e^{T_s L_x}$ and $s > 0$ the system will have the parameterized ISS property for some β and γ if it can be guaranteed that $\|\boldsymbol{x}\| < l_x$ and $\|\boldsymbol{w}\| < l_w$. For $|\boldsymbol{x}(0)| < x_{max}$ and $\|\boldsymbol{w}\| < w_{max}$ such that

$$\beta(x_{max}, 0; s) + \gamma(w_{max}; s) \le l_x \qquad \text{and} \qquad w_{max} \le l_w \tag{42}$$

this will be guaranteed and therefore the system will have the local practical parametrized ISS property. If the Lipschitz property holds globally, then the closed-loop system will have the parameterized ISS property.

A natural question would be what is the necessary number of quantizations regions needed to achieve ISS for a given bound on $|\boldsymbol{x}(0)|$ and $\|\boldsymbol{w}\|$. Unfortunately, the theorem does not give a direct answer to this question. Nevertheless, we can say the following: Given x_{max}, $l_w = w_{max}$, l_x, $\lambda_e = e^{T_s L_x}$, such that both (38) and

$$\beta_x(x_{max}, 0) + \gamma_{x,e}\left(\max\left\{\lambda_e\left(x_{max} + \frac{w_{max}}{\lambda_e - 1}\right), \frac{\lambda_e^3}{\lambda_e - 1} w_{max}\right\}\right) + \gamma_{x,w}(w_{max}) < l_x$$

hold, where β_x, $\gamma_{x,e}$ and $\gamma_{x,w}$ are the ISS gains of the state feedback control law, there exist appropriate design parameters P, Ω_{out}, α, N and s with which the closed-loop system will have the local practical parametrized ISS property.

The proof of Theorem 3 follows the same lines as the proof of Theorem 1 and it is therefore omitted. See also [11] for a similar result but without disturbances.

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VII. Conclusions

In this paper we showed how to achieve input-to-state stability with respect to external disturbances using measurements from a dynamic quantizer. We showed that our technique is applicable to output feedback, is robust to modeling errors, and can work with data rates arbitrarily close to the minimum data rate for unperturbed systems. We also showed that our technique can be extended to nonlinear systems.

The following are some problems which were raised by this work, and should be addressed in future research. In the state feedback case, we show what is the necessary and sufficient number of quantization regions required for Input-to-State Stability. In the output feedback case, however, we can only show that a given number of quantization regions is sufficient based on which observer is implemented. It is possible that using another observer a smaller number of quantization regions will be sufficient. This raises the question of what is the optimal observer. When addressing this question one usually needs to consider also the computational resources that are available for the observer.

Our analysis only considers the worst-case scenario, defined by the bound on the magnitude of the actual disturbance. In many applications the disturbance can be modeled to follow a certain distribution which rarely produces the worst-case disturbance. By utilizing the knowledge of the underlying distribution, it might be possible to get a more accurate description of the behavior of the system. It should also be possible in this case to provide better tools for choosing the design parameters under different performance requirements.

Appendix A: Proofs of the Technical Lemmas

Proof of Lemma 1: Assume α satisfies $\sigma_{pi} + \frac{a}{N} \leq 1$ and for simplicity assume also that P is a multiple of r. Then for all $l \in \{1 \dots P/r - 1\}$:

$$\|\mu'\|_{lr\dots(l+1)r-1} \leq \sigma_{pi}^{l} + \sum_{m=0}^{l-1} \sigma_{pi}^{m} \frac{\alpha}{N} \doteq V(l) .$$

Because $\sigma_{pi} < 1$ we have that $V(l)$ converges to $\frac{\alpha}{N(1-\sigma)}$ as $l \to \infty$. We also have

$$\|\mu'\|_{P-r\dots P-1} \leq \max\left\{ \frac{N}{N-2}\sigma_{pi} V(P/r-1) + \frac{\alpha}{N-2}, \right.$$
$$\left. \left(\frac{N}{N-2}\sigma_{pi}\right)^{r} V(P/r-1) + \sum_{m=0}^{r-1} \left(\frac{N}{N-2}\sigma_{pi}\right)^{m} \frac{\alpha}{N-2} \right\} .$$

