

Optimization and the Price of Anarchy in a Dynamic Newsboy Model

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Abstract

This paper examines a dynamic version of the newsboy problem in which a decision maker must maintain service capacity from several sources to attempt to meet uncertain demand for a perishable good, subject to the cost of providing sufficient capacity, and penalties for not meeting demand.

A complete characterization of the optimal outcome is obtained when normalized demand is modeled as Brownian motion. The centralized optimal solution is a function of variability in demand, production variables, and the cost of insufficient capacity.

A closed form expression is obtained for the unique (state dependent) market clearing prices that allows the decentralized market to sustain the centralized optimal solution. The prices are a non-smooth function of normalized demand and reserves. Consequently, prices show extreme volatility in the efficient decentralized market outcome.

Standard policy designs are examined to improve the behavior of the decentralized market, including price caps and partial decentralization where the buyer owns a portion of the total service capacity. It is shown that these remedies fail: The market outcome in a partially decentralized model is shown to be inefficient even if the buyer owns a substantial portion of the service capacity. Under the presence of a price cap, a market equilibrium, efficient or not, fails to exist under very general conditions.

KEYWORDS: inventory theory; newsboy model; pricing; service allocation; reliability; efficiency; optimization; networks.

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1 Introduction

Any industry that produces perishable goods over time is faced with some version of the *dynamic newsboy's* problem. The decision maker must maintain service capacity to attempt to meet uncertain demand subject to two conflicting sources of cost: the real cost of providing sufficient capacity, and penalties for not meeting demand. The decision maker would like to balance these costs in order to maximize profits subject to uncertainty and time-constraints on the rate of production.

Workforce management. In a large organization such as a hospital or a call center one must maintain a large workforce to ensure effective delivery of services (see [40] for a recent academic treatment, or IBM's website [3].) To increase capacity of service one must bring new employees to work. Proper talent is usually identified and trained to be placed in position, which takes time. Alternatively, the organization can hire an employee through a temporary service agency which can offer talent at short notice, but at a higher price than the "usual" hiring process. In an example such as a hospital where the cost of not meeting demand has high social cost, it is crucial to secure a reliable channel of workers through these means, and through on-call staff.

In a complex organization these decision processes can be correspondingly complex. Moreover there can be high variability in demand, and in the quality and availability of workers. This is especially true in today's workplace since individual workers are more likely to quit the company.

Fashion manufacture and retail. In supplying a seasonal fashion product, the retailer maintains a small inventory which is available at short notice, while maintaining a contract with the supplier for deliverables in case of an unexpected surge in demand. It is more economical to have such a contract than to maintain inventory, but it takes some time to deliver the product from the producer to the retailer if demand increases unexpectedly [21]. In some markets demand can be highly inflexible due to "herding behavior" [6, 12, 72].

Note that this paper treats what might be a small part of a larger supply chain in the case of manufacturing. In this case the 'buyer' engages the 'seller' to secure capacity for goods that will be sold to end customers.

Electric power. The electric industry is probably the largest, and arguably best known, example of a dynamic newsboy problem. The system operator is continuously facing the challenge of meeting rapidly changing demand through an array of generators that can ramp up rather slowly due to constraints on generation as well as the complex dynamics of the power grid [36, 23]. The electricity cannot be stored economically in large quantities, yet the cost of not meeting demand is astronomical. The massive black-out seen in the Northeast on August 14, 2003, which cost \$4-6 billion dollars according to the US Department of Energy, reveals the tremendous cost of service disruption [39].

In broad terms, we can state the two main objectives of investigation as follows.

1. **PORTFOLIO MANAGEMENT.** It is a common practice in many industries to obtain multiple sources for service. The products may be very similar, but the sources of service are distinguished by their rate of response. The central problem is to maintain an appropriate balance between the various sources of reserve.

In the single-period newsboy problem the determination of the optimal service capacity based on forecast demand can be computed through a static calculus exercise. The resulting threshold is dependent upon cost, penalties, and the distribution of demand (see e.g. the early work of H. Scarf [66].) Reserves have an additional benefit in a dynamic model. Positive reserves today make it easier to respond to a surge in demand in the future if service capacity can increase only slowly when compared with demand.

2. DECENTRALIZATION. Decision-making is decentralized in any of the applications envisioned in this paper. It is important to understand whether the optimal solution (or some socially desirable solution) can be sustained by the decentralized market in which the buyers and sellers make decisions in a decentralized manner conditioned on the market clearing price. Our central question is “*Can a decentralized market internalize through a market clearing price the various benefits furnished by available service capacity?*”

1.1 Summary of models and conclusions

We examine an idealized model in which a firm produces and delivers a good to the consumer in continuous time. The good is perishable, in that it must be consumed immediately. This is the case in electric power or temporary services, and to a lesser extent fashionable clothing. The firm has access to a number of sources of service that can produce the good.

The social value is realized as the good is delivered and consumed. On the other hand, if some demand is not met, the social cost is proportional to the size of the excess demand. It is assumed that the mean demand is met by *primary service* (the cheapest source of service capacity) through a long term contract.

On-going demand can be met through primary service, as well as $K \geq 1$ sources of *ancillary service*.

Other constraints and costs assumed in the model are,

- **CONSTRAINED PRODUCTION AND FREE DISPOSAL:** Service capacity at time t from primary and ancillary services are denoted $\{G^p(t), G^{a_1}(t), \dots, G^{a_K}(t)\}$. Capacity is rate constrained: $G^p(t)$ can increase at maximal rate ζ^{p+} , and $G^{a_k}(t)$ can increase at maximal rate $\zeta^{a_k+} < \infty$. Increasing capacity takes time, but the decision maker can freely (and instantaneously) dispose of excess capacity if he chooses to do so.

- **CONSTANT MARGINAL CAPACITY COST:** c^p is the cost per unit capacity for primary service, and c^{a_k} the cost of building one additional unit of capacity from the k th source of ancillary service. The cost parameters are ordered, $c^p < c^{a_1} < \dots < c^{a_K}$.

- **CONSTANT MARGINAL DIS-UTILITY OF SHORTAGE:** Let $G^a(t) := \sum_i G^{a_i}(t)$, and

$$Q(t) = G^p(t) + G^a(t) - D(t), \quad t \geq 0. \quad (1.1)$$

be the excess capacity at time t . The marginal cost of excess demand is denoted $c^{bo} > 0$, so that the social cost of excess demand is given by $c^{bo} \max(-Q(t), 0)$. The larger the shortage, the greater the damage to society. It is assumed that

$$c^{bo} > c^{a_K} > \dots > c^{a_1} > c^p.$$

When normalized demand is modeled as a driftless Brownian motion the system is viewed as a $(K + 1)$ -dimensional constrained diffusion model. A complete characterization of the optimal portfolio of service capacities is computed, thereby resolving the first issue presented above. The optimal policy is constructed explicitly through a detailed analysis of the dynamic programming equations for the multidimensional model, and is found to be of the precise affine form introduced in [18]. The optimal solution clearly reveals how volatility of demand and the production technology of service influences the size, the composition, and the dynamics of optimal service capacity.

Our assumptions do lead to an idealized model. In particular, in a real electric power system the cost of generation is non-linear and even non-convex. As we suppress non-convexity of the production cost we eliminate one source that is known to prevent the decentralized market from sustaining the centralized optimal solution. Consequently, this idealization makes the analysis tractable, but also strengthens the key policy implications of this paper. The decentralized *spot market* will not function as envisioned by policy makers whose intention is to protect the interests of the consumers. That is, even if the spot market can sustain the optimal solution, the payoff to the consumers in the efficient outcome is precisely what they would obtain if they chose to purchase no reserve, resulting in frequent black-outs. In fact, the generator extracts the entire gains from trading in the efficient market outcome.

These conclusions are based upon a calculation of the unique *equilibrium price functional* that can possibly sustain the optimal centralized solution as the market outcome. The price at time t is expressed as a piecewise smooth function of the excess capacity $Q(t) = q$ and the demand $D(t) = d$,

$$p^p(q, d) = p^{a_k}(q, d) = (c^{bo} + v) \frac{\max(d, 0) - \max(-q, 0)}{d + q} \quad k = 1, \dots, K \quad (1.2)$$

where c^{bo} is the marginal cost of not meeting demand, and v is the marginal value of the product.

The plot shown in Figure 1 shows a simulation of prices, demand D , and reserves Q based on the Markovian price functional (1.2) for the “controlled random-walk” model introduced in Section 2.3.

The price functional is simple and intuitive. Because the buyer does not differentiate which service produced the good, the market prices of primary and ancillary services coincide. The price reaches its maximum value $c^{bo} + v$ whenever $Q(t) < 0$ and $D(t) > 0$, and is zero when these inequalities are reversed. Volatility of the price as a function of time will sometimes be greater than the volatility in demand.

The conclusions obtained for the idealized model considered here demonstrate that volatility and high prices can be expected in a deregulated market whenever the market achieves an efficient allocation, even without market manipulation.

These conclusions offer new insight to the events in California following deregulation of the ancillary service market. Brien in [15] gives a summary of the ancillary services market in California as it existed in 1999, and lists several benefits of ancillary services, including stabilization of voltage and frequency, and the option to extract or dump energy at short notice. She notes that on July 12, 1998, prices for one source of reserve reached \$9,999/MWh for several hours, more than one-hundred times the marginal cost (see also [15, 74, 63].) It appears that high prices were due in part to deliberate manipulation, but high prices and volatility have been observed in other electric power markets. Extreme volatility and unprecedented high prices for power were seen in the partially deregulated market in Illinois in the summer of 1998. High

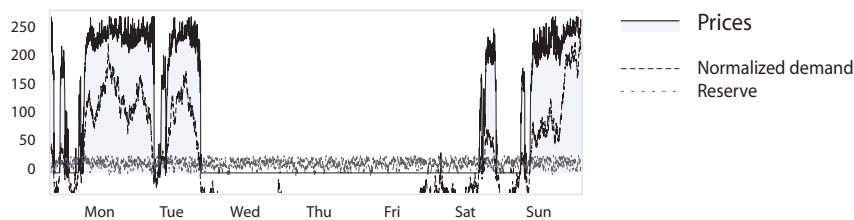


Figure 1: Prices as a function of demand and reserves in the efficient equilibrium, based on the Markovian price functional (1.2).

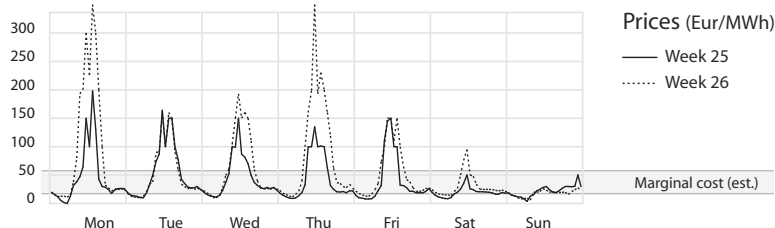


Figure 2: Price volatility in deregulated markets. Prices of power rise with demand at the Amsterdam Power Exchange during two typical weeks in July, 2003.

prices for power and volatility can be seen today at the Amsterdam Power Exchange website where demand and prices are tracked on an hourly basis [1]. Figure 2 shows two typical weeks in July, 2003.

In today’s power market in California, the ISO monitors power reserves hourly, and sends out emergency warnings whenever reserves fall below prescribed limits [2].

Beneath the apparent pessimism towards deregulation, this paper provides powerful support for a number of policy initiatives to improve market design. First, it is worth remembering that this paper concerns the *spot market*, and shows that the outcome of the spot market might not be what is expected by policy makers. Consumers can be better served through long term contracts to improve price stability and reliability of service. This is only a partial solution in the case of electric power since it is unlikely that the spot market can be eliminated entirely.

Second, again in the context of electric power, this paper provides theoretical justification for the policy to install “smart meters” which help consumers control their demand for power in response to evolving prices. One can increase reserve capacity by increasing on-line capacity, but precisely the same goal is achieved by reducing demand. The social value of the smart meter must be evaluated from the perspective of increasing the reserve capacity by rendering the demand responsive to the market price. Our analysis points out the tremendous value of flexible instantaneous demand for power, which the conventional static analysis underestimates. The installation of smart meters often faces opposition because of its high cost, but the analysis presented in this paper shows the social benefit of smart meters can be far larger than the one-time installation cost.

Some of the results reported in Section 2.2 are generalized to a more complex network setting in the companion paper [17] based on an aggregate relaxation, similar to the workload relaxations employed in the analysis of queueing models [41, 55]. The results of the present paper were presented in part in [19], and are surveyed in the SIAM news article [63].

1.2 Background

At the close of the 1950s, Herbert Scarf obtained the optimal policy for a single period newsboy problem and showed that it is of a threshold form [66, 67], following previous research on inventory models by Arrow et. al. [4] and by Bellman ([10] and [9, Chapter 5].) Scarf points out in [66] that the conclusion that the solution is defined by a threshold follows from the convexity of the value function with respect to the decision variables. These structural issues are also developed in [9], and in dozens of papers published over the past fifty years.

Following these results there was an intense research program concerning the control of one-dimensional inventory models, e.g. [35, 75, 64, 69, 7, 24, 58]. More recently there have been efforts in various directions to develop hedging (or safety stocks) in multidimensional inventory models to improve the performance of the system [33, 28, 29, 68] (especially in terms of responsiveness [31]), or obtain approximate optimality [41, 46, 32, 8, 55, 51, 52, 18, 70].

The results on centralized optimal control obtained here are most closely related to results reported in [18, 56]. It is shown that for a large class of multiclass, multidimensional network models, an optimal policy can be approximated in “workload space” by a generalized threshold policy. It is called an *affine policy* since it is constructed as an affine translation of the optimal policy for a fluid model. In particular, [18, Theorem 4.4] establishes affine approximations under the discounted cost criterion, and [18, Proposition 4.5 and Theorem 4.7] establishes similar results under the average cost criterion for a diffusion model. The method of proof reduces the optimization problem to a static optimization calculation based on a one-dimensional reflected Brownian motion (see also the discussion surrounding the height process (A.7) below.) An analogous calculation is performed in [71] in the analysis of a one-dimensional inventory model. Consequently, the formula for the optimal affine parameter obtained in [18, Proposition 4.5 and Theorem 4.7] coincides with the formula presented in [71, Proposition 3], and is similar to the threshold values given in (2.17).

The newsboy’s problem has been investigated from the perspective of a central decision maker who can control the size of inventory dynamically. The decentralized dynamic newsboy’s problem has received much less attention. The bulk of research has concentrated on joint pricing and inventory control in a single-period model, based on a functional model of demand vs. price - see surveys in [47, 62, 11]. Dynamic versions of this problem are treated in recent work (see [61, 43, 59, 11] and the references therein.) In some special cases it is found that price is roughly independent of state [61], but this conclusion cannot be expected to hold in general [43, 11].

