The number of neighbors needed for connectivity of wireless networks *†

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Abstract

Unlike wired networks, wireless networks do not come with links. Rather, links have to be fashioned out of the ether by nodes choosing neighbors to connect to. Moreover the location of the nodes may be random.

The question that we resolve is: How many neighbors should each node be connected to in order that the overall network is connected in a multi-hop fashion? We show that in a network with n randomly placed nodes, each node should be connected to $\Theta(\log n)$ nearest neighbors. If each node is connected to less than $0.074 \log n$ nearest neighbors then the network is asymptotically disconnected with probability one as n increases, while if each node is connected to more than $5.1774 \log n$ nearest neighbors then the network is asymptotically connected with probability approaching one as n increases. It appears that the critical constant may be close to one, but that remains an open problem.

These results should be contrasted with some works in the 1970s and 1980s which suggested that the "magic number" of nearest neighbors should be six or eight.

1 Introduction

Unlike wired networks, wireless networks do not come with ready-made links. Rather, links are formed by nodes choosing the power levels at which they transmit. This raises the question: How many neighbors should each node be connected to, in order that the overall network then becomes connected? This question arises naturally in mobile multi-hop radio networks, also known as *ad hoc* networks. All nodes cooperate in routing each other's packets, so that packets are transported in a multi-hop fashion from source to destination.

The problem of how many neighbors is desirable was considered in a series of papers [1, 2, 3, 4, 5, 6, 7] beginning in the 1970s. The wireless network is modeled as nodes located randomly on the plane according to a Poisson point process [1, 3, 4, 5, 6, 7] or on a line [2]. The focus in [1, 2, 4, 5, 6] is on the issue of maximizing the one hop progress of a packet in the desired direction under different transmission protocols. Based on

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†This material is based upon work partially supported by USARO under contract Nos. DAAD 19-00-1-0466 and DAAD 19-01010-465, DARPA under contract Nos. F33615-01-C-1905 and N00014-01-1-0576, AFOSR under contract Nos. F49620-02-1-0217 and AF-DC-5-36128, DARPA|AFOSR under contract No. F49620-02-1-0325, NSF under contract No. NSFANI-02-21357, and ONR under contract No. N00014-99-1-0696. Any opinions, findings and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the above agencies.

the analysis of the slotted ALOHA protocol and requiring that the transmission powers are the same for all nodes, it was first proposed by Kleinrock and Silvester in [1] that six was the "magic number," i.e., on average every node should connect itself to its six nearest neighbors. Later, the magic number was revised to eight in [5]. In the same paper [5], Takagi and Kleinrock also considered other transmission protocols, which resulted in some other magic numbers five and seven. Hou and Li [6] considered the situation when each node is allowed to adjust its transmission range individually, and obtained the magic numbers six and eight. When considering the maximization of the transmission efficiency, defined as the ratio between the expected progress and the area covered by the transmission, Hajek [7] suggested that each node should adjust its power to cover about three neighbors on average. Mathar and Mattfeldt [2] analyzed the wireless network generated by a Poisson point process on a line, and also obtained some magic numbers.

However, the above analyses did not resolve the issue of connectivity. Although simulations in [3] suggested that six to eight neighbors can make a small size network connected with high probability, it turns out that as the number of nodes in the network increases, the network becomes disconnected with probability one whether one connects to six or eight neighbors, as we will show. In fact, we show that there are no magic numbers if one takes connectivity also into consideration. Specifically, for every finite number, the probability of network disconnectivity converges to one as the number of nodes in the network increases.

We also show that the number of neighbors of each node needs to grow like $\Theta(\log n)$ if the network is to be connected, where n is the number of nodes in the network. We can even bound the constant involved. If each node connects with less than $0.074 \log n$ nearest neighbors, then the network is asymptotically disconnected. However, if each node connects to greater than $5.1774 \log n$ nearest neighbors, then the network is asymptotically connected.

The problem of choosing how many neighbors to connect to affects not only the connectivity of the network, but also the capacity of the network, i.e., how much traffic it can carry. In wireless networks, the presence of a link (i, j) can be an advantage in that it enables node i to send a packet on one hop to node j. However, it can also be a disadvantage in that, when i broadcasts, it causes interference at j. Thus the presence of a large number of links means the possibility for lots of interference, which reduces the capacity of the network.