Since we can make $V(P/r-1)$ arbitrarily small by taking P to be large enough and α to be small enough, we can make $\|\mu'\|_{P-r\ldots P-1} < 1$ which satisfies the convergence property. ■

We will use the following definition in the proofs below:

$$\widetilde{D} \doteq \begin{bmatrix} -C & -CA_d^{-1} & \cdots & -CA_d^{-r+2} \\ 0 & -C & \cdots & -CA_d^{-r+3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -C \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Proof of Lemma 3: Set $\lambda = \|\mu\|_{\{k'-r+1\ldots k'\}}$. Between time steps $k'+1$ and $k'+P$, μ is updated according to (16) or (17). Note that $F(\mu;k)$ depends linearly, with positive coefficients, on $\mu_{k-r}\ldots\mu_{k-1}$. Therefore, it is easy to see by induction from $k=k'+1$ to $k=k'+P$ that $\mu_k \le \lambda\mu'_{k-k'+r-1}$. As we have that condition (19) holds, the result of the lemma follows. ■

Proof of Lemma 4: The settings $mode(k'+1) = update$ and $p=0$ imply that for $m \in \{k'-r+1\ldots k'\}$ we had $saturated(m) =$ **false** and either $mode(m) = capture$ or $mode(m) = detect$. The structure of our quantizer is such that if $saturated(m) =$ **false** for some m then $|\tilde{\boldsymbol{y}}_m| < \mu_m$ where $\tilde{\boldsymbol{y}}_m \doteq \boldsymbol{z}_m - \boldsymbol{y}_m$ denotes the quantization error. The observations can be written as

$$\boldsymbol{z}_{k-l} = CA_d^l\boldsymbol{x}_k - C\sum_{i=1}^{l} A_d^{-i}\boldsymbol{u}^d_{k-l+i-1} - C\sum_{i=1}^{l} A_d^{-i}\boldsymbol{w}^d_{k-l+i-1} + \tilde{\boldsymbol{y}}_{k-l}. \tag{43}$$

Since the state estimate (13) was chosen so that $\hat{\boldsymbol{x}}_k = \boldsymbol{x}_k$ in the absence of measurement errors and disturbances, we get together with (43) that

$$\tilde{\boldsymbol{x}}_k^+ = G\begin{bmatrix} \tilde{\boldsymbol{y}}_{k-r+1} \\ \vdots \\ \tilde{\boldsymbol{y}}_k \end{bmatrix} + G\widetilde{D}\begin{bmatrix} \boldsymbol{w}^d_{k-r+1} \\ \vdots \\ \boldsymbol{w}^d_{k-1} \end{bmatrix}. \tag{44}$$

When taking the next measurement at time step $k+1$, the distance between the real output, $\boldsymbol{y}_{k+1}$, and the center of the quantizer $C\hat{\boldsymbol{x}}^-_{k+1}$ is

$$\left|\boldsymbol{y}_{k+1} - C\hat{\boldsymbol{x}}^-_{k+1}\right| = \left|CA_d\tilde{\boldsymbol{x}}^+_k + C\boldsymbol{w}^d_k\right| \le F(\mu;k+1) + \left\|\left[CA_dG\widetilde{D}\,|\,C\right]\right\|\left\|\boldsymbol{w}^d\right\|_{[k'-r+1,k]}. \tag{45}$$

Given that (23) holds with

$$\zeta_D \doteq \left\|\left[CA_dG\widetilde{D}\,|\,C\right]\right\|,$$

we have from (16) that

$$\left|\boldsymbol{y}_{k+1} - C\hat{\boldsymbol{x}}_{k+1}^{-}\right| \leq N\mu_{k+1}. \tag{46}$$

The structure of our quantizer guarantees in this case that $\left|\tilde{\boldsymbol{y}}_{k+1}\right| \leq \mu_{k+1}$. We can now repeat these arguments and show that (44)–(46) holds for all $k \in \{k' \ldots k' + P - r\}$.