Recently Van Mieghem has provided a complete analysis of a two-period newsboy model with two sources of service [57]. The framework is different, but some conclusions are similar to those obtained here. In particular, under certain conditions a unique Markovian price functional is obtained that can support the centralized optimal outcome.

The “price of anarchy” in the title refers to recent work of Papadimitriou, Tardos, Tsitsiklis, and Johari concerning Nash equilibria in static networks intended to model Internet routing [44, 65, 38, 37], and Johari also considers power networks in the thesis [37]. These results extend Braess’s celebrated “paradox” on efficiency of networks by demonstrating that the worst-case cost of a Nash flow is at most 33% above the optimal solution. These recent results concern general networks, and even contain extensions to nonlinear cost on links. However, one purpose of this paper is to send out a warning: The cost of deregulation may be enormous for some of the players in even a very simple dynamic network.

The remainder of the paper is organized as follows. Section 2 describes the diffusion model

in which normalized demand is a Brownian motion. The main results of this paper are collected in Sections 2.2 and 2.4. The development of optimal centralized control is contained in the first half of Section 3. The decentralized problem is the focus of Sections 3.4 and 3.5, where the Gaussian assumption on demand is relaxed, and the price functional (1.2) is constructed. Proofs of the major results are contained in the Appendix. Section 4 concludes the paper.

2 Dynamic Newsboy Model and Main Results

This section summarizes the main results of this paper. A diffusion model is considered in the simplest case in which there is a single customer (referred to as a buyer) that is served by primary and ancillary services. For the moment it is assumed that the buyer can access only a single source of ancillary service. It is assumed that the two sources of service are owned by the same firm, simply called the supplier (or the seller).

The analysis is extended to multiple sources of service in Section 3.3.

2.1 Diffusion Model

Recall that service capacity at time t from the primary and ancillary services are denoted $\{G^p(t), G^a(t)\}$. Demand is denoted $D(t)$, and reserve at time t is defined by $Q(t) = G^p(t) + G^a(t) - D(t)$ as expressed in (1.1). The event $Q(t) < 0$ is interpreted as the failure of reliable services. In the application to electric power this represents *black-out* since the demand for power exceeds supply.

We impose the constraint that $G^a(t) \geq 0$ for all t , but $G^p(t)$ is not sign-constrained.

We sometimes refer to $G^p(t) + G^a(t)$ as the on-line capacity, since the seller can offer primary and ancillary services $G^p(t)$ and $G^a(t)$ instantaneously at time t . Capacity is subject to ramping constraints: For finite, positive constants ζ^{p+} , ζ^{a+} ,

$$\frac{G^p(t') - G^p(t)}{t' - t} \leq \zeta^{p+} \quad \text{and} \quad \frac{G^a(t') - G^a(t)}{t' - t} \leq \zeta^{a+} \quad \text{for all } t' > t \geq 0.$$

We assume the free disposal of the capacity, which implies that $G^p(t)$ and $G^a(t)$ can decrease infinitely quickly. The ramping constraints can be equivalently expressed through the equations,

$$G^p(t) = G^p(0) - I^p(t) + \zeta^{p+}t, \quad G^a(t) = G^a(0) - I^a(t) + \zeta^{a+}t, \quad t \geq 0, \quad (2.1)$$

where the idleness processes $\{I^p, I^a\}$ are non-decreasing. It is assumed that $D(0)$ is given as an initial condition, and primary service is initialized using the definition (1.1),

$$G^p(0) = Q(0) + D(0) - G^a(0).$$

Throughout most of the paper it is assumed that $D(0) = 0$. Essentially, we assume that the buyer has a long term contract with the seller of primary service to take care of the mean demand. In the context of the electricity market, the supplier is the power generator, and the buyer is a “representative” consumer, who takes care of the “expected” demand for power through a long term contract, but wants to procure additional capacity to ensure reliability of service. Under this interpretation, the “outside” option for the buyer is not to purchase any reserve capacity. Our focus is how to maximize “social surplus” by balancing the size of the capacities of services in response to uncertain demand.

Until Section 3.4 it is assumed that \mathbf{D} is a driftless Brownian motion, with instantaneous variance denoted $\sigma_D^2 > 0$. The model (1.1) is then called the *controlled Brownian motion* (CBM) model, with two-dimensional state process $\mathbf{X} := (\mathbf{Q}, \mathbf{G}^a)^T$. A Gaussian model for demand might be justified by considering a Central Limit Theorem scaling of a large number of individual demand processes, as in [42]. Rather than attempt to justify a limiting model, here we choose a Gaussian demand model for the purposes of control design.

Under this assumption, the state process \mathbf{X} evolves according to the Itô equation,

$$d\mathbf{X} = \delta_X - B dI(t) - dD(t), \quad t \geq 0, \quad (2.2)$$

where $\delta_X = (\zeta^{p+} + \zeta^{a+}, \zeta^{a+})^T$, $X(0) = (q, g^a)^T \in \mathbf{X} = \mathbb{R} \times \mathbb{R}_+$ is given as an initial condition, and the 2×2 matrix B is defined by,

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \quad (2.3)$$

It is assumed that the process $\mathbf{I} = (\mathbf{I}^p, \mathbf{I}^a)^T$ appearing in (2.1) and (2.2) is adapted to \mathbf{D} , and that the resulting state process \mathbf{X} is constrained to the state space $\mathbf{X} = \mathbb{R} \times \mathbb{R}_+$. A process \mathbf{I} satisfying these constraints is called *admissible*.

In what follows we restrict to *stationary Markov policies* defined as a family of admissible idleness processes $\{\mathbf{I}^x\}$, parameterized by the initial condition $x \in \mathbf{X}$, with the defining property that the controlled process \mathbf{X} is a strong Markov process on \mathbf{X} .

For $x \in \mathbb{R}$ we denote,

$$x_+ = \max(x, 0), \quad x_- = \max(-x, 0) = (-x)_+.$$

An *affine policy* for the CBM model (2.2) is based on a pair of thresholds (\bar{q}^p, \bar{q}^a) :

- (i) For a given initial condition $X(0) = x = (q, g^a)^T \in \mathbf{X}$,

$$X(0+) = \begin{cases} \begin{pmatrix} q \\ g^a \end{pmatrix} - \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & g^a \geq q - \bar{q}^p \\ \begin{pmatrix} \bar{q}^p \\ 0 \end{pmatrix}, & g^a \leq q - \bar{q}^p. \end{cases} \quad (2.4)$$

where $\mu := \min((q - \bar{q}^a)_+, g^a) \geq 0$. The potential jump at time $t = 0$ reflects the assumption that the seller can freely reduce capacity instantaneously.

Consequently, for $t > 0$ the state process \mathbf{X} is restricted to the smaller state space given by $\mathcal{R}(\bar{q}) := \text{closure}(\mathcal{R}^p \cup \mathcal{R}^a)$, where

$$\mathcal{R}^a = \{x \in \mathbf{X} : x_1 < \bar{q}^a, x_2 \geq 0\}, \quad \mathcal{R}^p = \mathcal{R}^a \cup \{x \in \mathbf{X} : x_1 < \bar{q}^p, x_2 = 0\}. \quad (2.5)$$

- (ii) For any $t > 0$, if $Q(t) < \bar{q}^p$, then $\frac{d}{dt}G^p(t) = \zeta^{p+}$, and if $Q(t) < \bar{q}^a$ then $\frac{d}{dt}G^a(t) = \zeta^{a+}$. Consequently, the following boundary constraints hold with probability one,

$$\int_0^\infty \mathbb{I}\{X(t) \in \mathcal{R}^p\} dI^p(t) = \int_0^\infty \mathbb{I}\{X(t) \in \mathcal{R}^a\} dI^a(t) = 0. \quad (2.6)$$

A sketch of a typical sample path of \mathbf{X} under an affine policy is shown in Figure 3. From the initial condition shown, the process has mean drift δ_X up until the first time that $Q(t)$ reaches the threshold \bar{q}^a . The subsequent downward motion shown is a consequence of reflection at the boundary of \mathcal{R}^a . Since $Q(t)$ remains near \bar{q}^a yet primary service is ramped up at maximum rate ζ^{p+} , it follows that ancillary service has long-run average drift of $-\zeta^{p+}$ up until the first time that $G^a(t)$ reaches zero. For the *fluid model* in which $\sigma_D^2 = 0$ we have $\frac{d}{dt}G^a(t) = \zeta^{a+}$ when $Q(t) < \bar{q}^a$, $\frac{d}{dt}G^a(t) = -\zeta^{p+}$ whenever $G^a(t) > 0$ and $Q(t) = \bar{q}^a$.

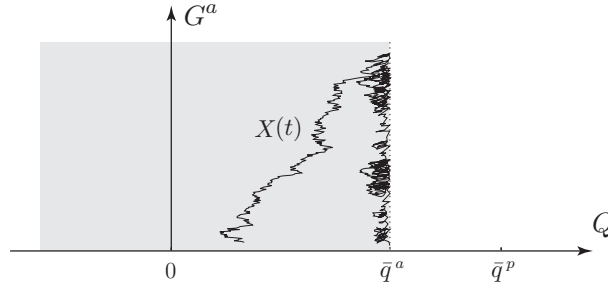


Figure 3: Trajectory of the two-dimensional model under an affine policy

2.2 Optimization

Recall that c^p , c^a denote the cost for maintaining one additional unit of capacity for primary and ancillary services. It is assumed that the marginal cost of production is higher for ancillary service,

$$c^p < c^a.$$

Welfare functions for the supplier and consumer are defined respectively by,

$$\mathcal{W}_s(t) := (p^p - c^p)G^p(t) + (p^a - c^a)G^a(t) \quad (2.7)$$

$$\mathcal{W}_D(t) := v \min(D(t), G^p(t) + G^a(t)) - (p^p G^p(t) + p^a G^a(t) + c^{bo} Q_-(t)).$$

The supplier is paid for the “on-line” capacity rather than the services delivered. For example, in an application to power the generator may have to burn coal in order to maintain a certain level of on-line capacity. On the other hand, the consumer obtains surplus only from the power delivered.

The welfare function for the consumer can be simplified,

$$\mathcal{W}_D(t) = vD(t) - (p^p G^p(t) + p^a G^a(t) + (c^{bo} + v)Q_-(t)), \quad (2.8)$$

where we have used the identity,

$$\min(D(t), G^p(t) + G^a(t)) = \min(D(t), Q(t) + D(t)) = D(t) - Q_-(t).$$

In the centralized problem the network is controlled by an impartial agent, called the social planner, who is given full authority over the buyer and the seller to achieve the best possible outcome. The *social surplus* at time t is given by,

$$\mathcal{W}(t) = \mathcal{W}_s(t) + \mathcal{W}_D(t)$$

We have $G^p(t) = Q(t) + D(t) - G^a(t)$, so that the social surplus is equivalently expressed,

$$\begin{aligned} \mathcal{W}(t) &= vD(t) - [c^p G^p(t) + c^a G^a(t) + (c^{bo} + v)Q_-(t)] \\ &= (v - c^p)D(t) - [c^p Q(t) + (c^a - c^p)G^a(t) + (c^{bo} + v)Q_-(t)] \\ &= (v - c^p)D(t) - \mathcal{C}(t), \end{aligned}$$

where

$$\mathcal{C}(t) := c(X(t)) := c^p Q(t) + (c^a - c^p)G^a(t) + (c^{bo} + v)Q_-(t). \quad (2.9)$$

For a given initial condition $D(0) = d$ for demand we have $d = \mathbf{E}[D(t)]$, and hence,

$$\mathbf{E}[\mathcal{W}(t)] = (v - c^p)d - \mathbf{E}[\mathcal{C}(t)], \quad t \geq 0. \quad (2.10)$$

Motivated by the representation (2.10), we consider each of the two cost-criteria,

$$\text{Average cost: } \phi := \limsup_{T \rightarrow \infty} \mathbf{E}_x \left[\frac{1}{T} \int_0^T c(X(t)) dt \right] \quad (2.11)$$

$$\text{Discounted cost: } K(x) := \mathbf{E}_x \left[\int_0^\infty e^{-\eta t} c(X(t)) dt \right], \quad (2.12)$$

where $\eta > 0$ is the discount parameter, and $x \in \mathbf{X}$ is the initial condition of \mathbf{X} . Our goal is to minimize the given criterion over all stationary policies.

Before pursuing optimization, let us first consider some simple sub-optimal policies.

One approach is ‘open-loop’: The buyer can trust the ‘long term contract’ already secured to meet mean-demand, and then choose $\mathbf{G}^p = \mathbf{G}^a \equiv 0$. Based on (2.7) this leads to $\mathcal{W}_s(t) = 0$ and

$$\mathcal{W}_D(t) = -(vD_-(t) + c^{bo}D_+(t)). \quad (2.13)$$

If $D(0) = 0$ this then gives,

$$\mathbf{E}[\mathcal{W}_D(t)] = -\frac{1}{2}(c^{bo} + v)\mathbf{E}[|D(t)|] = -\frac{1}{2}(c^{bo} + v)\sigma_D\sqrt{t}. \quad (2.14)$$

This is precisely the payoff of the consumer at t if no reserve capacity is procured: $G^a(t) = G^p(t) = 0$. We consider this value as the ‘outside’ option for the buyer in the market. Thus, in order to check the individual rationality of a market outcome, we use this value as the benchmark. Note that regardless of the initial condition we see that the mean welfare for the buyer and the social surplus simultaneously diverge to $-\infty$ as $t \rightarrow \infty$.

Consider next an arbitrary affine policy. The steady-state social surplus can be computed based on the following result of [17].