Hence one needs to examine the tradeoff between the presence and absence of links in a more careful manner. When many links are present, a packet can get to its destination in a fewer number of hops. This means that the relaying burden on other nodes is reduced. Thus while interference is larger when the number of links is larger, the relaying burden is smaller. This problem has been addressed in [8] for the case of connectivity based on distance. It is shown there that if r is the range of a broadcast, then the relaying burden grows like $O\left(\frac{1}{r}\right)$, but the interference grows only like $O(r^2)$. Thus, the net effect, the product, is a growth of $O(r)$, implying that the smaller the range the better. However, if one chooses too small a range, then the network can get disconnected. This motivates the study of connectivity based on distance. This is a different method of connectivity where one simply connects to all nodes within a range r. This connectivity problem was studied in [9] when the points are distributed as a Poisson process on a square. It was shown that if n is the intensity of the Poisson process on a unit square, then a choice of radius $r(n) = \sqrt{\frac{(1-\epsilon) A \log n}{\pi n}}$ would lead to probability of connectedness converging to zero, for every $\epsilon > 0$. In [10], for the case of n points uniformly iid distributed in a disk of area A it was shown that a range chosen as $r(n) = \sqrt{\frac{A(\log n + \gamma_n)}{\pi n}}$ will lead to the probability of connectedness converging to one as $n \to \infty$ if and only if $\gamma_n \to +\infty$. This result can also be deduced from a result in Penrose [11] where the length of the longest edge in a minimum spanning tree, suitably centered and normalized, is shown to converge to a double exponential distribution.

Yet another area where the connectivity issue arises is in random graphs. Specifically, the problem has been considered for Bernoulli random graphs where an edge (i, j) is inserted with probability p. It is shown (Theorem VII.3 in [12]) that if $p(n) = \frac{\log n + \gamma_n}{n}$, then the probability that the graph is connected goes to one as $n \to +\infty$ iff $\gamma_n \to +\infty$. It should be noted that Bernoulli random graphs are not appropriate models for connectivity in wireless networks since edges are introduced independent of the distance between nodes. Thus such a graph may have a link from a node to a faraway node, without a link to a nearer node.

Connectedness has also been considered in the field of continuum percolation theory [13, 14]. There, the model for the points is a Poisson point process on the infinite plane, and the focus is on the existence of an infinite size connected component under different models of connections. Recently, [15, 16] have addressed wireless networks and covering algorithms by the methods of continuum percolation. Under connection based on the number of neighbors, [17] considers the central limit theorem. The issue of uniqueness of the infinite component is addressed in [18].

The rest of the paper is organized as follows. In Section 2, we provide the problem formulation and present the main result. In Sections 3 and 4, we give the proof of the main result. We provide some simulation results in Section 5, and conclude in Section 6.

2 Formulation and main result

Let $\mathcal{S}$ be a unit square in $\mathcal{R}^2$, and suppose that n nodes are placed uniformly and independently in $\mathcal{S}$. Denote by $\mathcal{G}(n, \phi_n)$ the network formed when each node is connected to its ϕ_n nearest neighbors. More precisely, there exists an edge (i, j) if either j is one of the ϕ_n nearest neighbors of i, or i is one of the ϕ_n nearest neighbors of j.

We are interested in the following question:

For what choice of the number of nearest neighbors ϕ_n will the resulting graph $\mathcal{G}(n, \phi_n)$ be connected as n goes to ∞?

Our main result is the following theorem:

Theorem 1. *For $\mathcal{G}(n, \phi_n)$ to be asymptotically connected, $\Theta(\log n)$ neighbors are necessary and sufficient. Specifically, there are two constants $0 < c_1 < c_2$ such that:*

$$\lim_{n\to\infty} Pr\{\mathcal{G}(n, c_1 \log n) \text{ is disconnected}\} = 1, \text{ and}$$

$$\lim_{n\to\infty} Pr\{\mathcal{G}(n, c_2 \log n) \text{ is connected}\} = 1.$$

Remark 1. From the calculations in the proof of the Theorem 1 one can choose $c_1 = 0.074$ and any $c_2 > 2/\log(4/e) = 5.1774$.

3 $\Theta(\log n)$ neighbors are necessary for connectivity

In this section we will give the proof of the necessity part of Theorem 1.