At time steps $k' + P - r$ the controller will switch to $mode(k' + P - r + 1) = detect$, and we will have for $l = P - r + 1$ that $\left|\boldsymbol{y}_{k'+l} - C\hat{\boldsymbol{x}}_{k'+l}^{-}\right| \leq (N-2)\,\mu_{k'+l}$. This guarantees that both $\left|\tilde{\boldsymbol{y}}_{k'+l}\right| \leq \mu_{k'+l}$ and $saturated(k'+l) =$ **false**, thus $mode(k'+l+1) = detect$. Again, we can repeat these arguments for $l \in \{P - r + 2 \ldots P\}$ with the exception that for $l = P$ the controller will set $mode(k' + \ l + \ 1) = update$.

Based on (44) we can bound the estimation error for $l \in \{0 \ldots P-1\}$ as:

$$\left|\tilde{\boldsymbol{x}}_{k'+l}^{+}\right| \leq \|G\| \, \|\mu\|_{\{k'-r+1\ldots k'+l\}} + \left\|G\widetilde{D}\right\| \left\|\boldsymbol{w}^{d}\right\|_{\{k'-r+1\ldots k'+l-1\}} \leq \zeta_{\mu} \, \|\mu\|_{\{k'-r+1\ldots k'\}}$$

where

$$\zeta_{\mu} \doteq \|G\| \, \|\mu'\|_{\{0\ldots r+P-2\}} + \left\|G\widetilde{D}\right\| \frac{\alpha}{\zeta_D}.$$

Note that in the definition of ζ_{μ} we used the constants μ''s defined in (18). ■

Proof of Lemma 5: If (23) does not hold, then it will not necessarily be true that $\|\tilde{\boldsymbol{y}}_k\| \leq \mu_k$, $\forall k \in \{k'+1 \ldots k'+P-r\}$. However, since now we have that

$$\|\tilde{\boldsymbol{y}}\|_{\{k'-r+1\ldots k'\}} \leq \|\mu\|_{\{k'-r+1\ldots k'\}} \leq \frac{1}{\alpha}\zeta_D \left\|\boldsymbol{w}^{d}\right\|_{\{k'-r+1,k'+P\}} \tag{47}$$

we can still bound the estimation error as follows. For $k \in \{k' \ldots k_2\}$ we have

$$\begin{aligned} \left\|\tilde{\boldsymbol{x}}_{k}^{+}\right\| &\leq \|G\| \, \|\tilde{\boldsymbol{y}}\|_{\{k-r+1\ldots k\}} + \left\|G\widetilde{D}\right\| \left\|\boldsymbol{w}^{d}\right\|_{\{k-r+1\ldots k-1\}} \\ \left\|\tilde{\boldsymbol{y}}_{k+1}\right\| &\leq \|CA_d\| \left\|\tilde{\boldsymbol{x}}_{k}^{+}\right\| + \|C\| \left|\boldsymbol{w}_{k}^{d}\right|. \end{aligned}$$

Iterating these two inequalities and combining with (47) we get $\left|\tilde{\boldsymbol{x}}_{k}^{+}\right| \leq \zeta_w \left\|\boldsymbol{w}^{d}\right\|_{k'-r+1\ldots k-1}$ where

$$\zeta_w \doteq \|G\| \, \|CA_dG\|^{P} \frac{1}{\alpha}\zeta_D + \sum_{m=1}^{P} \|G\| \, \|CA_dG\|^{m-1} \left(\left\|CA_dG\widetilde{D}\right\| + \|C\|\right) + \left\|G\widetilde{D}\right\|.$$