Theorem 2.1 *For any affine policy, the Markov process \mathbf{X} is exponentially ergodic [25]. The unique stationary distribution π on \mathbf{X} satisfies,*

(i) *The first marginal of π is given by the distribution function,*

$$\mathbf{P}_\pi\{Q(t) \leq q\} = \begin{cases} e^{-\gamma_p(\bar{q}^p - q)} & \bar{q}^a \leq q \leq \bar{q}^p \\ e^{-\gamma_p(\bar{q}^p - \bar{q}^a) - \gamma_a(\bar{q}^a - q)} & q \leq \bar{q}^a, \end{cases}$$

where

$$\gamma_a = 2\frac{\zeta^{p+} + \zeta^{a+}}{\sigma_D^2}, \quad \gamma_p = 2\frac{\zeta^{p+}}{\sigma_D^2}. \quad (2.15)$$

(ii) *The steady state mean of the cost $c: \mathbf{X} \rightarrow \mathbb{R}_+$ defined in (2.9) is explicitly computable,*

$$\phi(\bar{q}) := \pi(c) = \gamma_a^{-1} \left(\frac{\zeta^{a+}}{\zeta^{p+}} c^a + e^{-\gamma_a \bar{q}^a} (c^{bo} + v) \right) e^{-\gamma_p(\bar{q}^p - \bar{q}^a)} + (\bar{q}^p - \gamma_p^{-1})c^p. \quad (2.16)$$

□

Consequently, the steady-state mean welfare functions of the seller and buyer are computable when prices are fixed:

Corollary 2.2 *For any affine policy the steady-state mean social surplus is given by*

$$\lim_{t \rightarrow \infty} \mathbb{E}[\mathcal{W}(t)] = -\phi(\bar{q}),$$

where ϕ is given in (2.16), and the convergence is exponentially fast. Moreover, if the prices (p^p, p^a) are fixed, then the individual welfare functions have finite steady-state means,

$$\lim_{t \rightarrow \infty} \mathbb{E}[\mathcal{W}_s(t)] = \gamma_a^{-1} \frac{\zeta^{a+}}{\zeta^{p+}} (p^a - c^a) e^{-\gamma_p(\bar{q}^p - \bar{q}^a)} + (\bar{q}^p - \gamma_p^{-1})(p^p - c^p)$$

$$\lim_{t \rightarrow \infty} \mathbb{E}[\mathcal{W}_D(t)] = - \left[\gamma_a^{-1} \left(\frac{\zeta^{a+}}{\zeta^{p+}} p^a + e^{-\gamma_a \bar{q}^a} (c^{bo} + v) \right) e^{-\gamma_p(\bar{q}^p - \bar{q}^a)} + (\bar{q}^p - \gamma_p^{-1}) p^p \right].$$

When $D(t)$ has zero-mean and the prices are fixed with $p^p \leq p^a \leq c^{bo} + v$ then necessarily $\mathbb{E}[\mathcal{W}_D(t)] < 0$: Exactly as in the derivation of (2.10) we have,

$$\mathbb{E}[\mathcal{W}_D(t)] = -\mathbb{E}[p^p Q(t) + (p^a - p^p) G^a(t) + (c^{bo} + v) Q_-(t)], \quad t \geq 0.$$

This is the inevitable ‘cost of variability’, i.e. risk, as seen by the buyer. Since we have assumed that demand is normalized, the buyer sees additional value arising from the contract purchase of mean-demand at time 0–.

In conclusion, although the residual mean welfare seen by the buyer is always negative, there remains much benefit to engage the seller for services if the prices are not too high. As clearly shown in (2.13), the alternative ‘open-loop’ strategy is not sustainable.

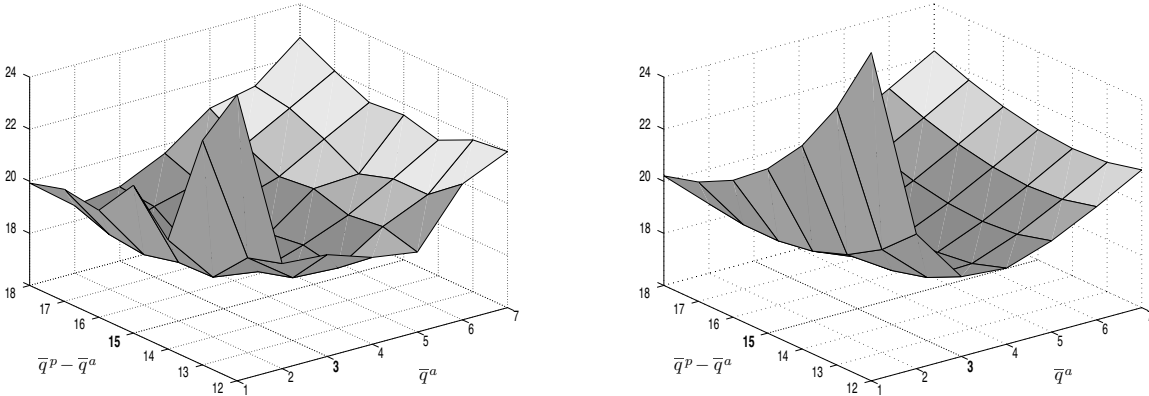


Figure 4: Shown at left is the average cost obtained using simulation for the network considered in Section 2.3 with Bernoulli noise using various affine parameters. At right is the average cost obtained from Theorem 2.1 (ii) for the CBM model.

The following two theorems describe the optimal policy for the social planner under the two cost criteria (2.11), (2.12). The proof of Theorem 2.3 is contained in Section 3.2. It is remarkable that the optimal policy is computable for this multi-dimensional model, and that the optimal solution is of this simple affine form.

Theorem 2.3 *The average-cost optimal policy (over all stationary policies) is affine, with specific parameter values given by,*

$$\bar{q}^{a*} = \gamma_a^{-1} \ln \left(\frac{c^{bo} + v}{c^a} \right) \quad \bar{q}^{p*} = \bar{q}^{a*} + \gamma_p^{-1} \ln \left(\frac{c^a}{c^p} \right). \quad (2.17)$$

□

From Theorem 2.1 (ii) it can be shown that the average cost is *convex* as a function of (\bar{q}^p, \bar{q}^a) , with unique minimum given in (2.17). This is plainly illustrated in the plot shown at right in Figure 4.

The proof of Theorem 2.4 is similar to the proof of Theorem 2.3. Computation of the parameters in (2.18) is provided in [16].

Theorem 2.4 *The optimal policy under the discounted-cost control criterion is affine for a unique pair of parameters given by*

$$\begin{aligned} \bar{q}_\eta^{p*} &= \frac{\sigma_D^2}{\zeta^{p+} + m^p} \left(\ln \frac{c^a}{c^p} + \frac{\zeta^{p+} + m^p}{\zeta^+ + m} \ln \frac{c^{bo} + v}{c^a} \right), \\ \bar{q}_\eta^{a*} &= \frac{1}{2} \frac{\sigma_D^2}{\zeta^{p+} + \zeta^{a+}} \left(\ln \frac{c^{bo} + v}{c^a} + \ln \frac{\zeta^{a+}}{\zeta^{p+}} + \ln \frac{-\zeta^{a+} + m^p + m}{2\zeta^{a+}} \right), \end{aligned} \quad (2.18)$$

where

$$m^p = \sqrt{(\zeta^{p+})^2 + 2\sigma_D^2\eta}, \quad m = \sqrt{(\zeta^+)^2 + 2\sigma_D^2\eta}.$$

Furthermore $\bar{q}_\eta^{p*} \rightarrow \bar{q}^{p*}$ and $\bar{q}_\eta^{a*} \rightarrow \bar{q}^{a*}$ as $\eta \rightarrow 0$, where \bar{q}^{p*} and \bar{q}^{a*} are given in (2.17). □

Theorem 2.3 and Theorem 2.4 have clear implications to the decentralized market problem: if the prices $\{c^p, c^a\}$ are the prices charged to the buyer by the seller, the seller will assume that the buyer will optimize based on what it is charged. The formulae given in these two theorems quantify the observation: *When γ_p is small, then the fair price for ancillary service may be extremely high.* The parameter γ_p is small if there is significant variability in demand, or if the maximum ramp-up rate for primary service is small. This is an important observation critical for interpreting a market outcome sustaining the optimal allocation.

Before turning to the decentralized model we present numerical results illustrating the conclusions of Theorem 2.3.

2.3 Numerical Examples

In this section we present some numerical results based on simulation and dynamic programming experiments. These plots are taken from [16] where the reader can find further numerical results.

Simulation and optimization will be performed for a two-dimensional *controlled random-walk* (CRW) model which evolves in discrete time. The forecasted excess capacity at time $t \geq 1$ is again defined by (1.1), where $D(t)$ is the demand at time t , and $(G^p(t), G^a(t))$ are current capacity levels from primary and ancillary service. It is assumed that \mathbf{D} is a random walk

$$D(t) = \sum_{s=1}^t \mathcal{E}(s), \quad t = 1, 2, \dots,$$

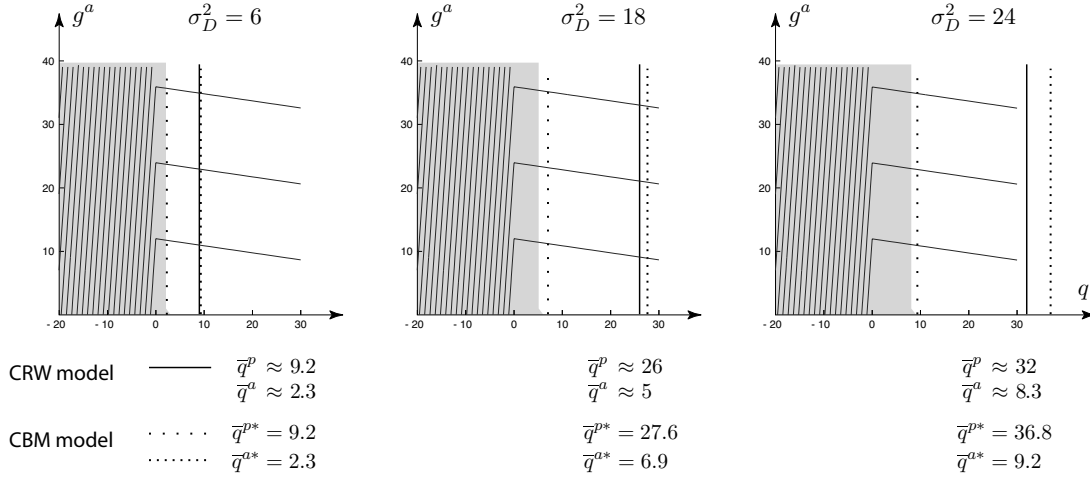


Figure 5: Optimal policies for the controlled random-walk model described in Section 2.3 obtained using value iteration. The grey region indicates those states for which ancillary service ramps up at maximum rate in the random-walk model, and the constant \bar{q}^p is the value such that primary ramps up at maximum rate when $Q(t) < \bar{q}^p$. The optimal policy for the CBM model closely matches the optimal policy for the discrete-time model in each case.

where the increment process \mathcal{E} is i.i.d., with zero mean and bounded support. The state process \mathbf{X} is constrained to the state space \mathbf{X} , and obeys the recursion,

$$X(t+1) = X(t) + BU(t) - \mathcal{E}_X(t+1), \quad t = 0, 1, \dots, \quad (2.19)$$

where the two dimensional process \mathcal{E}_X is defined as $\mathcal{E}_X(t) := (\mathcal{E}(t), 0)^T$. It is assumed that $U(t) \in \mathbf{U}(X(t))$ for all $t \in \mathbb{Z}_+$, where

$$\begin{aligned} \mathbf{U} &:= \{u = (u^p, u^a)^T \in \mathbb{R}^2 : -\infty \leq u^p \leq \zeta^{p+}, -\infty \leq u^a \leq \zeta^{a+}\}, \\ \mathbf{U}(x) &:= \{u \in \mathbf{U} : x + Bu \in \mathbf{X}\}, \quad x \in \mathbf{X}. \end{aligned}$$

An average-cost optimal policy is defined by a feedback law $f_*: \mathbf{X} \rightarrow \mathbf{U}$, which is used to define the pair of regions,

$$\mathcal{R}^p = \{x \in \mathbf{X} : f_*(x) = u^{p+}\}, \quad \mathcal{R}^a = \{x \in \mathbf{X} : f_*(x) = u^{a+}\}, \quad (2.20)$$

where u^{p+} and u^{a+} are defined in (3.26).

Consider a simple instance of the CRW model in which an optimal policy can be computed using value iteration. The marginal distribution of the increment process \mathcal{E} is symmetric, with support contained in the finite set $\{0, \pm 3, \pm 6\}$. The cost parameters are taken to be $c^{bo} = 100$, $c^a = 10$, $c^p = 1$, and we take $\zeta^{p+} = 1$, $\zeta^{a+} = 2$. The state process \mathbf{X} is restricted to an integer lattice to facilitate computation.

Three cases are considered to show how an optimal policy is influenced by variability: In each case the marginal distribution of \mathcal{E} is uniform on its respective support. The support and respective variance values are given by,

$$(a) \quad \{-3, 0, 3\}, \sigma_a^2 = 6 \quad (b) \quad \{-6, -3, 0, 3, 6\}, \sigma_b^2 = 18 \quad (c) \quad \{-6, 0, 6\}, \sigma_c^2 = 24.$$

The marginal distribution has zero mean since the support is symmetric in each case.

The average-cost optimal policy was computed for the three different models using value iteration. Results from these experiments are illustrated in Figure 5: The constant \bar{q}^p is defined as the maximum of $q \geq 0$ such that $U^p(t) = 1$ when $X(t) = (q, 0)^T$. The grey region represents \mathcal{R}^a , and the constant \bar{q}^a is an approximation of the value of q for x on the right-hand boundary of \mathcal{R}^a .

Also shown in Figure 5 is a representation of the optimal policy for the CBM model with first and second order statistics consistent with the CRW model. That is, the demand process \mathbf{D} was taken to be a drift-less Brownian motion with variance σ_D^2 equal to 6, 18, or 24 as shown in the figure. The constants \bar{q}^{p*} , \bar{q}^{a*} indicated in the figure are the optimal parameters for the CBM model given in (2.17). The optimal policy for the CBM model closely matches the optimal policy for the discrete-time model in each case.

We consider now a simulation experiment based on a family of affine policies for the CRW model.

The model parameters used in this simulation are as follows: $v = 0$, $c^p = 1$, $c^a = 20$, and $c^{bo} = 400$. The ramp-up rates were taken as $\zeta^{p+} = 1/10$ and $\zeta^{a+} = 2/5$. The marginal distribution of the increment distribution was taken symmetric on $\{\pm 1\}$.

Shown at right in Figure 4 is the average cost obtained from Theorem 2.1 (ii) for the CBM model with first and second order statistics identical to those of the CRW model. For these numerical values, (2.17) gives $(\bar{q}^{p*}, \bar{q}^{a*}) = (17.974, 2.996)$.

Affine policies for the CRW model were constructed based on threshold values $\{\bar{q}^p, \bar{q}^a\}$. The average cost was approximated under several values of (\bar{q}^p, \bar{q}^a) based on the (unbiased) *smoothed estimator* of [34] (for details see [16].) In the simulation shown at left in Figure 4 the time-horizon was $n = 8 \times 10^5$. Among the affine parameters considered, the best policy for the discrete time model is given by $(\bar{q}^{p*}, \bar{q}^{a*}) = (19, 3)$, which almost coincides with the values obtained using (2.17). The thesis [16] contains similar simulations in which \mathcal{E} is Markov rather than i.i.d.. Similar solidarity is seen in these experiments when σ_D^2 is taken to be the asymptotic variance appearing in the Central Limit Theorem for \mathcal{E} .