3.1 A scenario for disconnection

Definition 3.1. *Square tessellation $\mathcal{T}_S^n$. We split the unit square equally into $M_n = \lceil\sqrt{\frac{n}{K\log n}}\rceil^2$ small squares as depicted in Figure 1, where $K > 0$ is a tunable parameter, and $\lceil x\rceil$ is the smallest integer larger*

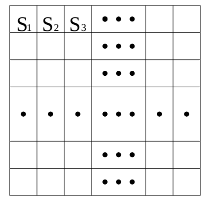

Figure 1: The square tessellation $\mathcal{T}_s^n$.

than or equal to x. This tessellation of the unit square will be denoted by $\mathcal{T}_S^n$. We name the small squares as S_i^n, $i = 1, 2, \cdots, M_n$, from left to right, and from top to bottom.

Denote by N_i^n, the number, among the n nodes, that fall into square S_i^n.

Definition 3.2. *Trap of size d. We call the structure in Figure 2 a trap of size d. It is composed of 21*

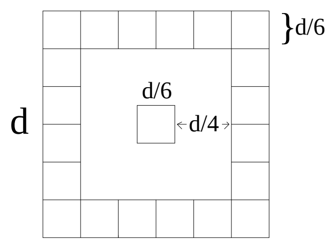

Figure 2: A trap of size d.

small squares all of the same size placed in a larger square of size d. Twenty of the small squares are at the periphery of the square, while one is located at the center.

Definition 3.3. *k-filling event. We say a k-filling event occurs if there are at least k nodes in each of the twenty-one small squares, and there is no node in the space between the center and the other twenty squares; see Figure 3.*

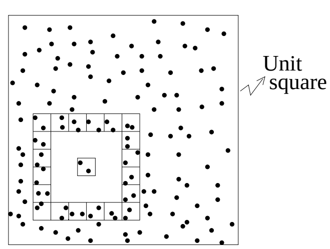

Figure 3: A k-filling event.

The significance of a $(\phi_n + 1)$-filling event is that the nodes in the central small square are disconnected from the other nodes. This is because the ϕ_n nearest neighbors of the nodes in the center square are all within the central square itself. Also, the nodes in the side squares cannot have a node in the center as one of their ϕ_n nearest neighbors since they will have enough neighbors close to them. Finally, the nodes outside the square of size d are also blocked away from the nodes in the central square by the nodes in the side squares.

Thus, if a $(\phi_n + 1)$-filling event occurs in a trap in $\mathcal{G}(n, \phi_n)$, then the center nodes contain a connected component. A similar, but not the same, localization construction is used in [17] in their study of the Central Limit Theorem for certain geometric random variables.

Before presenting the full proof of the *necessity* part of Theorem 1, we sketch the basic ideas involved and outline it. We first show in Section 3.2 that, if we tessellate the unit square by small squares with area of order $\frac{\log n}{n}$, then, with a very high probability, there are $\Theta(\log n)$ nodes simultaneously in *every* such small square. Then we calculate the probability that a k-filling event happens within such a small square containing about $\Theta(\log n)$ nodes. This allows us to calculate a lower bound on the probability that a k-filling event happens within the unit square. Our calculation shows that, if $k < c_1 \log n$, then a k-filling event will happen in the unit square with high probability, thus proving the necessity part.

3.2 Not quite the Law of Large Numbers: A uniformity result

What we now show is that every square S_i^n in the tessellation $\mathcal{T}_S^n$ has about $\Theta(\log n)$ nodes in it with high probability. This result is uniformly true for all squares S_i^n. However, it is not quite a law of large numbers for the following reason. The mean number of nodes in any S_i^n is

$$E[N_i^n] \quad = \quad K \log n.$$

However the bound allows for a deviation in the actual number of nodes of order $\Theta(\log n)$, and not $o(\log n)$. In fact we cannot apply the Vapnik-Chervonenkis Uniform Law of Large Numbers to deduce the result, since it deals with a resolution higher than what the uniform law of large numbers can provide.