■

Proof of Lemma 6: Let k_3 be the first time step after k_2 such that $mode(k_3+1) = update$ (let $k_3 = \infty$ if no such time step exists). We now have that for all $l \in \{0 \dots k_3 - k_2\}$

$$\left|\tilde{\boldsymbol{x}}_{k_2+l}^{-}\right| \leq \|A_d\|^l \left|\tilde{\boldsymbol{x}}_{k_2}\right| + \sum_{m=0}^{l-1} \|A_d\|^{l-m-1} \left|\boldsymbol{w}_{k_2+m}^d\right| \leq \|A_d\|^l \left|\tilde{\boldsymbol{x}}_{k_2}\right| + \frac{\|A_d\|^l - 1}{\|A_d\| - 1} \left\|\boldsymbol{w}^d\right\|$$
$$\leq \|A_d\|^l \left(\left|\tilde{\boldsymbol{x}}_{k_2}\right| + \zeta_C \left\|\boldsymbol{w}^d\right\|\right) \tag{48}$$

where $\zeta_C \doteq \frac{1}{\|A_d\|-1}$. Now, the zoom factor grows as $\mu_{k_2+l} = \mu_{k_2}\Omega_{out}^l$. Define

$$T_1^*(\nu;\rho) \doteq \max\left\{0, \log_{\Omega_{out}/\|A_d\|}\left(\frac{\|C\|\,\nu}{\rho\,(N-2)}\right) + 1\right\} + r - 1$$

and note that when ρ is fixed, $T_1^*(\cdot;\rho)$ is a nondecreasing function. Assuming $mode(k) = capture$ $\forall k \in \left\{k_2+1 \dots k_2 + \left\lfloor T_1^*\left(\left|\boldsymbol{x}_{k_2}\right| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_2}\right)\right\rfloor\right\}$, we will have

$$\left|\boldsymbol{y}_{k_2+\lfloor T_1^*\rfloor - r+1} - C\hat{\boldsymbol{x}}_{k_2+\lfloor T_1^*\rfloor-r+1}^{-}\right| \leq \|C\| \left|\tilde{\boldsymbol{x}}_{k_2+\lfloor T_1^*\rfloor - r+1}^{-}\right| \leq (N-2)\,\mu_{k_2+\lfloor T_1^*\rfloor-r+1}.$$

Thus $saturated(k_2 + \lfloor T_1^*\rfloor - r + 1) =$ **false** as well as $saturated(k_2 + \lfloor T_1^*\rfloor + l) =$ **false** for $l = -r+2 \dots 0$ which guarantees that $k_3 \leq k_2 + T_1^* < \infty$ where k_3 is the first time step after k_2 such that $mode\,(k_3+1) = update$ and $p\,(k_3) = 0$. Using (48) we can bound the estimation error until the controller switches to the *measurement update* mode at k_3 by:

$$\|\tilde{\boldsymbol{x}}\|_{\{k_2 \dots k_3\}} \leq \tilde{\delta}_1\left(\left|\tilde{\boldsymbol{x}}_{k_2}\right| + \zeta_C\left\|\boldsymbol{w}^d\right\|; \mu_{k_2}\right), \qquad \tilde{\delta}_1(\nu;\rho) \doteq \|A_d\|^{T_1^*(\nu;\rho)}\nu.$$

Note also that $\tilde{\delta}_1\left(\left|\tilde{\boldsymbol{x}}_{k_2}\right| + \zeta_C\left\|\boldsymbol{w}^d\right\|; \mu_{k_2}\right) \leq \mu_{k_2}\zeta_b\Omega_{out}^{T_1^*\left(\left|\tilde{\boldsymbol{x}}_{k_2}\right| + \zeta_C\left\|\boldsymbol{w}^d\right\|; \mu_{k_2}\right)}$ where $\zeta_b \doteq \frac{(N-2)}{\|C\|}$. ■

Proof of Lemma 7: Assume first that $mode(k_2) = capture$ and consider the case $\left|\tilde{\boldsymbol{x}}_{k_2}\right| + \zeta_C\left\|\boldsymbol{w}^d\right\| \leq \frac{\mu_{k_2}}{\|C\|}$. Following the same arguments as in Lemma 6 which led to (48), we can write for $l \in \{1 \dots r\}$:

$$\|C\|\left|\tilde{\boldsymbol{x}}_{k_2+l}\right| \leq \|C\|\,\|A_d\|^l\left(\left|\tilde{\boldsymbol{x}}_{k_2}\right| + \zeta_C\left\|\boldsymbol{w}^d\right\|\right) \leq \mu_{k_2}\Omega_{out}^l = \mu_{k_2+l}.$$