In conclusion, in spite of the drastically different demand statistics, the best affine policy for the discrete-time model is remarkably similar to the average-cost optimal policy for the continuous-time model with Gaussian demand. Moreover, the optimal average cost for the two models are in close agreement.

2.4 Dynamic equilibria

A fundamental question is whether we can implement the centralized solution through a decentralized market mechanism. For analytic convenience we restrict to a *Markovian price mechanism* in which the market prices at time t are completely determined by $(Q(t), G^a(t), G^p(t))$ for $t \geq 0$. This is Markovian with respect to the three-dimensional CBM model $\mathbf{X}^\dagger := (\mathbf{Q}, \mathbf{G}^a, \mathbf{D})^T$.

While the market clearing price may depend upon the entire history of the market outcome in general, this restriction is reasonable given the Markovian nature of the model, as we demonstrate in the far more general setting of Section 3.4.

A decentralized, discounted optimal control problem is formulated as follows. Let $p(x, d) = \{p^a(x, d), p^p(x, d)\}$ denote the pair of market clearing prices given $X(t) = (Q(t), G^a(t)) = (q, g^a)$ and $D(t) = d$ at time t . Given the functional $p: (\mathbf{X} \times \mathbb{R})^2 \rightarrow \mathbb{R}^2$, the buyer and seller solve the

respective optimization problems,

$$\max \mathbf{E} \left[\int e^{-\eta t} \mathcal{W}_S(t) dt \right], \quad \max \mathbf{E} \left[\int e^{-\eta t} \mathcal{W}_D(t) dt \right],$$

where the definitions (2.7) are maintained based on $p = (p^a, p^p)$.

The optimization problem of the central planner and that of the consumer are very similar: $\mathcal{W}_D(t)$ is obtained by replacing (c^a, c^p) by p . However, there are two important differences that have profound economic implications. First, in the decentralized market ramping rates are not considered in the optimization problem posed by the buyer. Second, while the cost parameters (c^a, c^p) are assumed constant, the prices can fluctuate with the evolving state process. The first observation greatly simplifies the optimization problem of the consumer. Unfortunately, the second observation prevents us from applying directly the analysis developed for the centralized problem.

For a given price functional p , let $(\mathbf{G}_d^a(p), \mathbf{G}_d^p(p))$ and $(\mathbf{G}_s^a(p), \mathbf{G}_s^p(p))$ denote the respective solutions of the optimization problems for the consumer and supplier. We then define,

- (i) The pair $(\mathbf{G}^a, \mathbf{G}^p)$ is called a *centralized solution* for a given initial state $x \in \mathbf{X}$ if it achieves the optimal cost (discounted or average, depending upon the context.)
- (ii) The triple $(p, \mathbf{G}^a, \mathbf{G}^p)$ is called a *decentralized market outcome* if p clears the market:

$$\mathbf{G}^a = \mathbf{G}_d^a(p) = \mathbf{G}_s^a(p), \quad \mathbf{G}^p = \mathbf{G}_d^p(p) = \mathbf{G}_s^p(p).$$

If this holds for each initial condition then p is called an *equilibrium price functional*.

- (iii) Let $(\mathbf{G}^a, \mathbf{G}^p)$ be a centralized solution for a given initial state $x \in \mathbf{X}$. A decentralized market outcome *sustains the centralized solution* if there exists a functional p such that the triple $(p, \mathbf{G}^a, \mathbf{G}^p)$ forms a decentralized market outcome for the same initial condition x . We say that the *second welfare theorem holds* if it is possible to construct a decentralized market outcome that sustains the centralized solution from each initial condition.

Two real-valued functions f, g on \mathbf{X} are regarded as equal if $f(x) = g(x)$ for a.e. $x \in \mathbf{X}$ with respect to Lebesgue measure. Similarly, when an equilibrium price functional is declared to be unique, it is understood that this uniqueness holds almost everywhere.

The calculation of the price functional provides important insights about the operation of the decentralized market, and also allows us to calculate the expected payoff in the decentralized market outcome, which reveals a serious problem in the decentralized market:

Theorem 2.5 *The second welfare theorem holds. There is a unique equilibrium price functional that is Markovian with respect to (\mathbf{X}, \mathbf{D}) , and it is expressed by (1.2). This price functional sustains the centralized optimal solution as the decentralized market outcome, and the resulting optimized objective function of the buyer results in (2.13):*

$$\mathcal{W}_D(t) = -(vD_-(t) + c^{bo}D_+(t)).$$

□

Theorem 2.5 has several subtle consequences. In the following paragraphs we explore several policies that are commonly proposed to improve the market. We conclude with comments on the impact of responsive demand.

We begin with a few remarks on the unique dynamic equilibrium obtained in Theorem 2.5.

2.4.1 Fairness

While the second welfare theorem offers rationale for deregulation, the same result does not tell us whether the market outcome is “fair” in any reasonable sense.

Theorem 2.5 states that the welfare seen by the buyer is precisely what the buyer would attain using the open-loop policy $\mathbf{G}^p = \mathbf{G}^a \equiv 0$, as shown in (2.13). By the same token, the mean surplus of the seller in period t is

$$\mathbb{E}[\mathcal{W}_s(t)] \gg \mathbb{E}[\mathcal{W}(t)].$$

In particular, in the CBM model in which demand has zero mean, the expected payoff at time t given in (2.14) is $\mathbb{E}[\mathcal{W}_D(t)] = -\frac{1}{2}(c^{bo} + v)\sigma_D\sqrt{t} < 0$. Hence, the buyer can never generate positive surplus by participating in the decentralized market [22, 60, 45, 73].

If demand is normalized so that $\mathbb{E}[D(t)] = 0$ for all $t \geq 0$ then one might claim that the negative expected payoff for the buyer is a natural consequence of this normalization. A careful examination shows otherwise. The theorem states that $\mathcal{W}_D(t) < 0$ whenever $D(t) \neq 0$, even if \mathbf{D} is initialized at some value $D(0) > 0$.

Recall that the buyer has no alternative way to procure reserve capacity to improve the reliability of service. Under this restriction, the consumer is essentially exploited by the generators in the efficient market, paying an enormous price for reliability.

2.4.2 Price caps

In Proposition 3.6 we show that (1.2) is the only candidate equilibrium price functional. Consequently,

Corollary 2.6 *If a price cap \bar{p} is imposed with $\bar{p} < v + c^{bo}$ then the resulting market does not admit any Markovian equilibrium price functional.*

Note that the corollary says that there is no equilibrium of any kind, efficient (in the sense that it sustains the centralized solution) or not. The proof follows from Proposition 3.6 and the fact that $\bar{p} < \max_{x,d} p(x,d)$ under the assumptions of the corollary.

2.4.3 Semi-Unified ownership

The fundamental problem of the decentralized market is that the buyer must pay a very high price in order to internalize the social benefit of sufficient service capacity. A possible remedy would be to allow the buyer control service capacity, which would bring us back to the centralized regime.

Here we examine an intermediate setting in which the buyer owns primary service, while ancillary services remain under the ownership of a separate entity. For simplicity, we consider the case where there is a single source of ancillary service, and we restrict to the case where the price functional is Markovian with respect to $\mathbf{X}^\dagger = (\mathbf{X}, \mathbf{D})$ with \mathbf{D} Brownian motion.

If the buyer controls primary service then we can consider the decentralized problem with welfare functions

$$\begin{aligned} \mathcal{W}_s(t) &:= (p^a - c^a)G^a(t) \\ \mathcal{W}_D(t) &:= v \min(D(t), G^p(t) + G^a(t)) - (c^p G^p(t) + p^a G^a(t) + c^{bo} Q_-(t)) \\ &= vD(t) - (c^p G^p(t) + p^a G^a(t) + (c^{bo} + v)Q_-(t)) \end{aligned} \tag{2.21}$$

In the final equation we have substituted $G^p + G^a = Q + D$.

Theorem 2.7 *In the decentralized market in which the buyer owns primary supply there is a unique equilibrium price functional given by,*

$$p^a(x, d) = p^a(x) = (c^{bo} + v) \frac{\max(g^a - q, 0) - \max(-q, 0)}{g^a} \quad (2.22)$$

This conclusion holds for both average-cost and discounted-cost. In the average-cost case the resulting market outcome is defined uniquely by the pair of thresholds:

$$\bar{q}^a = \bar{q}^{a*} = \gamma_a^{-1} \ln \left(\frac{c^{bo} + v}{c^a} \right) \quad \bar{q}^p = \gamma_p^{-1} \ln \left(\frac{c^{bo} + v}{c^p} \right). \quad (2.23)$$

The inequality $\bar{q}^p > \bar{q}^{p}$ holds whenever $\zeta^{a+} > 0$. Hence the decentralized market outcome is not the centralized solution, and hence the second welfare theorem fails.*

The threshold \bar{q}^p given in (2.23) is precisely what would be obtained in (2.17) with $\zeta^{a+} = 0$ and all other parameters unchanged,

Consider for example the CBM model with steady-state cost illustrated at right in Figure 4, and optimal thresholds $(\bar{q}^{p*}, \bar{q}^{a*}) \approx (18, 3)$ (see Section 2.3 for a precise description of this example.) In this case the threshold \bar{q}^p obtained from (2.23) is 50% larger than \bar{q}^{p*} ,

$$\bar{q}^p = \gamma_p^{-1} \ln \left(\frac{c^{bo} + v}{c^p} \right) \approx 27.$$

2.4.4 Long-term contracts

If the buyer and seller agree to a long term contract in which they act according to the optimal policy then we arrive at the centralized optimal outcome.

Moving beyond this cooperative setting is an open problem. Suppose for example that an individual supplier agrees to a certain dispatch schedule at an agreed upon price that is significantly lower than the maximum $(v + c^{bo})$. The remaining power is procured through the spot market.

We thus arrive at a new optimization problem in which the supplier seeks to compute an optimal schedule of power from the long-term contract, while anticipating that supply will be obtained from a spot market to ensure reliability. We conjecture that the solution will be similar to what is obtained in a model with semi-unified ownership: Reserves will be far higher than found in the centralized optimal solution.

2.4.5 Price responsiveness

In Theorem 2.5 and Theorem 2.7 it is assumed that demand is exogenous, and shows no responsiveness to price, which essentially captures today's wholesale electricity markets. Severin Borenstein [13] summarizes the fundamental difficulties in deregulation:

The difficulties that have appeared in California and elsewhere are intrinsic to the design of current electricity markets: demand exhibits virtually no price responsiveness and supply faces strict production constraints and very costly storage. Such a structure will necessarily lead to periods of surplus and of shortage, the latter resulting from both real scarcity of electricity and from sellers exercising market power. Extreme volatility in prices and profits will be the outcome.

The main results of this paper provide a formal proof that these outcomes can occur even when the decentralized market is efficient.

To improve matters, recall that in the efficient equilibrium with price functional given in (1.2), prices rise only when demand is relatively high, and reserves are relatively low.

Suppose that in an electricity market price signals for electric power are sent to end-consumers in real time, and that home and business owners use ‘smart meters’ that are able to respond to current prices for power. Assuming demand responds appropriately to price, demand for power will ramp down precisely when reserves become low. The benefits to the network are very similar to those gained through the introduction of a highly responsive source of ancillary service.

The overall system can be modeled as a network with several sources of ancillary service. Some are real sources of power, and other are “virtual generators” that arise from the aggregate affect of thousands of smart power meters.

Section 3.3 treats models with multiple levels of ancillary service, where it is shown that the centralized outcome is entirely analogous to the results obtain in the case of two suppliers. The conclusions of Theorem 2.5 remain the same, so that prices can potentially show extreme price volatility in the efficient equilibrium. However, the addition of a highly responsive source of ancillary service has tremendous benefit since the probability that the price takes on a high value in the efficient equilibrium is reduced substantially. The same conclusions can be reached for a power market with flexible demand.

We now consider in further detail the diffusion model, Markovian generalizations, and the decentralized outcome.

3 Diffusion Model

Here we prove Theorem 2.3, as well as necessary background that may be of independent interest. Without loss of generality we take $D(0) = 0$ throughout this section.

3.1 Poisson’s equation

It is convenient to introduce two “generators” for \mathbf{X} under a given Markov policy. The *extended generator*, denoted \mathcal{A} , is defined as follows: We write $\mathcal{A}f = g$ and say that f is in the domain of \mathcal{A} if the stochastic process M_f defined below is a local martingale for each initial condition,

$$M_f(t) := f(X(t)) - f(X(0)) + \int_0^t g(X(s)) ds, \quad t \geq 0. \quad (3.24)$$

That is, there exists a sequence of stopping times $\{\tau^n\}$ satisfying $\tau^n \uparrow \infty$, and for each n the stochastic process $\{M_f^n(t) = M_f(t \wedge \tau^n) : t \geq 0\}$ satisfies the martingale property,

$$\mathbb{E}[M_f^n(t+s) | \mathcal{F}_t] = M_f^n(t), \quad t, s \geq 0,$$

where $\mathcal{F}_t = \sigma(X(s), D(s) : s \leq t)$. See [26, 25] for background.

The *differential generator* is defined on C^2 functions $f : \mathbf{X} \rightarrow \mathbb{R}$ via,

$$\mathcal{D}f := \langle \nabla f, Bu^{a+} \rangle + \frac{1}{2} \sigma_D^2 \frac{\partial^2}{\partial q^2} f, \quad (3.25)$$

with

$$u^{p+} = (\zeta^{p+}, 0)^T, \quad u^{a+} = (\zeta^{p+}, \zeta^{a+})^T. \quad (3.26)$$

Suppose that \mathbf{X} is controlled using an affine policy, and that the C^2 function f satisfies the boundary conditions,

$$\langle \nabla f(x), B1^1 \rangle = 0, \quad q = \bar{q}^p, \quad g^a = 0, \quad \langle \nabla f(x), B1^2 \rangle = 0, \quad q = \bar{q}^a, \quad g^a \geq 0. \quad (3.27)$$

It then follows from Itô's formula that f is in the domain of \mathcal{A} with $\mathcal{A}f = \mathcal{D}f$.