Lemma 3.1. *Let* $K > \frac{1}{\log\left(\frac{4}{e}\right)}$, *and let* $\mu^* \in (0,1)$ *be the sole root of the equation*

$$-\mu^* + (1+\mu^*)\log(1+\mu^*) \quad = \quad \frac{1}{K}. \tag{1}$$

If we tessellate $\mathcal{S}$ *by* $\mathcal{T}_S^n$, *then the following limit holds for any* $\mu > \mu^*$*:*

$$\lim_{n\to\infty} Pr\{\max_i |N_i^n - K\log n| \le \mu K \log n\} \quad = \quad 1.$$

Proof. The basic idea of the proof is to consider the problem under a new assumption that the nodes are generated in the unit square by a planar Poisson point process with intensity $\lambda_n = n$, instead of having a deterministic numbers of n nodes thrown into it. The main reason for doing so is that the numbers of nodes in disjoint sets are then *independent* random variables. This will allow us to compute the probability that *all* squares have $K \log n \pm \mu K \log n$ nodes by taking the product of the probabilities. To compute the result we will show that the number of nodes in a square is more or less the same whether we have a fixed number of nodes n in the unit square, or a random number given by a Poisson process of intensity n.

So consider a unit square on the plane with a Poisson process with intensity $\lambda_n = n$. Denote the number of the nodes that fall into square S_i^n by $\widetilde{N}_i^n$. The total number of the nodes that fall into the unit square is $\widetilde{M}_n$, which is also a random variable.

We also introduce a sequence of iid rv's $\{X_k,\ k = 1, 2, \cdots\}$, which are uniformly distributed in a unit square.

Remark 2. We have used the ceil function $\lceil\cdot\rceil$ to truncate a real number into an integer. For clarity of presentation we ignore this and treat $\sqrt{\frac{n}{K\log n}}$ as an integer since it is nearly so, i.e.,

$$\left\lceil\sqrt{\frac{n}{K\log n}}\right\rceil^2 = \frac{n}{K\log n}(1+o(1)), \quad \text{as } n\to+\infty.$$

Lemma 3.2.1. *Suppose we tessellate the unit square by $\mathcal{T}_S^n$. Then for any deterministic integer sequence $\{m_1^n, m_2^n, n=1,2,\cdots\}$ satisfying $0\le m_2^n - m_1^n \le \sqrt{n\log n}$, $\forall n$, we have:*

$$\text{(i) } Pr\left\{\max\nolimits_{1\le i\le M_n}\sum_{k=m_1^n}^{m_2^n}\mathrm{I}_{S_i^n}(X_k)\le 2\right\}\ge Pr\left\{\max\nolimits_i\sum_{k=1}^{\sqrt{n\log n}}\mathrm{I}_{S_i^n}(X_k)\le 2\right\}, \tag{2}$$

$$\text{(ii)} \quad \lim_{n\to\infty} Pr\{\max\nolimits_i\sum_{k=1}^{\sqrt{n\log n}}\mathrm{I}_{S_i^n}(X_k)\le 2\}=1. \tag{3}$$

Proof. The statement (i) is obvious. Let us prove (ii). Let $\widehat{n}\triangleq\sqrt{n\log n}$ and $m\triangleq M_n=\frac{n}{K\log n}$. First we want to show that

$$\Pr\left\{\sum_{k=1}^{\hat{n}}\mathrm{I}_{S_1^n}(X_k)\le 2\right\} = 1+o\left(\frac{\log n}{n}\right), \quad \text{as } n\to\infty. \tag{4}$$

We have

$$\begin{aligned}\Pr\left\{\sum_{k=1}^{\hat{n}}\mathrm{I}_{S_1^n}(X_k)\le 2\right\} &= C_{\widehat{n}}^0\left(\frac{1}{m}\right)^0(1-1/m)^{\widehat{n}}+C_{\widehat{n}}^1\left(\frac{1}{m}\right)^1(1-1/m)^{\widehat{n}-1}+C_{\widehat{n}}^2\left(\frac{1}{m}\right)^2(1-1/m)^{\widehat{n}-2}\\ &\triangleq p_0+p_1+p_2.\end{aligned}$$