This implies that if $mode(k_2) = capture$, then at $k_2 + r - p\,(k_2) \leq k_2 + r$ the controller will switch to the *measurement update* mode. If for some time step k the following holds

$$\left|\boldsymbol{y}_k - C\hat{\boldsymbol{x}}_k^{-}\right| \leq \|C\|\left|\tilde{\boldsymbol{x}}_k^{-}\right| \leq \mu_k \tag{49}$$

then the output from the quantizer will be such that $\boldsymbol{z}_k = \boldsymbol{c} = C\hat{\boldsymbol{x}}_k^{-}$. If for some time step k' (49) is true $\forall k \in \{k'-p \dots k'\}$, and $\hat{\boldsymbol{x}}_{k'}^{+}$ is updated with $G(\boldsymbol{z};\boldsymbol{u}^d;k')$, then we will have $\hat{\boldsymbol{x}}_{k'}^{+} = \hat{\boldsymbol{x}}_{k'}^{-}$. This implies that $\tilde{\boldsymbol{x}}_{k'} = A_d\tilde{\boldsymbol{x}}_{k'-1} + \boldsymbol{w}_{k'-1}^d$. In turn, this means that if the estimation error is

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sufficiently small compared to the zoom factor, then (48) continues to hold for $l \in \{0 \dots k_3 - k_2\}$ even if we pick $k_3 > k_2$ such that $mode(k_3) \neq capture$.

Now define

$$\xi(\nu;\rho) \doteq \left(\frac{1}{\rho\varsigma}\right)^{\frac{\log\left(\|A_d\|^P\right)}{\log(\sigma)-\log\left(\|A_d\|^P\right)}} \left(\|C\|\,\nu\right)^{\frac{\log(\sigma)}{\log(\sigma)-\log\left(\|A_d\|^P\right)}}$$
$$T_2^*(\nu;\rho) \doteq P\left\lfloor \log_\sigma\left(\frac{\xi(\nu;\rho)}{\rho\varsigma}\right)\right\rfloor, \qquad \varsigma \doteq \min_{k\in\{r\dots r+P-1\}} \mu'(k) \leq \sigma.$$

Note that in the definition of ς we use the μ''s defined in (18) and we assume without loss of generality that $\varsigma > 0$. Assume also that $|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C \left\|\boldsymbol{w}^d\right\|$ is sufficiently small such that $T_2^*\left(|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C \left\|\boldsymbol{w}^d\right\|;\mu_{k_2}\right) \geq r+P$. We defined ξ and T_2^* such that we will have for all $k \in \{k_2 \dots k_2 + T_2^*\left(\|\boldsymbol{w}\|\right)\}$

$$\mu_k \geq \mu_{k_2}\varsigma\sigma^{T_2^*\left(|\tilde{\boldsymbol{x}}_{k_2}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_2}\right)/P} > \xi\left(|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_2}\right) \tag{50}$$

and

$$\|C\|\,\|\tilde{\boldsymbol{x}}\|_{\left\{k_2\dots k_2+T_2^*\left(|\tilde{\boldsymbol{x}}_{k_2}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_2}\right)\right\}} \leq \|A_d\|^{T_2^*\left(|\tilde{\boldsymbol{x}}_{k_2}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_2}\right)}\|C\|\left(|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C\left\|\boldsymbol{w}^d\right\|\right)$$
$$\leq \left(\frac{\xi\left(|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_2}\right)}{\mu_{k_2}\varsigma}\right)^{\frac{\log\left(\|A_d\|^P\right)}{\log(\sigma)}} \|C\|\left(|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C\left\|\boldsymbol{w}^d\right\|\right) = \xi\left(|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_2}\right). \tag{51}$$