Suppose that \mathbf{X} is defined by a Markov policy with steady-state cost $\phi := \pi(c) < \infty$, where c is defined in (2.9). *Poisson's equation* is then defined to be the identity,

$$\mathcal{A}h = -c + \phi \quad (3.28)$$

The function $h: \mathbf{X} \rightarrow \mathbb{R}$ is known as the *relative value function*. If (3.28) holds then the stochastic process defined below is a local martingale for each initial condition,

$$M_h(t) = h(X(t)) - h(X(0)) + \int_0^t (c(X(s)) - \phi) ds, \quad t \geq 0. \quad (3.29)$$

The following result is an extension of results of [17], following [53, 30] and [54, Chapter 17].

Proposition 3.1 *Suppose that \mathbf{X} is controlled using an affine policy, and define the stopping time,*

$$\tau_p = \inf\{t \geq 0 : X(t) = (\bar{q}^p, 0)^T\}. \quad (3.30)$$

Then,

- (i) *The following bound holds for each $m \geq 2$, some constant $b_m < \infty$, and any stopping time τ satisfying $\tau \leq \tau_p$,*

$$\mathbb{E}_x \left[\|X(\tau)\|^m + \int_0^\tau \|X(t)\|^{m-1} dt \right] \leq b_m (\|x\|^m + 1) \quad x \in \mathbf{X}.$$

- (ii) *One solution to Poisson's equation is given by,*

$$h(x) = \mathbb{E}_x \left[\int_0^{\tau_p} (c(X(t)) - \phi) dt \right], \quad x \in \mathbf{X}. \quad (3.31)$$

Moreover, for this solution the stochastic process \mathbf{M}_h is a martingale.

- (iii) *The function h given in (3.31) satisfies for some $b_0 < \infty$,*

$$-b_0 \leq h(x) \leq b_0 (\|q - g^a\|^2 + 1), \quad x \in \mathbf{X}.$$

Proof: Part (i) is a minor extension of the proof of Proposition A.2 in [17]. Parts (ii) and (iii) are given in [17, Proposition A.2].

Consider for $m \geq 2$ the C^2 function $V_m(x) := m^{-1}|q - g^a - \bar{q}^p|^m$, $x = (q, g^a)^T \in \mathbf{X}$. Applying the differential generator (3.25) we obtain,

$$DV_m(x) = \zeta^{p+} (q - g^a - \bar{q}^p)^{m-1} + \sigma_D^2 (m-1) (q - g^a - \bar{q}^p)^{m-2}, \quad x \in \mathbf{X}.$$

This function also satisfies the boundary conditions given in (3.27) since $m \geq 2$, so that V_m is in the domain of \mathcal{A} and $\mathcal{A}V_m = \mathcal{D}V_m$.

Consequently, one can find a compact set $S_m \subset \mathbf{X}$, $c_m < \infty$, and $\varepsilon_m > 0$ such that,

$$\mathcal{A}V_m \leq -\varepsilon_m V_{m-1} + c_m \mathbb{1}_{S_m}, \quad \text{on } \mathcal{R}(\bar{q}).$$

The bound in (i) then follows from standard arguments (see [17, Proposition A.2] and also [53]). \square

3.2 Optimization

In this section we apply Proposition 3.1 to show that the affine policy described in Theorem 2.3 is average-cost optimal. The treatment of the discounted case is identical - we omit the details.

The dynamic programming equations for the CBM model are written as follows,

$$\text{Average cost: } \left(\mathcal{D}h_* + c - \phi_* \right) \wedge \left(\inf \{ \langle \nabla h_*(x), -Bu \rangle : u \in \mathbb{R}_+^2 \} \right) = 0 \quad (3.32)$$

$$\text{Discounted cost: } \left(\mathcal{D}K_* + c - \eta K_* \right) \wedge \left(\inf \{ \langle \nabla K_*(x), -Bu \rangle : u \in \mathbb{R}_+^2 \} \right) = 0 \quad (3.33)$$

where the differential generator \mathcal{D} is defined in (3.25). The function $h_*: \mathbf{X} \rightarrow \mathbb{R}_+$ in (3.32) is known as the relative value function, and ϕ_* is the optimal average cost. For the models considered here however, we do not know if these value functions are C^2 on all of \mathbf{X} . Hence the dynamic programming equation (3.32) or (3.33) is interpreted in the viscosity sense [27, 20] (alternatively, one can replace \mathcal{D} by \mathcal{A} in these definitions.)

The relative value function defines a constraint region for \mathbf{X} as follows: define in analogy with (2.5),

$$\mathcal{R}^p = \{x \in \mathbf{X} : \langle \nabla h_*(x), B1^1 \rangle < 0\}, \quad \mathcal{R}^a = \{x \in \mathbf{X} : \langle \nabla h_*(x), B1^2 \rangle < 0\}.$$

Then, with $\mathcal{R}^* := \text{closure} \{ \mathcal{R}^a \cup \mathcal{R}^p \}$, the optimal policy maintains for each initial condition,

$$(i) X(t) \in \mathcal{R}^* \text{ for all } t > 0, \quad (ii) \text{ With probability one (2.6) holds.} \quad (3.34)$$

A representation for h_* can be obtained through a generalization of [50, Theorem 1.7] or [49, Theorem 3.3] to the continuous time case (see also [48].) Consider for any $x \in \mathbf{X}$,

$$h_o(x) := \inf \mathbb{E}_x \left[\int_0^{\tau_p} (c(X(t)) - \phi_*) dt \right],$$

where the stopping time τ_p is defined for a general policy by,

$$\tau_p = \inf \{ t \geq 0 : I^p(t) > 0 \}, \quad (3.35)$$

and the infimum is over all admissible \mathbf{I} . Under the optimal policy, the value function h_o solves the same martingale problem as h_* , that is $\mathcal{A}h_o = -c + \phi_*$. It is the unique solution to (3.32) (up to an additive constant) over all functions with quadratic growth.¹

Conversely, if a solution to (3.32) can be found with quadratic growth then this defines an optimal policy:

¹Uniqueness is established in [48, Theorem A3]. Although stated in discrete time, Section 6 of [48] describes how to translate to continuous time. Related results are obtained for constrained diffusions in [5].

Proposition 3.2 *Suppose that (3.32) holds for a function h_* satisfying for some $b_0 < \infty$,*

$$-b_0 \leq h_*(x) \leq b_0(1 + \|x\|^2), \quad x \in \mathbf{X}.$$

Then for any Markov policy that gives rise to a positive recurrent process \mathbf{X} with invariant probability measure π we have $\int c(x)\pi(dx) \geq \phi_$. Moreover, this lower bound is attained for the process defined in \mathcal{R}^* satisfying (3.34).*

Proof: Proposition 3.2 is a minor extension of [48, Theorem 5.2]. We sketch the proof here.

The essence of the dynamic programming equation (3.32) is that the process defined by

$$M_{h_*}(t) = h_*(X(t)) - h_*(X(0)) + \int_0^t (c(X(s)) - \phi_*) ds, \quad t \geq 0,$$

is a local submartingale for any solution \mathbf{X} obtained using an admissible idleness process \mathbf{I} : There exists a sequence of stopping times $\{\tau^n\}$ satisfying $\tau^n \uparrow \infty$, and the stochastic process defined by $M_{h_*}^n(t) = M_{h_*}(t \wedge \tau^n)$ satisfies the sub-martingale property,

$$\mathbf{E}[M_{h_*}^n(t+s) \mid \mathcal{F}_t] \geq M_{h_*}^n(t), \quad t, s \geq 0.$$

We can in fact take $\tau_n = \min\{t \geq 0 : h_*(X(t)) \geq n\}$. From the Monotone Convergence Theorem we obtain the bound,

$$\mathbf{E}_x \left[h_*(X(t)) + \int_0^t c(X(s)) ds \right] \geq t\phi_* + h_*(x), \quad t \geq 0, x \in \mathbf{X}. \quad (3.36)$$

That is, the modifier ‘local’ can be removed: \mathbf{M}_{h_*} is a sub-martingale.

Arguments used in [48, Theorem 5.2] imply that the following limit holds for a.e. $x \in \mathbf{X}$ [π] whenever $\pi(c) < \infty$,

$$\lim_{t \rightarrow \infty} t^{-1} \mathbf{E}_x[h_*(X(t))] = \lim_{t \rightarrow \infty} t^{-1} \mathbf{E}_x[\|X(t)\|^2] = 0.$$

Consequently, for a.e. $X(0) = x \in \mathbf{X}$,

$$\phi = \lim_{t \rightarrow \infty} t^{-1} \mathbf{E}_x \left[\int_0^t c(X(s)) ds \right] \geq \phi_*.$$

Moreover, if \mathbf{X} is defined under the optimal policy then \mathbf{M}_{h_*} is a local martingale since Poisson’s equation holds,

$$\mathcal{A}h_* = -c + \phi_*. \quad (3.37)$$

If h_* has quadratic growth then \mathbf{M}_{h_*} is a martingale, and hence (3.36) can be strengthened to an equality,

$$\mathbf{E}_x \left[h_*(X(t)) + \int_0^t c(X(s)) ds \right] = t\phi_* + h_*(x), \quad t \geq 0, x \in \mathbf{X}.$$

This shows that $\int c(x)\pi(dx) = \phi_*$ under the policy defined in (3.34). \square

We can now state the main result of this section. Recall that the thresholds \bar{q}^{p*} and \bar{q}^{a*} are defined in (2.17). A proof of Proposition 3.3 is contained in the Appendix.

Proposition 3.3 *The following hold for the CBM model under an affine policy:*

(i) *Suppose that primary service is specified using the threshold $\bar{q}^p > \bar{q}^{a*}$. If $\bar{q}^a = \bar{q}^{a*}$ then the solution to Poisson's equation (3.31) satisfies,*

$$\langle \nabla h(x), B1^2 \rangle < 0, \quad x \in \mathcal{R}^a.$$

(ii) *If $\bar{q}^p = \bar{q}^{p*}$ and $\bar{q}^a = \bar{q}^{a*}$ then h satisfies in addition,*

$$\langle \nabla h(x), B1^1 \rangle < 0, \quad x \in \mathcal{R}^p.$$

Consequently, h solves the dynamic programming equation (3.32).

□

3.3 Multiple Levels of Ancillary Service

The extension to multiple sources of ancillary service is now straightforward. Suppose that there are K classes of ancillary service, with reserve capacities at time t denoted $\{G^{a_1}(t), \dots, G^{a_K}(t)\}$. The reserve remains defined as (1.1) with $G^a(t) := \sum_i G^{a_i}(t)$. The associated cost parameters and ramping rate constraints are denoted $\{c^{a_i}, \zeta^{a_i+} : 1 \leq i \leq K\}$ where

$$\frac{G^{a_i}(t') - G^{a_i}(t)}{t' - t} \leq \zeta^{a_i+} \quad \text{for all } t' > t.$$

It is assumed that the cost parameters are strictly increasing in the index i , with $c^{a_K} < c^{b_0}$.

The state process for control is the $K+1$ -dimensional process $X(t) := (Q(t), G^{a_1}(t), \dots, G^{a_K}(t))^T$, which is constrained to $\mathsf{X} := \mathbb{R} \times \mathbb{R}_+^K$. An affine policy is defined using the natural extension of the previous definition: For given parameters $\{\bar{q}^p > \bar{q}^{a_1} > \dots > \bar{q}^{a_K}\}$ we denote

$$\mathcal{R}^{a_i} := \{x = (q, g^{a_1}, \dots, g^{a_K}) \in \mathbb{R} \times \mathbb{R}_+^m : q < \bar{q}^{a_i}, g^{a_j} = 0 \text{ for } j > i\}.$$

The affine policy is defined so that $X(t) \in \text{closure}(\mathcal{R}^{a_i})$ whenever $G^{a_i}(t) > 0$ and moreover,

$$\int_0^\infty \mathbb{I}\{X(t) \in \mathcal{R}^{a_i}\} dI^{a_i}(t) = 0.$$

The cost function for the centralized planner is given by,

$$c(x) := c^p q + \sum_{i=1}^K (c^{a_i} - c^p) g^{a_i} + (c^{b_0} + v) q_-.$$

We present an extension of Theorem 2.3 for the model with K levels of ancillary service. It is found that the average cost optimal policy is again affine. The analogous result in the case of discounted cost is also valid.

Observe that in an optimal solution capacity will sometimes be sought from a supplier that is very inefficient in the sense that c^{a_j} is very large, yet ζ^{a_j+} is very small. It can be shown that the introduction of a new generator will strictly reduce the value of \bar{q}^{p*} obtained in (3.38) whenever $\zeta^{a_j+} > 0$.

A proof of Theorem 3.4 is contained in the Appendix.

Theorem 3.4 *The average-cost optimal policy for \mathbf{X} is affine, with specific parameter values given in the following modification of (2.17),*

$$\bar{q}^{a_i^*} = \bar{q}^{a_{i+1}^*} + \frac{1}{2} \frac{\sigma_D^2}{\zeta_i^+} \ln \left(\frac{c^{a_{i+1}}}{c^{a_i}} \right), \quad 1 \leq i \leq K, \quad \bar{q}^{p^*} = \bar{q}^{a_1^*} + \frac{1}{2} \frac{\sigma_D^2}{\zeta^{p^+}} \ln \left(\frac{c^{a_1}}{c^p} \right), \quad (3.38)$$

where $\zeta_i^+ := \zeta^{p^+} + \sum_{j \leq i} \zeta^{a_j^+}$, and we denote $c^{K+1} := c^{b_0}$ and $\bar{q}^{a_{K+1}^*} := 0$. \square

3.4 Second welfare theorem

Here we prove Theorem 2.5, and a far stronger result characterizing possible decentralized market outcomes. We are forced to restrict to the discounted criterion since, as we shall see, the average welfare for the buyer is always $-\infty$ in any decentralized market outcome.

Recall that we introduced the three dimensional process $\mathbf{X}^\dagger := (\mathbf{Q}, \mathbf{G}^a, \mathbf{D})^T$ in consideration of the decentralized model in Section 2.4. To obtain the most general possible results in this section we relax our assumptions on the demand process \mathbf{D} and redefine \mathbf{X}^\dagger accordingly.

We do not assume that \mathbf{D} is Brownian motion in this section. Suppose that demand is simply a function of a strong Markov process Υ evolving on a general topological state space \mathbf{Y} ,

$$D(t) = d(\Upsilon(t)), \quad t \geq 0.$$

It is assumed that Υ is *CADLAG* (the sample paths are right continuous, with left-hand limits), and that $d: \mathbf{Y} \rightarrow \mathbb{R}$ is continuous. We then redefine $\mathbf{X}^\dagger := (\mathbf{Q}, \mathbf{G}^a, \Upsilon)^T$, and extend the definition of a Markov policy as follows: \mathbf{X}^\dagger is a strong Markov process, and the associated idleness process \mathbf{I} is admissible for each initial condition.