We can compute that:

$$\begin{aligned}p_0 &= \exp\left\{n^{1/2}(\log n)^{1/2}\log\left(1-\frac{K\log n}{n}\right)\right\}\\
&= \exp\left\{n^{1/2}(\log n)^{1/2}\left(-\frac{K\log n}{n}-\frac{K^2(\log n)^2}{2n^2}+o\left(\frac{(\log n)^2}{n^2}\right)\right)\right\}\\
&= \exp\left\{-\frac{K(\log n)^{3/2}}{n^{1/2}}-\frac{K^2(\log n)^{5/2}}{2n^{3/2}}+o\left(\frac{\log n}{n}\right)\right\}\\
&= 1-\frac{K(\log n)^{3/2}}{n^{1/2}}-\frac{K^2(\log n)^{5/2}}{2n^{3/2}}+o\left(\frac{\log n}{n}\right)+\frac{1}{2}\left(\frac{K^2(\log n)^3}{n}+o\left(\frac{\log n}{n}\right)\right)+o\left(\frac{\log n}{n}\right)\\
&= 1-\frac{K(\log n)^{3/2}}{n^{1/2}}+\frac{K^2(\log n)^3}{2n}+o\left(\frac{\log n}{n}\right),\\
p_1 &= \sqrt{n\log n}\cdot\frac{K\log n}{n}\exp\left\{\left(\sqrt{n\log n}-1\right)\log\left(1-\frac{K\log n}{n}\right)\right\}\\
&= \frac{K(\log n)^{3/2}}{n^{1/2}}\exp\left\{\left(\sqrt{n\log n}-1\right)\left(-\frac{K\log n}{n}-\frac{K^2(\log n)^2}{2n^2}+o\left(\frac{(\log n)^2}{n^2}\right)\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{K(\log n)^{3/2}}{n^{1/2}} \exp\left\{-\frac{K(\log n)^{3/2}}{n^{1/2}} + \frac{K\log n}{n} + o\left(\frac{\log n}{n}\right)\right\} \\
&= \frac{K(\log n)^{3/2}}{n^{1/2}} \left(1 - \frac{K(\log n)^{3/2}}{n^{1/2}} + \frac{K\log n}{n} + \frac{K^2(\log n)^3}{2n} + o\left(\frac{\log n}{n}\right)\right) \\
&= \frac{K(\log n)^{3/2}}{n^{1/2}} - \frac{K^2(\log n)^3}{n} + o\left(\frac{\log n}{n}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
p_2 &= \frac{\sqrt{n\log n}\,(\sqrt{n\log n}-1)}{2} \cdot \frac{K^2(\log n)^2}{n^2} \exp\left\{\left(\sqrt{n\log n}-2\right)\log\left(1-\frac{K\log n}{n}\right)\right\} \\
&= \frac{n\log n - \sqrt{n\log n}}{2} \cdot \frac{K^2(\log n)^2}{n^2} \exp\left\{\left(\sqrt{n\log n}-2\right)\log\left(1-\frac{K\log n}{n}\right)\right\} \\
&= \frac{K^2(\log n)^3}{2n} \exp\left\{\left(\sqrt{n\log n}-2\right)\log\left(1-\frac{K\log n}{n}\right)\right\} + o\left(\frac{\log n}{n}\right) \\
&= \frac{K^2(\log n)^3}{2n} \exp\left\{\left(\sqrt{n\log n}-2\right)\left(-\frac{K\log n}{n} + o\left(\frac{\log n}{n}\right)\right)\right\} + o\left(\frac{\log n}{n}\right) \\
&= \frac{K^2(\log n)^3}{2n} \exp\left\{-\frac{K(\log n)^{3/2}}{n^{1/2}} + o\left(\frac{(\log n)^{3/2}}{n^{1/2}}\right)\right\} + o\left(\frac{\log n}{n}\right) \\
&= \frac{K^2(\log n)^3}{2n} + o\left(\frac{\log n}{n}\right).
\end{aligned}
$$

Adding them up, we get (4).