In deriving the first inequality in (51) we used (48) to bound the estimation error – even though it is not true that $mode(k_2+l) = capture\ \forall l < T_2^*$, we can still use (48) since (50) and (51) imply that (49) holds. The proof is completed by setting

$$\tilde{\delta}_2\left(\nu;\rho\right) \doteq \frac{\xi\left(\nu;\rho\right)}{\|C\|}, \qquad \zeta_s \doteq \Omega_{out}^r/\varsigma. \tag{52}$$

Note that the function $\tilde{\delta}_2\left(\cdot;\rho\right)$ is a class $\mathcal{K}_\infty$ function for each fixed ρ. The bound $\|\tilde{\boldsymbol{x}}\|_{k_2\dots T_2^*} \leq \tilde{\delta}_2\left(|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_2}\right)$ holds if $|\tilde{\boldsymbol{x}}_{k_2}|+\zeta_C\left\|\boldsymbol{w}^d\right\| < \varepsilon\left(\mu_{k_2}\right)$ where $\varepsilon\left(\rho\right) > 0$ is any real number such that $\varepsilon\left(\rho\right) \leq \frac{\rho}{\|C\|}$ and $T_2^*\left(\varepsilon\left(\rho\right);\rho\right) \geq r+P$. Note also that $\tilde{\delta}_2\left(|\tilde{\boldsymbol{x}}_{k_2}| + \zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_2}\right) \leq \mu_{k_2}\zeta_s\sigma^{T_2^*\left(|\tilde{\boldsymbol{x}}_{k_2}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_2}\right)/P}/\left\|C\right\|$. ∎

APPENDIX B: SMALL-GAIN THEOREM FOR LOCAL PRACTICAL PARAMETERIZED ISS

The following is a modification of the Small Gain Theorem ([25, Theorem 2.1]). It states that the interconnection of an ISS system and a parameterized ISS system, under a small-gain condition, results in a local practical parameterized ISS system.

Theorem 4: Consider two systems whose state variables, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, satisfy the ISS and the parameterized ISS properties, respectively:

$$\begin{aligned}
&|\boldsymbol{x}_1(t)| \le \beta_1(|\boldsymbol{x}_1(t_0)|, t-t_0) + \gamma_1\left(\|\boldsymbol{x}_2\|_{[t_0,t]}\right) + \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right)\\
&|\boldsymbol{x}_2(t)| \le \beta_2(|\boldsymbol{x}_2(t_0)|, t-t_0; \mu(t_0)) + \gamma_2\left(\|\boldsymbol{x}_1\|_{[t_0,t]}; \mu(t_0)\right) + \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}; \mu(t_0)\right)\\
&\quad \mu(t) \le \gamma_\mu\left(\|\boldsymbol{x}_2\|_{[t_0,t]}, \mu(t_0)\right), \qquad\qquad \forall t > t_0 > 0.
\end{aligned} \tag{53}$$

Assume the first trajectory, $\boldsymbol{x}_1$, is continuous and that the following local small-gain condition,

$$\gamma_1(\gamma_2(r,\mu)) < r \text{ and } \gamma_2(\gamma_1(r), \mu) < r, \qquad \forall r \in [r_0, r_1] \subset \mathbb{R}_{\ge 0}, \quad \forall \mu \in [0, \bar{\mu}], \tag{54}$$

holds. There exists a function $s_\infty(\nu_1, \nu_2, \rho, \nu_3) : \mathbb{R}^4_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that if all the variables are small enough to satisfy

$$\begin{aligned}
&s_\infty(|\boldsymbol{x}_1(0)|, |\boldsymbol{x}_2(0)|, \mu(0), \|\boldsymbol{w}\|) < s_{max} < r_1 \text{ and}\\
&\gamma_\mu(\beta_2(|\boldsymbol{x}_2(0)|, 0; \mu(0)) + \gamma_2(s_{max}; \mu(0)) + \gamma(\|\boldsymbol{w}\|; \mu(0)), \mu(0)) < \bar{\mu}
\end{aligned}$$

for some $s_{max} > 0$, then the interconnected system will satisfy the local practical parameterized Input-to-State Stability property:

$$\left|\begin{pmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{pmatrix}\right| \le \beta_{ic}\left(\left|\begin{pmatrix} \boldsymbol{x}_1(t_0) \\ \boldsymbol{x}_2(t_0) \end{pmatrix}\right|, t-t_0\right) + \gamma_{ic}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right) + d \tag{55}$$

for all $t \ge t_0 \ge 0$ where β_{ic} is of class $\mathcal{KL}$ and γ_{ic} is of class $\mathcal{K}_\infty$. The function s_∞ is continuous in all its variables and satisfies $s_\infty(0,0,\rho,0) = 0$ $\forall \rho \ge 0$.

Remark 3: The proof below follows the first part of the proof of [25, Theorem 2.1] with necessary modifications. Two things make the setting of Theorem 4 different from the setting of [25, Theorem 2.1]. These are the additional third state μ, and the fact that the small-gain condition only holds locally. We need to show that our system can be written in the formulation of [25, Theorem 2.1]. We achieve this by showing that as long as the small-gain condition holds, the signal $\boldsymbol{x}_1$, and subsequently all the other signals, are bounded in the interior of the region in

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which the small-gain condition holds. Since the signal $\boldsymbol{x}_1$ is continuous, it cannot jump to where the small-gain condition does not hold, and we can conclude that the small-gain condition must hold indefinitely. Note that the second signal, $\boldsymbol{x}_2$, corresponding to our state estimation error, may not be continuous at the sampling times. In the state feedback case it can be shown that the norm of the estimation error only decreases at these instances of discontinuity, making it possible to use the same argument that it cannot jump to where the small-gain condition does not hold. In the output feedback case, however, the norm of the estimation error may in fact increase at these instances of discontinuity, making this argument not applicable (it is still applicable on $\boldsymbol{x}_1$ even in this case).

Proof of Theorem 4: Since the inequalities in (54) are strict, there exist $\alpha > 0$ and $\delta < 1$ such that

$$\gamma_1\left(\left(1+\alpha\right)\gamma_2\left(r;\mu\right)\right) \leq \delta r \text{ and } \gamma_2\left(\left(1+\alpha\right)\gamma_1\left(r\right);\mu\right) \leq \delta r, \quad \forall r \in \left[r_0, r_1\right], \quad \forall \mu \in \left[0, \bar{\mu}\right]. \tag{56}$$

Assume that $\|\boldsymbol{x}_1\|_{[0,t]} < r_1$. For all nondecreasing functions γ and all $\alpha > 0$, $a > 0$ and $b > 0$ we have $\gamma\left(a+b\right) \leq \gamma\left(\left(1+\alpha\right)a\right) + \gamma\left(\left(1+1/\alpha\right)b\right)$. Using this and (56) we can derive

$$\begin{aligned}
\|\boldsymbol{x}_1\|_{[0,t]} \leq\ & \beta_1\left(\left|\boldsymbol{x}_1\left(0\right)\right|, 0\right) + \gamma\left(\|\boldsymbol{w}\|\right) + \\
& \gamma_1\left(\beta_2\left(\left|\boldsymbol{x}_2\left(0\right)\right|, 0; \mu\left(0\right)\right) + \gamma_2\left(\|\boldsymbol{x}_1\|_{[0,t]}\right) + \gamma\left(\|\boldsymbol{w}\|; \mu\left(0\right)\right)\right) \\
\leq\ & \beta_1\left(\left|\boldsymbol{x}_1\left(0\right)\right|, 0\right) + \gamma_1\left(\left(1+\alpha\right)\gamma_2\left(\|\boldsymbol{x}_1\|_{[0,t]}; \mu\left(0\right)\right)\right) + \\
& \gamma_1\left(\left(1+\frac{1}{\alpha}\right)\left(\beta_2\left(\left|\boldsymbol{x}_2\left(0\right)\right|, 0; \mu\left(0\right)\right) + \gamma\left(\|\boldsymbol{w}\|; \mu\left(0\right)\right)\right)\right) + \gamma\left(\|\boldsymbol{w}\|\right) \\
\leq\ & \beta_1\left(\left|\boldsymbol{x}_1\left(0\right)\right|, 0\right) + \delta\|\boldsymbol{x}_1\|_{[0,t]} + \\
& \gamma_1\left(\left(1+\frac{1}{\alpha}\right)\left(\beta_2\left(\left|\boldsymbol{x}_2\left(0\right)\right|, 0; \mu\left(0\right)\right) + \gamma\left(\|\boldsymbol{w}\|; \mu\left(0\right)\right)\right)\right) + \gamma\left(\|\boldsymbol{w}\|\right),
\end{aligned}$$