A pair of prices (p_t^p, p_t^a) is called *Markovian with respect to \mathbf{X}^\dagger* if each price can be expressed as a fixed function of \mathbf{X}^\dagger ,

$$p_t^p = p^p(X^\dagger(t)), \quad p_t^a = p^a(X^\dagger(t)), \quad t \geq 0.$$

An example is $\Upsilon(t) = D_{t-T}^t$, $t \geq 0$, where \mathbf{D} is two-sided Brownian motion and $\mathbf{Y} = C[0, T]$. A Markovian price at time t thus depends upon $X(t)$ and the history of demand over $[t - T, t]$.

In this section we characterize possible equilibrium price functionals that are Markovian with respect to \mathbf{X}^\dagger .

Recall that the consumer faces no ramping rate constraints. As a result, upon optimizing the consumer behaves in a myopic fashion: If at time t the prices satisfy $p^a > p^p$ then $G_d^a = 0$. Similarly, if $p^a < p^p$ then $G_d^a = -\infty$. We thus arrive at the following conclusion:

Lemma 3.5 *In a decentralized outcome $p^{a_i}(x, y) = p^p(x, y)$ for each $i = 1, \dots, K$.* \square

The value of p^{a_i} is irrelevant when $g^{a_i} = 0$. We henceforth simplify notation by defining on all of $\mathbf{X} \times \mathbf{Y}$,

$$p^e := p^{a_i} = p^p \quad \text{for all } i = 1, \dots, K.$$

Lemma 3.5 also allows us to restrict analysis to $K = 1$ since the extension to multiple ancillary sources is merely notational.

We can now state the main result of this section. Note that the main assumption of Proposition 3.6 is that there exists a decentralized market outcome with Markovian price $p^e = p^p = p^a$. It is not assumed that this market outcome sustains the centralized solution.

Proposition 3.6 *Suppose that there exists an equilibrium price functional p^e . Then p^e can be expressed as a function of (q, d) only,*

$$p^e(x, y) = p^e(q, d) = (c^{bo} + v) \frac{d_+ - q_-}{d + q}, \quad (x, y) \in \mathbf{X} \times \mathbf{Y}. \quad (3.39)$$

To prove the proposition we first consider the consumer's optimization problem. Applying (2.8) we can write $\mathcal{W}_D(t) = vD(t) - \mathcal{C}_D(t)$, with $\mathcal{C}_D(t) = p^e(X^\dagger(t))(G^a(t) + G^p(t)) + (c^{bo} + v)Q_-(t)$. Consequently, the maximization problem posed by the consumer can be converted to a minimization problem with value function,

$$K_*^D(x^\dagger) := \inf \mathbb{E}_{x^\dagger} \left[\int_0^\infty e^{-\eta t} c_D(X^\dagger(t)) dt \right], \quad (3.40)$$

where

$$c_D(x, y) := p^e(x, y)(q + d) + (c^{bo} + v)q_-, \quad (x, y) \in \mathbf{X} \times \mathbf{Y}.$$

The principle of optimality is expressed as follows: For each $x^\dagger = (x, y) \in \mathbf{X} \times \mathbf{Y}$, and any stopping time τ ,

$$K_*^D(x^\dagger) = \min \mathbb{E}_{x^\dagger} \left[\int_0^\tau e^{-\eta t} c_D(X^\dagger(t)) dt + e^{-\eta \tau} K_*^D(X^\dagger(\tau)) \right], \quad (3.41)$$

where the minimum is over all admissible $(\mathbf{G}^p, \mathbf{G}^a)$.

Lemma 3.7 *The value function for the consumer is independent of x :*

$$K_*^D(x, y) = K_*^D(y), \quad (x, y) \in \mathbf{X} \times \mathbf{Y}$$

Proof: The following derivative conditions are consequences of optimality:

$$\langle \nabla_x K_*^D(x, y), B1^1 \rangle = 0, \quad \langle \nabla_x K_*^D(x, y), B1^2 \rangle = 0, \quad (x, y) \in \mathbf{X} \times \mathbf{Y}, \quad (3.42)$$

It follows that $K^u(x, y)$ is constant on \mathbf{X} for each y . \square

Because the supply of service is subject to ramping constraints while demand is not, the supply is “almost constant” over an infinitesimal time interval, while the demand can take any value. Thus, the market price is completely determined by the demand side of the market.

Next we recall the following resolvent equation (see e.g. [26, 53].)

Lemma 3.8 *Letting \mathcal{A} denote the extended generator for the Markov process \mathbf{X}^\dagger ,*

$$\mathcal{A}K_*^D = \eta K_*^D - c_D, \quad (x, y) \in \mathbf{X} \times \mathbf{Y}.$$

\square

Proof of Proposition 3.6. From the preceding two lemmas we conclude that c_D is also independent of x : For $(x, y) \in \mathbf{X} \times \mathbf{Y}$,

$$p^e(x, y)(q + d) + (c^{bo} + v)q_- = c_D(x, y) = -\mathcal{A}K_*^D(y) + \eta K_*^D(y).$$

Setting $q = -d$ then gives $-\mathcal{A}K_*^D(y) + \eta K_*^D(y) = (c^{bo} + v)d_+$, and hence

$$p^e(x, y)(q + d) + (c^{bo} + v)q_- = (c^{bo} + v)d_+.$$

Rearranging terms completes the proof of (3.39). \square

Proof of Theorem 2.5 We have shown in Proposition 3.6 that *if* there exists a Markovian price functional, then necessarily it is given by (3.39). We now show that when \mathbf{D} is Brownian motion, the optimal policy (G^a, G^p) from the centralized problem is indeed the profit maximizing policy for the seller and for the buyer with respect to the price functional (3.39).

Recall that $\mathcal{W}_s(t) = (p_t^p - c^p)G^p(t) + (p_t^a - c^a)G^a(t)$, where p_t^p and p_t^a are the market clearing prices at t . Since $p_t^e = p_t^p = p_t^a$, we can re-write

$$\begin{aligned}\mathcal{W}_s(t) &= p_t^e(G^p(t) + G^a(t)) - (c^p G^p(t) + c^a G^a(t) + (c^{bo} + v)Q_-(t)) + (c^{bo} + v)Q_-(t) \\ &= p_t^e(G^p(t) + G^a(t)) - vD(t) + \mathcal{W}(t) + (c^{bo} + v)Q_-(t)\end{aligned}$$

where we have used the representation of $\mathcal{W}(t) = \mathcal{W}_s(t) + \mathcal{W}_D(t)$ shown above (2.9). From the form of the price functional (1.2) and the identity $G^p(t) + G^a(t) = Q(t) + D(t)$, we conclude that for each t ,

$$\begin{aligned}\mathcal{W}_s(t) &= D_+(t)(c^{bo} + v) - vD(t) + \mathcal{W}(t), \\ \mathcal{W}_D(t) &= -D_+(t)(c^{bo} + v) + vD(t).\end{aligned}\tag{3.43}$$

Note that

$$D_+(t)(c^{bo} + v) - vD(t) = c^{bo}D_+(t) + vD_-(t) \geq 0$$

with equality holding only if $D(t) = 0$, and that this term is not controllable by the supplier. Hence, $(G^a(t), G^p(t))$ maximizes

$$\mathbb{E}_{x^\dagger} \left[\int_0^\infty e^{-\eta t} \mathcal{W}(t) dt \right],$$

if and only if it maximizes

$$\mathbb{E}_{x^\dagger} \left[\int_0^\infty e^{-\eta t} \mathcal{W}_s(t) dt \right].$$

Therefore, the price functional (1.2) sustains the centralized solution as the outcome of the decentralized solution for the supplier.

We have yet to check whether $(G^a(t), G^p(t))$ also maximizes the discounted utility

$$\mathbb{E}_{x^\dagger} \left[\int_0^\infty e^{-\eta t} \mathcal{W}_D(t) dt \right],$$

under the same price functional. From (3.43) we see that this expectation is completely independent of $(G^a(t), G^p(t))$, so that any allocation is optimal, with value given in Theorem 2.5:

$$\max \mathbb{E}_{x^\dagger} \left[\int_0^\infty e^{-\eta t} \mathcal{W}_D(t) dt \right] = -\mathbb{E}_{x^\dagger} \left[\int_0^\infty e^{-\eta t} (c^{bo}D_+(t) + vD_-(t)) dt \right].$$

□

3.5 Semi-Unified ownership

We now turn to the setting of Theorem 2.7. The value function for the consumer is defined in (3.40), with cost function $c_D(X(t)) = c^p Q(t) + (p^a - c^p)G^a(t) + (c^{bo} + v)Q_-(t)$. The value function has the equivalent form,

$$K_*^D(x^\dagger) := -\sup \mathbb{E}_{x^\dagger} \left[\int_0^\infty e^{-\eta t} \mathcal{W}_D(t) dt \right] = \inf \mathbb{E}_{x^\dagger} \left[\int_0^\infty e^{-\eta t} c_D(X(t)) dt \right], \tag{3.44}$$

Lemma 3.9 *The value function for the consumer (3.44) is constant on any 45° line in X :*

$$K_*^D(q, g^a, d) = K_*^D(q - g^a, 0, d), \quad x \in \mathsf{X}, \quad d \in \mathbb{R}.$$

Proof: Ramping rate constraints for ancillary service are disregarded in the optimization problem posed by the buyer. Hence, exactly as in the proof of Lemma 3.7 we obtain the following derivative condition:

$$\langle \nabla_x K_*^D(x, d), B1^2 \rangle = 0, \quad x \in \mathsf{X}, \quad d \in \mathbb{R}.$$

□

Proof of Theorem 2.7 Lemma 3.9 combined with the resolvent equation in Lemma 3.8 implies that any equilibrium price functional $p^a(x, d)$ will result in a functional equation of the form,

$$c_D(x) = c^p q + (p^a - c^p)g^a + (c^{bo} + v)q_- = F(q - g^a)$$

The function F can be computed on setting $g^a = 0$,

$$F(q) = c^p q + (c^{bo} + v)q_-.$$

Substituting this into the previous identity then gives,

$$c_D(x) = c^p q + (p^a - c^p)g^a + (c^{bo} + v)q_- = F(q - g^a) = c^p(q - g^a) + (c^{bo} + v)(q - g^a)_-.$$

Canceling terms and dividing by g^a then gives a formula equivalent to (2.22),

$$p^a(x) = (c^{bo} + v)[(q - g^a)_- - q_-]/g^a. \quad (3.45)$$

We conclude that if a Markovian price functional exists, it is uniquely expressed as (2.22).

Next we demonstrate that (2.22) is indeed an equilibrium price functional for the CBM model. Consider first the optimization problem of the consumer. On substituting the equivalent expression (3.45) for p^a into the expression for \mathcal{W}_D given in (2.21) we obtain,

$$\begin{aligned} \mathcal{W}_D(t) &= vD(t) - (c^p G^p(t) + p^a G^a(t) + (c^{bo} + v)Q_-(t)) \\ &= vD(t) - (c^p G^p(t) + (c^{bo} + v)[(Q(t) - G^a(t))_- - Q_-(t)] + (c^{bo} + v)Q_-(t)) \\ &= vD(t) - (c^p G^p(t) + (c^{bo} + v)(Q(t) - G^a(t))_-) \end{aligned}$$

Write $Q^0(t) = Q(t) - G^a(t) = G^p(t) - D(t)$, so that the welfare function of the consumer can be written explicitly as a function independent of \mathbf{G}^a :

$$\mathcal{W}_D(t) = vD(t) - \mathcal{C}_D(t),$$

with $\mathcal{C}_D(t) = c^p G^p(t) + (c^{bo} + v)Q_-^0(t)$, and the discounted cost (3.40) is equivalently expressed,

$$\mathbb{E}_x \left[\int_0^\infty e^{-\eta t} \mathcal{C}_D(t) dt \right].$$

The process Q^0 is precisely the one-dimensional controlled RBM analyzed in [71], which on optimizing gives the threshold value \bar{q}^p shown in (2.23).

Consider now the supply side,

$$\begin{aligned} \mathcal{W}_s(t) &= (p^a - c^a)G^a(t) = (c^{bo} + v)[(Q(t) - G^a(t))_- - Q_-(t)] - c^a G^a(t) \\ &= (c^{bo} + v)Q_-^0(t) - (c^a G^a(t) + (c^{bo} + v)Q_-(t)) \end{aligned}$$

where we have again used (3.45). The first positive term on the right hand side is uncontrollable by the ancillary supplier. Hence, maximization of the discounted welfare is equivalent to minimizing the discounted cost,

$$\mathbb{E}_x \left[\int_0^\infty e^{-\eta t} \mathcal{C}_s(t) dt \right],$$

with $\mathcal{C}_s(t) = c^a G^a(t) + (c^{bo} + v)Q_-(t)$. Given any threshold $\bar{q}^p > \bar{q}^{a*}$ for primary service, the optimal G^a minimizing this objective function is affine with threshold \bar{q}^{a*} . \square

4 Conclusions

Theorem 2.3 establishes an explicit formula for reserves in the dynamic newsboy model. Optimal reserves are high whenever there is high variability in demand, or significant ramping constraints on production. These conclusions are (qualitatively) consistent with the high reserves maintained in today's power market in California [2].

This paper also formally proves that volatility and high prices can be expected in a deregulated market whenever the market achieves an efficient allocation, even without market manipulation. These conclusions have direct implication to any of the current industries that require high reliability and are undergoing rapid deregulation.

The next step is to relax some of the restrictive assumptions imposed here to investigate the impact of, for example, nonconvex cost or general demand statistics. Unfortunately, in this generality we are left with numerical rather than mathematical analysis.

Perhaps more important than analysis is design. While it is well known that dynamics are important in economic systems, economic analysis and design typically begins and ends with a static equilibrium model. Simple models can give much insight, but they do not tell the entire story. Greater attention to dynamics in the form of delay, constraints, and variability will lead to more robust market designs that can withstand unexpected events such as drought, outages due to repair or bad weather, or unexpected surges in demand. It is a task of fundamental importance to build robust market rules that can withstand considerable volatility and possibly, strategic manipulation by the players. We need to move beyond a static analysis in order to address issues surrounding reliability and dynamics.

Similar issues arise in the field of automatic control. Robustness has been a focus in this area since its inception, but its importance became most clear after the infamous crash of the X-15 aircraft on November 15, 1967, which caused the death of its pilot Major M. J. Adams. The crash was caused by instability arising from a highly sensitive adaptive control system which could not have been predicted based on prevailing idealized models.