Now we have

$$
\begin{aligned}
\Pr\left\{\max_i \sum_{k=1}^{\sqrt{n\log n}} \mathrm{I}_{S_i^n}(X_k) \le 2\right\} &= 1 - \Pr\{\max_i \sum_{k=1}^{\widehat{n}} \mathrm{I}_{S_i^n}(X_k) > 2\} \\
&\ge 1 - \sum_i \Pr\{\sum_{k=1}^{\widehat{n}} \mathrm{I}_{S_i^n}(X_k) > 2\} \\
&= 1 - M_n \cdot \Pr\{\sum_{k=1}^{\widehat{n}} \mathrm{I}_{S_1^n}(X_k) > 2\} \\
&= 1 - \frac{n}{K\log n}\left(1 - \left(1 + o\left(\frac{\log n}{n}\right)\right)\right) \\
&= 1 + o(1), \quad \text{as } n \to \infty. \qquad \square
\end{aligned}
$$

Lemma 3.2.2. $\lim_{n\to\infty} Pr\left\{|\widetilde{M}_n - n| \le \sqrt{n\log n}\right\} = 1.$

Proof. *Since $E\widetilde{M}_n = n$ and $\mathrm{var}(\widetilde{M}_n) = n$, by Chebyshev's inequality,*

$$
Pr\left\{|\widetilde{M}_n - n| > \sqrt{n\log n}\right\} \le \frac{\mathrm{var}(\widetilde{M}_n)}{n\log n} = \frac{n}{n\log n} = \frac{1}{\log n} \longrightarrow 0, \quad \text{as } n \to \infty.
$$

Lemma 3.2.3. *Consider any $K > 0$ and $\mu \in (0,1)$. If $\lim_{n\to\infty}$ $Pr\left\{\max_i|\widetilde{N}_i^n - K\log n| \le \mu K \log n\right\} = 1$, then for any $\mu_1 > \mu$, we have*

$$\lim_{n\to\infty} Pr\{\max_i|N_i^n - K\log n| \le \mu_1 K\log n\} \quad = \quad 1. \tag{5}$$

If $\lim_{n\to\infty}$ Pr $\left\{\max_i|\widetilde{N}_i^n - K\log n| \le \mu K\log n\right\} = 0$, then for any $\mu_2 < \mu$, we have

$$\lim_{n\to\infty} Pr\{\max_i|N_i^n - K\log n| \le \mu_2 K\log n\} \quad = \quad 0. \tag{6}$$

Proof. First we prove (5). By Lemma 3.2.2, we know

$$\begin{aligned}
&\Pr\left\{\max_i|\widetilde{N}_i^n - K\log n| \le \mu K\log n\right\} \\
&= \sum_{j=0}^{\infty} \Pr\left\{\widetilde{M}_n = j\right\} \cdot \Pr\left\{\max_i|\widetilde{N}_i^n - K\log n| \le \mu K\log n|\widetilde{M}_n = j\right\} \\
&= \left(\sum_{|j-n|\le\sqrt{n\log n}} + \sum_{otherwise}\right) \Pr\left\{\widetilde{M}_n = j\right\} \cdot \Pr\left\{\max_i|\widetilde{N}_i^n - K\log n| \le \mu K\log n|\widetilde{M}_n = j\right\} \\
&= \sum_{|j-n|\le\sqrt{n\log n}} \Pr\left\{\widetilde{M}_n = j\right\} \cdot \Pr\left\{\max_i|\sum_{k=1}^{j} \mathrm{I}_{S_i^n}(X_k) - K\log n| \le \mu K\log n\right\} + o(1).
\end{aligned}$$

By Lemma 3.2.1, we know that for any j such that $n + \sqrt{n\log n} \ge j \ge n$, and n large enough,

$$\begin{aligned}
&\Pr\left\{\max_i|\sum_{k=1}^{j} \mathrm{I}_{S_i^n}(X_k) - K\log n| \le \mu K\log n\right\} \\
&= \Pr\left\{\max_i|\sum_{k=1}^{j} \mathrm{I}_{S_i^n}(X_k) - K\log n| \le \mu K\log n;\ \max_i \sum_{k=n+1}^{j} \mathrm{I}_{S_i^n}(X_k) \le 2\right\} \\
&\quad +o(1) \\
&\le \Pr\left\{\max_i|\sum_{k=1}^{n} \mathrm{I}_{S_i^n}(X_k) - K\log n| \le \mu_1 K\log n\right\} + o(1), \quad \text{as } n\to\infty.
\end{aligned}$$

Similarly, for any j such that $n > j \ge n - \sqrt{n\log n}$, and n large enough, we have

$$\begin{aligned}
&\Pr\left\{\max_i|\sum_{k=1}^{j} \mathrm{I}_{S_i^n}(X_k) - K\log n| \le \mu K\log n\right\} \\
&\le \Pr\left\{\max_i|\sum_{k=1}^{n} \mathrm{I}_{S_i^n}(X_k) - K\log n| \le \mu_1 K\log n\right\} + o(1), \quad \text{as } n\to\infty.
\end{aligned}$$