from which we can bound the trajectory for all $t \geq 0$ as:

$$\begin{aligned}
\|\boldsymbol{x}_1\|_{[0,t]} \leq & \frac{1}{1-\delta}\left(\beta_1\left(\left|\boldsymbol{x}_1\left(0\right)\right|, 0\right) + \gamma\left(\|\boldsymbol{w}\|\right)\right) + \\
& \frac{1}{1-\delta}\gamma_1\left(\left(1+\frac{1}{\alpha}\right)\left(\beta_2\left(\left|\boldsymbol{x}_2\left(0\right)\right|, 0; \mu\left(0\right)\right) + \gamma\left(\|\boldsymbol{w}\|; \mu\left(0\right)\right)\right)\right).
\end{aligned} \tag{57}$$

Denote the RHS of (57) by $s_\infty\left(\left|\boldsymbol{x}_1\left(0\right)\right|, \left|\boldsymbol{x}_2\left(0\right)\right|, \mu\left(0\right), \|\boldsymbol{w}\|\right)$. The bound (57) was derived using the assumption that $\|\boldsymbol{x}_1\|_{[0,t]} < r_1$. However, if $s_\infty\left(\left|\boldsymbol{x}_1\left(0\right)\right|, \left|\boldsymbol{x}_2\left(0\right)\right|, \mu\left(0\right), \|\boldsymbol{w}\|\right) < s_{max} < r_1$,

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and using the fact that $\boldsymbol{x}_1$ is continuous, this assumption must hold. We can use the bound on $\boldsymbol{x}_1$ to bound the $\boldsymbol{x}_2$,

$$|\boldsymbol{x}_2(t)| \leq \beta_2(|\boldsymbol{x}_2(0)|, 0; \mu(0)) + \gamma_2(s_{max}; \mu(0)) + \gamma(\|\boldsymbol{w}\|; \mu(0))$$

and with the assumptions in the theorem, derive the bound $\|\mu\| < \bar{\mu}$. With this, we can write

$$|\boldsymbol{x}_1(t)| \leq \beta_1(|\boldsymbol{x}_1(t_0)|, t-t_0) + \gamma_1\left(\|\boldsymbol{x}_2\|_{[t_0,t]}\right) + \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right)$$
$$|\boldsymbol{x}_2(t)| \leq \max_{\mu\in[0,\bar{\mu}]} \beta_2(|\boldsymbol{x}_2(t_0)|, t-t_0; \mu) + \max_{\mu\in[0,\bar{\mu}]} \gamma_2\left(\|\boldsymbol{x}_1\|_{[t_0,t]}; \mu\right) + \max_{\mu\in[0,\bar{\mu}]} \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}; \mu\right)$$

for all $t > t_0 > 0$. Note that for every fixed $\mu \in \mathbb{R}_{\geq 0}$ the function $\beta_2(\cdot,\cdot;\mu)$ is a function of class $\mathcal{KL}$ and the functions $\gamma_2(\cdot;\mu)$ and $\gamma(\cdot;\mu)$ are of class $\mathcal{K}_\infty$. They are also all continuous in μ. Thus taking the maximum of these functions over μ is well defined and does not change their $\mathcal{KL}/\mathcal{K}_\infty$ characteristics. This formulation now conforms to the assumptions of the small-gain theorem from [25], which we can use to derive (55). ■

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