Simple models remain a valuable starting point in control design. However, once an initial design is constructed, it is refined and tested using increasingly complex models of the physical system. Finally, the control system is tested on the physical system to be controlled.

One can argue that the financial market for airline tickets is at least as complex as the equations describing a single airplane. Moreover, a civilian aircraft is not typically subject to

manipulation for profit as may happen in the financial market. Rather, it flies in a carefully managed system with oversight by a pilot, co-pilot, and air-traffic control system.

This form of centralized management is not suitable for an economy, which is the typical rationale for pursuing decentralized market mechanisms. However, this does not imply that the only alternative is a completely deregulated market. This paper shows that in a completely decentralized market it is possible to achieve one goal, *efficiency*, but one fails to achieve another, *fairness*. The grand challenge is to achieve better compromises in market design to encourage fairness among participants, to create incentives for investment, and discourage deliberate manipulation of shared resources.

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A Appendices

We begin with some general results required in the analysis of the CBM model.

A.1 Height process

Let \mathbf{H} be a reflected Brownian motion on \mathbb{R}_+ satisfying the the Itô equation,

$$dH = -\delta_H - dI(t) + dD(t), \quad t \geq 0, \quad (\text{A.1})$$

where \mathbf{D} is a driftless Brownian motion, and the reflection process \mathbf{I} is non-decreasing and satisfies,

$$\int_0^\infty \mathbb{I}\{H(t) > 0\} dI(t) = 0.$$

When $\delta_H > 0$ the Markov process \mathbf{H} is positive recurrent, and its unique invariant probability measure is exponential with parameter $\gamma_H := 2\delta_H/\sigma_H^2$.

For a given constant $r_0 \geq 0$ define

$$\tau_{r_0} = \min\{t \geq 0 : H(t) = r_0\}$$

and consider the convex, C^1 function defined by,

$$\Psi(r) = \begin{cases} e^{\gamma_H r} - \gamma_H r & r < r_0 \\ e^{\gamma_H r_0} - \gamma_H r_0 + m_H(r - r_0) & r \geq r_0, \end{cases} \quad (\text{A.2})$$

where $m_H = \Psi'(r_0) = \gamma_H(e^{\gamma_H r_0} - 1)$.

Proposition A.1 *Suppose that the reflected Brownian motion has negative drift so that $\delta_H > 0$. Then for each initial condition $H(0) = r \geq 0$,*

(i) $\mathbb{E}[\tau_0] = \delta_H^{-1}r$.

(ii) For any $r_0 \geq r$,

$$\mathbb{P}\{\tau_{r_0} < \tau_0\} = (e^{\gamma_H r_0} - 1)^{-1}(e^{\gamma_H r} - 1).$$

(iii) For any constant $r_0 > 0$,

$$\mathbb{E}_x \left[\int_0^{\tau_0} \mathbb{I}\{H(t) \geq r_0\} dt \right] = \left(\Psi(r) - 1 + \gamma_H r \right) \left(\delta_H \gamma_H e^{\gamma_H r_0} \right)^{-1}. \quad (\text{A.3})$$

Proof: These formulae can be found in or derived from results in [14]. In particular (ii) is given as formula 3.0.4 (b), p. 309, and (iii) follows from formula 3.46 (a), p. 313 of [14]. We provide a brief proof based on invariance equations for the differential generator,

$$\mathcal{D}_H = -\delta_H \nabla + \frac{1}{2} \sigma_H^2 \nabla^2.$$

To prove (i) let $g(r) = \delta_H^{-1}r$ so that $\mathcal{D}_H g = -1$. It follows that $M_i(t) = t \wedge \tau_0 + g(H(t \wedge \tau_0))$ is a martingale,

$$\mathbb{E}[t \wedge \tau_0 + \delta_H^{-1}H(t \wedge \tau_0)] = \mathbb{E}[M_i(t)] = \mathbb{E}[M_i(0)] = \delta_H^{-1}H(0).$$

Letting $r \rightarrow \infty$ and applying the Dominated Convergence Theorem gives (i).

The function $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by $g(r) := e^{\gamma_H r}$, $r \in \mathbb{R}$, is in the null space of the differential generator for \mathbf{H} , which implies that $M_{ii}(t) = e^{\gamma_H H(t \wedge \tau_0)}$ is a local martingale. Uniform integrability can be established to conclude that with $\tau = \min(\tau_{r_0}, \tau_0)$,

$$\mathbb{E}[M_{ii}(\tau)] = M_{ii}(0), \quad 0 \leq r \leq r_0.$$

Rearranging terms gives (ii).

To see (iii) we apply the differential generator to Ψ to obtain,

$$\mathcal{D}_H \Psi = -\delta_H (-\gamma_H \mathbb{I}_{r < r_0} + m_H \mathbb{I}_{r \geq r_0}).$$

Here we have again used the fact that $\mathcal{D}_H g = 0$. Substituting the definition of m_H and writing $\mathbb{I}_{r < r_0} + \mathbb{I}_{r \geq r_0} = 1$ then gives,

$$\mathcal{D}_H \Psi = \delta_H (\gamma_H - (m_H + \gamma_H) \mathbb{I}_{r \geq r_0}).$$

The function Ψ satisfies $\Psi'(0) = 0$, so we can replace \mathcal{D}_H by \mathcal{A}_H in the expression above, which means that the process defined below is a local martingale,

$$M(t) := \Psi(H(t)) - \delta_H \int_0^t (\gamma_H - (m_H + \gamma_H) \mathbb{I}_{H(s) \geq r_0}) ds, \quad t \geq 0.$$

It is in fact a martingale since it is uniformly integrable on any bounded interval. It is also uniformly integrable on $[0, \tau_0]$ which implies that,

$$\Psi(r) = M(0) = \mathbb{E}[M(\tau_0)] = \Psi(0) - \delta_H \mathbb{E} \left[\int_0^{\tau_0} (\gamma_H - (m_H + \gamma_H) \mathbb{I}\{H(s) \geq r_0\}) ds \right]$$

Rearranging terms gives for any $r_0 > 0$,

$$\mathbb{E} \left[\int_0^{\tau_0} \mathbb{I}\{H(s) \geq r_0\} ds \right] = \left(\delta_H (m_H + \gamma_H) \right)^{-1} \left(\Psi(r) - \Psi(0) + \delta_H \gamma_H \mathbb{E}[\tau_0] \right).$$

The expression (A.3) then follows. □

A.2 Optimization

Recall that Proposition 3.1 asserts that the function h defined in (3.31) is a solution to Poisson's equation. To prove Proposition 3.3 we construct representations for $\langle \nabla h(x), B1^i \rangle$, $i = 1, 2$, within $\mathcal{R}(\bar{q})$ in terms the pair of sensitivity functions,

$$\lambda_p(x) = \langle \nabla c(x), B1^1 \rangle = c^p - \mathbb{I}\{q \leq 0\}c^{bo}, \quad \lambda_a(x) = \langle \nabla c(x), B1^2 \rangle = c^a - \mathbb{I}\{q \leq 0\}c^{bo}, \quad (\text{A.4})$$

and the stopping times

$$\tau_p := \inf\{t \geq 0 : Q(t) = \bar{q}^p\}, \quad \tau_a := \inf\{t \geq 0 : Q(t) \geq \bar{q}^a\}. \quad (\text{A.5})$$

The analysis is based upon a reduction to a pair of one-dimensional reflected processes: the *height processes* with respect to each of the thresholds \bar{q}^p, \bar{q}^a ,

$$H^p(t) := \bar{q}^p - Q(t) + G^a(t), \quad (\text{A.6})$$

$$H^a(t) := \bar{q}^a - Q(t), \quad t \geq 0. \quad (\text{A.7})$$

We have $H^p(t) \geq 0$ for all $t > 0$ under the affine policy, and moreover,

$$dH^p = -\zeta^{p+} + dI^p(t) + dD(t), \quad t \geq 0.$$

Hence \mathbf{H}^p is a one-dimensional RBM.

First we must establish that h is smooth. We consider the normalized function $h_\bullet(x) = h(x) - h(x^p)$, $x \in \mathbf{X}$, with $x^p := (\bar{q}^p, 0)^T$. We obtain a useful representation for h_\bullet through a particular construction of the state processes starting from various initial conditions. Based on a single Brownian motion \mathbf{D} , we define on the same probability space the entire family of solutions to (2.2), denoted $\{X(t; x) : t \geq 0, x \in \mathbf{X}\}$.

The processes $X(t; x)$ and $X(t; x^p)$ have corresponding height processes $H^p(t; x)$, $H^p(t; x^p)$ satisfying $H^p(0; x^p) = 0$ and $H^p(0; x) \geq 0$. Consequently, the two processes couple at time $\tau_p(x) = \min\{t : H^p(t; x) = 0\} = \min\{t : H^p(t; x) = H^p(t; x^p)\}$. This combined with (3.31) implies the representation,

$$h_\bullet(x) = \mathbb{E}_x \left[\int_0^\infty (c(X(t; x)) - c(X(t; x^p))) dt \right], \quad x \in \mathbf{X}. \quad (\text{A.8})$$

In Proposition A.2 the function (A.8) is compared to the function obtained when \mathbf{X} is replaced by the fluid model in which $\sigma_D^2 = 0$. This deterministic process is denoted $\mathbf{x} = (\mathbf{q}, \mathbf{g}^a)^T$, and satisfies for $t > 0$,

$$\frac{d}{dt} \mathbf{x}(t; x) = \begin{cases} Bu^{a+} & q(t; x) < \bar{q}^a; \\ Bu^{a-} & q(t; x) = \bar{q}^a, g^a(t; x) > 0; \\ Bu^{p+} & \bar{q}^a \leq q(t; x) < \bar{q}^p, g^a(t; x) = 0; \\ 0 & q(t; x) = \bar{q}^p, \end{cases}$$

where u^{p+} and u^{a+} are defined in (3.26), and $u^{a-} = (\zeta^{p+}, -\zeta^{p+})^T$. The potential jump at time $t = 0$ is identical to its stochastic counterpart (see (2.4).)

Proposition A.2 *Suppose that \mathbf{X} is controlled using an affine policy and that h_\bullet is defined by (A.8). Denote by h_0 the corresponding function when $\mathbf{D} = 0$,*

$$h_0(x) = \int_0^\infty (c(x(t; x)) - c(x^p)) dt \quad x \in \mathbf{X}. \quad (\text{A.9})$$

Then,

(i) *The function h_0 is C^1 , and satisfies,*

$$\mathcal{D}h_0 = -c + c(x^p) + \frac{1}{2}\sigma_D^2 \frac{\partial^2}{\partial q^2} h_0.$$

(ii) *The function h_\bullet has the explicit form,*

$$h_\bullet = h_0 + \ell + m$$

where ℓ is a continuous piecewise-linear function of $(q, g^a)^T$, and m is a piecewise exponential function of q . The following identities hold whenever the derivatives are defined.

$$\mathcal{D}\ell = -\left(c(x^p) + \frac{1}{2}\sigma_D^2 \frac{\partial^2}{\partial q^2} h_0\right) + \phi, \quad \mathcal{D}m = 0.$$

(iii) *h_\bullet is C^1 and satisfies the boundary conditions,*

$$\frac{\partial}{\partial q} h_\bullet(x) + \frac{\partial}{\partial g^a} h_\bullet(x) = 0, \quad \text{when } q \geq \bar{q}^a; \quad \frac{\partial}{\partial q} h_\bullet(x) = 0, \quad \text{when } q + g^a \geq \bar{q}^p. \quad (\text{A.10})$$

Proof: The proof of Part (i) is similar to Proposition 4.2 of [52]. We first establish that h_0 is smooth. A representation of the derivative is obtained by differentiating the expression for h_0 :

$$\nabla h_0(x) = \int_0^\infty \nabla c(x(t; x)) dt \quad x \in \mathbf{X},$$

where the derivative is with respect to the initial condition x . It can be shown that this expression is continuous and piecewise-linear on \mathbf{X} . The expression $\langle \nabla h_0(x), Bu^{a+} \rangle = -c(x) + c(x^p)$ follows from the Fundamental Theorem of Calculus and the representation,

$$h_0(x(s; x)) = \int_s^\infty (c(x(t; x)) - c(x^p)) dt \quad x \in \mathbf{X},$$

and this completes the proof of (i).

Parts (ii) and (iii) are proved in Appendix A of Chen's thesis [16]. The details are based on complex computations, but the main idea is as follows: We have from (i),

$$\mathcal{D}h_0 = -c + b_0,$$

where b_0 is piecewise constant. Following the proof of (i) we can give an explicit expression for ℓ ,

$$\ell(x) = \int_0^{\tau_p^0} (b_0(x(t; x)) - \phi) dt \quad x \in \mathbf{X},$$

where $\tau_p^0 = H^p(0; x)/\zeta^{p+} = (\bar{q}^p - q + g^a)/\zeta^{p+}$. This function is piecewise-linear and continuous, and satisfies,

$$\mathcal{D}\ell = -b_0 + \phi,$$

whenever ℓ is differentiable. Hence $h_0 + \ell$ satisfies Poisson's equation for the differential generator. The function m is in the null space of the differential generator, and is constructed so that $h_0 + \ell + m$ is C^1 . It is of the specific form,

$$m(q) = \begin{cases} A_p + B_p e^{-\gamma_p q} & \bar{q}^a \leq q \leq \bar{q}^p; \\ A_a + B_a e^{-\gamma_a q} & 0 \leq q < \bar{q}^a; \\ A_a + B_a & q < 0, \end{cases}$$

where the parameters $\{\gamma_a, \gamma_p\}$ are defined in (2.15), and (A_a, A_p, B_a, B_p) are constants. \square

We can now obtain representations of the derivatives:

Proposition A.3 *Under an affine policy using any thresholds \bar{q}^p, \bar{q}^a , the directional derivatives can be expressed as follows for $x \in \mathcal{R}(\bar{q})$,*

$$\langle \nabla h(x), I^2 \rangle = -\mathbb{E}_x \left[\int_0^{\tau_p} \lambda^p(X(t)) dt \right] = -c^p \mathbb{E}[\tau_p] + c^a \mathbb{E}_x \left[\int_0^{\tau_a} \mathbb{I}\{Q(t) \leq \bar{q}^a\} dt \right] \quad (\text{A.11})$$

$$\langle \nabla h(x), B I^2 \rangle = \mathbb{E}_x \left[\int_0^{\tau_a} \lambda^a(X(t)) dt \right] = c^a \mathbb{E}[\tau_a] - c^{bo} \mathbb{E}_x \left[\int_0^{\tau_a} \mathbb{I}\{Q(t) \leq 0\} dt \right]. \quad (\text{A.12})$$

Proof: We omit the proof of (A.11) since it is similar (and simpler) than the proof of (A.12). Equation (A.12) is based on the following representation of h ,

$$h(x) = \mathbb{E}_x \left[\int_0^{\tau_a} (c(X(t)) - \phi) dt + h(X(\tau_a)) \right], \quad x \in \mathcal{X}. \quad (\text{A.13})$$

This follows from the martingale property for \mathbf{M}_h and the bound Proposition 3.1 (i) with $m = 3$ (which implies that the martingale is uniformly integrable on $[0, \tau_a]$.)

Suppose that $X(0) = x$ lies in the interior of $\mathcal{R}^a(\bar{q})$ and consider the perturbation $X^\varepsilon(0) = x^\varepsilon := x - \varepsilon B1^2$ for small $\varepsilon > 0$. The state processes $\{X^\varepsilon : \varepsilon \geq 0\}$ are defined on a common probability space, with common demand process \mathbf{D} . We then have,

$$X^\varepsilon(t) = X(t) - \varepsilon B1^2, \quad 0 \leq t \leq \tau_a,$$

and from (A.13) it follows that for all $x \in \mathcal{X}$,

$$h(x) - h(x^\varepsilon) = \mathbb{E}_x \left[\int_0^{\tau_a} (c(X(t)) - c(X(t) - \varepsilon B1^2)) dt + h(X(\tau_a)) - h(X(\tau_a) - \varepsilon B1^2) \right]. \quad (\text{A.14})$$

Proposition A.2 and the Mean Value Theorem give,

$$|h(X(\tau_a)) - h(X(\tau_a) - \varepsilon B1^2)| = \varepsilon |\langle \nabla h(\tilde{X}), B1^2 \rangle|,$$

where $\tilde{X} = X(\tau_a) - \tilde{\varepsilon} B1^2$ for some $\tilde{\varepsilon} \in (0, \varepsilon)$. We have $Q(\tau_a) = \bar{q}^a$, which implies that $|\langle \nabla h(\tilde{X}), B1^2 \rangle| \rightarrow 0$ with probability one as $\varepsilon \rightarrow 0$.

For some constant b_0 we have,

$$\sup_{0 \leq \varepsilon \leq 1} \|\nabla h(y - \varepsilon B1^2)\| \leq b_0 \|y\|, \quad y = (\bar{q}^a, g^a), \quad g^a \geq 0,$$

where the constant is independent of g^a . Proposition 3.1 (i) with $m = 3$ implies uniform integrability, so that

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E}[\|\langle \nabla h(\tilde{X}), B1^2 \rangle\|] = 0.$$

Multiplying the identity (A.14) by ε^{-1} and applying Proposition 3.1 (i) once more gives,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} (h(x) - h(x^\varepsilon)) = \mathbf{E}_x \left[\int_0^{\tau_a} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} (c(X(t)) - c(X(t) - \varepsilon B1^2)) dt \right], \quad x \in \mathcal{R}^a(\bar{q}),$$

which gives (A.12). \square

Proof of Proposition 3.3. We now return to the pair of height processes defined in (A.6,A.7). Recall that under an affine policy, \mathbf{H}^p is a one-dimensional RBM. For an initial condition $x \in \mathcal{R}(\bar{q})$ satisfying $g^a = G^a(0) > 0$, the height process \mathbf{H}^p evolves as a one-dimensional RBM up to the first time $t > 0$ that $G^a(t) = 0$.

Part (i). We begin by considering the right hand side of (A.12). We show that this is strictly negative on $\mathcal{R}^a(\bar{q})$ when $\bar{q}^a = \bar{q}^{a*}$ through an analysis of the height process \mathbf{H}^a .

The gradient formula (A.12) can be expressed in terms of the height process (A.7) via,

$$\langle \nabla h(x), B1^2 \rangle = c^a \mathbf{E}[\tau_a] - c^{bo} \mathbf{E}_x \left[\int_0^{\tau_a} \mathbb{I}\{H^a(t) \geq \bar{q}^a\} dt \right]. \quad (\text{A.15})$$

The stopping time τ_a can be interpreted as the first hitting time to the origin for \mathbf{H}^a . The following identities are obtained in Proposition A.1:

$$\mathbf{E}[\tau_a] = \delta_H^{-1} r, \quad \mathbf{E}_x \left[\int_0^{\tau_a} \mathbb{I}\{H^a(t) \geq \bar{q}^a\} dt \right] = \left(\Psi(r) - 1 + \gamma_H r \right) \left(\delta_H \gamma_H e^{\gamma_H \bar{q}^a} \right)^{-1},$$

where Ψ is defined in (A.2) by

$$\Psi(r) = \begin{cases} e^{\gamma_H r} - \gamma_H r & r < r_0 \\ e^{\gamma_H r_0} - \gamma_H r_0 + m_H(r - r_0) & r \geq r_0, \end{cases}$$

with $r_0 = \bar{q}^a$, $r = H^a(0) = \bar{q}^a - q \geq 0$, and $\delta_H = \zeta^{p+} + \zeta^{a+}$. Consequently, (A.15) can be expressed,

$$\langle \nabla h(x), B1^2 \rangle = \Phi(r) := c^a \delta_H^{-1} r - c^{bo} \left(\Psi(r) - 1 + \gamma_H r \right) \left(\delta_H \gamma_H e^{\gamma_H \bar{q}^a} \right)^{-1}, \quad r \geq 0.$$

The function Ψ is convex, with $\Psi(0) = 1$ and $\Psi'(0) = 0$. Consequently, the function Φ defined above is *concave*, strictly concave on $[0, \bar{q}^a]$, with $\Phi(0) = 0$. To show that Φ is negative on $(0, \infty)$ it suffices to show that $\Phi'(0) \leq 0$.

The derivative at zero is expressed,

$$\Phi'(0) = c^a \delta_H^{-1} - c^{bo} \left(\Psi'(0) + \gamma_H \right) \left(\delta_H \gamma_H e^{\gamma_H \bar{q}^a} \right)^{-1} = \delta_H^{-1} \left(c^a - c^{bo} e^{-\gamma_H \bar{q}^a} \right).$$

When $\bar{q}^a = \bar{q}^{a*}$ we have $e^{-\gamma_H \bar{q}^a} = c^a / c^{bo}$, so that $\Phi'(0) = 0$.

Part (ii). We next consider $\langle \nabla h, B1^1 \rangle$ for $x \in \mathcal{R}^a$ when \bar{q}^{p*} and \bar{q}^{a*} are given by (2.17).

Consider the height process relative to \bar{q}^p defined in (A.6). By the foregoing analysis we have on \mathcal{R}^a ,

$$\langle \nabla h, 1^1 + 1^2 \rangle = \langle \nabla h, B1^2 \rangle < 0.$$

Consequently, to show that $\langle \nabla h, B1^1 \rangle < 0$ it is sufficient to show that $\langle \nabla h, 1^2 \rangle \geq 0$. This derivative is given in (A.11), which can be expressed in terms of the height process,

$$\langle \nabla h, 1^2 \rangle = c^a \mathbf{E} \left[\int_0^{\tau_p} \mathbb{I}\{H^p(t) \geq r_0\} dt \right] - c^p \mathbf{E}[\tau_p],$$

where $r_0 = \bar{q}^p - \bar{q}^a$. Proposition A.1 then gives, with $r = H^p(0)$,

$$\langle \nabla h, 1^2 \rangle = c^a \left(\Psi(r) - 1 + \gamma_H r \right) \left(\delta_H \gamma_H e^{\gamma_H r_0} \right)^{-1} - c^p \delta_H^{-1} r.$$

The drift parameter for \mathbf{H}^p is $\delta_H = \zeta^{p+}$. The parameter \bar{q}^{p*} is chosen so that $e^{\gamma_H r_0} = c^a / c^p$, which on substitution gives,

$$\langle \nabla h, 1^2 \rangle = c^p \delta_H^{-1} \left(\gamma_H^{-1} (\Psi(r) - 1 + \gamma_H r) - r \right) = c^p \delta_H^{-1} \gamma_H^{-1} (\Psi(r) - 1). \quad (\text{A.16})$$

For $r \geq r_0 > 0$ (equivalently, $q < \bar{q}^a$), we obtain from the definition of Ψ ,

$$\langle \nabla h, 1^2 \rangle \geq c^p \delta_H^{-1} \gamma_H^{-1} (e^{\gamma_H r_0} - \gamma_H r_0 - 1) > 0.$$

We conclude that $\langle \nabla h, 1^1 \rangle < 0$ on \mathcal{R}^a .

Finally we demonstrate that $\langle \nabla h, B1^1 \rangle < 0$ on $\mathcal{R}^p \setminus \mathcal{R}^a$. For an initial condition $x \in \mathcal{R}^p$ satisfying $\bar{q}^a \leq q < \bar{q}^p$ we can write,

$$h(x) = \mathbf{E}_x \left[\int_0^\tau (c(X(t)) - \phi) dt + h(X(\tau)) \right]$$

where $\tau = \min(\tau_a, \tau_p)$. Consequently,

$$\langle \nabla h(x), 1^1 \rangle = \mathbf{E} [c^p \tau + \langle \nabla h(X(\tau)), 1^1 \rangle] = \mathbf{E} [(c^p \tau_a + \langle \nabla h(x^a), 1^1 \rangle) \mathbb{I}\{\tau_a \leq \tau_p\} + c^p \tau_p \mathbb{I}\{\tau_a > \tau_p\}],$$

where $x^a = (\bar{q}^a, 0)^T$. On rearranging terms and applying the strong Markov property we obtain,

$$\langle \nabla h(x), 1^1 \rangle = c^p \mathbf{E}_x[\tau_p] + (\langle \nabla h(x^a), 1^1 \rangle - c^p \mathbf{E}_{x^a}[\tau_p]) \mathbf{P}\{\tau_a > \tau_p\}.$$

Consideration of the height process \mathbf{H}^p then gives,

$$\langle \nabla h(x), 1^1 \rangle = c^p \delta_H^{-1} r + (\langle \nabla h(x^a), 1^1 \rangle - c^p \delta_H^{-1} r_0) \mathbf{P}^H\{\tau_{r_0} > \tau_0\}, \quad (\text{A.17})$$

where $r_0 = \bar{q}^p - \bar{q}^a$ and the probability on the right hand side is with respect to the height process: Proposition A.1 gives $\mathbf{P}^H\{\tau_{r_0} < \tau_0\} = (e^{\gamma_H r_0} - 1)^{-1} (e^{\gamma_H r} - 1)$.

Equation (A.16) provides an expression for the derivative of h at x^a ,

$$\langle \nabla h(x^a), 1^1 \rangle = -\langle \nabla h(x^a), 1^2 \rangle = -c^p \delta_H^{-1} \gamma_H^{-1} (e^{\gamma_H r_0} - \gamma_H r_0 - 1).$$

Combining this identity with (A.17) we obtain,

$$\begin{aligned} \langle \nabla h, 1^1 \rangle &= c^p \delta_H^{-1} \left(r - \gamma_H^{-1} (e^{\gamma_H r_0} - 1) \mathbf{P}^H\{\tau_{r_0} > \tau_0\} \right) \\ &= c^p \delta_H^{-1} \left(r - \gamma_H^{-1} (e^{\gamma_H r} - 1) \right), \quad 0 \leq r \leq r_0. \end{aligned}$$

The right hand side is strictly negative for $r > 0$. □

A.3 Multiple Levels of Ancillary Service

The proof of Theorem 3.4 is identical to the simpler setting of Section 3.2. We demonstrate that the solution to Poisson's equation h under the affine policy solves the dynamic programming equation,

$$\left(\mathcal{D}h_*(x) + c - \phi_* \right) \wedge \left(\inf \{ \langle \nabla h_*(x), -Bu \rangle : u \in \mathbb{R}_+^{K+1} \} \right) = 0, \quad x \in \mathbf{X},$$

where here we let B denote the $(K+1) \times m$ matrix defined by $B(1, i) = 1 = B(i+1, i) = 1$ for each i , and $B(i, j) = 0$ for all other indices (i, j) .

Proof of Theorem 3.4. The proof that $\langle \nabla h(x), B1^i \rangle$ is non-positive for $i = 1$ and $i = K+1$ is identical to the proof of Theorem 2.3. To obtain the analogous result for $i \in \{2, \dots, m\}$ we apply similar reasoning. Exactly as in (A.12), it can be shown that for $x \in \mathcal{R}^{a_i}$,

$$\langle \nabla h(x), B1^{i+1} \rangle = c^{a_i} \mathbf{E}[\tau_{a_i}] - c^{a_{i+1}} \mathbf{E}_x \left[\int_0^{\tau_{a_i}} \mathbb{I}\{Q(t) \leq \bar{q}^{a_{i+1}}\} dt \right], \quad (\text{A.18})$$

where $\tau_{a_i} := \inf\{t \geq 0 : Q(t) = \bar{q}^{a_i}\}$.

To compute the right hand side of (A.18) we again construct a one-dimensional Brownian motion to represent these expectations. Define for $Q(0) = q < \bar{q}^{a_i}$,

$$H^{a_i}(t) := \bar{q}^{a_i} - Q(t) + \sum_{j>i} G^{a_j}(t), \quad t \geq 0.$$

This is described by,

$$dH^{a_i}(t) = -\zeta_i^+ dt + dI^{a_i}(t) + dD(t), \quad t \geq 0 \quad \text{while } G^{a_i}(t) > 0.$$

Consequently, we can write using (A.18),

$$\langle \nabla h(x), B1^{i+1} \rangle = c^{a_i} \mathbf{E}[\tau_{a_i}] - c^{a_{i+1}} \mathbf{E}_x \left[\int_0^{\tau_{a_i}} \mathbb{I}\{H^{a_i}(t) \geq (\bar{q}^{a_i} - \bar{q}^{a_{i+1}})\} dt \right], \quad (\text{A.19})$$

and τ_{a_i} coincides with the first hitting time to the origin for H^{a_i} . Consequently, applying Proposition A.1,

$$\langle \nabla h(x), B1^{i+1} \rangle = c^{a_i} \delta_H^{-1} r - c^{a_{i+1}} \left(\Psi(r) - 1 + \gamma_H r \right) \left(\delta_H \gamma_H e^{\gamma_H (\bar{q}^{a_i} - \bar{q}^{a_{i+1}})} \right)^{-1}, \quad r \geq 0,$$

where $\gamma_H = 2\zeta_i^+ / \sigma_D^2$, and the function Ψ is defined in (A.2). The remainder of the proof is identical to the proof of Proposition 3.3 using the formula $e^{-\gamma_H \bar{q}^{a_i}} = c^{a_i} / c^{a_{i+1}}$. \square

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