So we get

$$\Pr\left\{\max_i|\widetilde{N}_i^n - K\log n| \le \mu K\log n\right\}$$

$$
\begin{aligned}
&\leq \sum_{|j-n|\leq\sqrt{n\log n}} \Pr\left\{\widetilde{M}_n = j\right\} \cdot \left(\Pr\left\{\max_i |\sum_{k=1}^{n} \mathrm{I}_{S_i^n}(X_k) - K\log n| \leq \mu_1 K \log n\right\} + o(1)\right) + o(1) \\
&= \Pr\left\{\max_i |\widetilde{M}_n - n| \leq \sqrt{n\log n}\right\} \cdot (\Pr\{\max_i |N_i^n - K\log n| \leq \mu_1 K\log n\} + o(1)) + o(1) \\
&= (1+o(1)) \cdot (\Pr\{\max_i |N_i^n - K\log n| \leq \mu_1 K \log n\} + o(1)) + o(1).
\end{aligned}
$$

Since $\lim_{n\to\infty} \Pr\left\{\max_i |\tilde{N}_i^n - K\log n| \leq \mu K \log n\right\} = 1$, we deduce

$$
\lim_{n\to\infty} \Pr\{\max_i |N_i^n - K\log n| \leq \mu_1 K \log n\} \quad = \quad 1.
$$

The proof of (6) is similar and so we omit it. □

Lemma 3.2.4. *For $\mu \in \mathcal{R}^+$, define*

$$
\begin{aligned}
\psi_\alpha(\mu) &\triangleq \mu + (1-\mu)\log(1-\mu), \\
\psi_\beta(\mu) &\triangleq -\mu + (1+\mu)\log(1+\mu).
\end{aligned}
$$

Then

(i) $\psi_\alpha(\mu) > \psi_\beta(\mu) > 0$, $\forall \mu \in (0,1)$.

(ii) *For any $K > \frac{1}{\log(4/e)}$, there is one and only one root μ^* of the equation $\psi_\beta(\mu) = \frac{1}{K}$. Also $\psi_\beta(\mu) > \psi_\beta(\mu^*)$, $\forall \mu \in (\mu^*, 1)$.*

Proof.

(i) Let $\psi(\mu) \triangleq \psi_\alpha(\mu) - \psi_\beta(\mu)$. Then

$$
\psi'(\mu) \quad = \quad 1 - \log(1-\mu) - 1 - (-1 + \log(1+\mu) + 1) \quad = \quad -\log(1-\mu^2) > 0, \quad \forall \mu \in (0,1).
$$

Also $\lim_{\mu\to 0^+} \psi(\mu) = 0$. So we know $\psi_\alpha(\mu) > \psi_\beta(\mu) > 0, \forall \mu \in (0,1)$.

(ii) We have $\psi_\beta'(\mu) = -1 + \log(1+\mu) + 1 = \log(1+\mu) > 0$. Also $\lim_{\mu\to 0^+} \psi_\beta(\mu) = 0, \lim_{\mu\to 1^-} \psi_\beta(\mu) = \log\left(\frac{4}{e}\right)$. So $\psi_\beta(\mu)$ is strictly increasing on $(0,1)$ and its range is $\left(0, \log\left(\frac{4}{e}\right)\right)$. □

Lemma 3.2.5. *Suppose Y is a Poisson random variable with parameter λ, then for any $K > 0$ and $\mu \in (0,1)$, we have*

$$
\text{(i)} \quad P_1 \triangleq Pr\{Y - \lambda \leq \mu\lambda\} = \frac{e^{-\lambda}\cdot\lambda^{(1-\mu)\lambda}}{((1-\mu)\lambda)!} \cdot \frac{1}{\mu} \cdot (1+o(1)), \text{ as } \lambda\to\infty \tag{7}
$$

$$
\text{(ii)} \quad P_2 \triangleq Pr\{Y - \lambda \geq -\mu\lambda\} = \frac{e^{-\lambda}\cdot\lambda^{(1+\mu)\lambda}}{((1+\mu)\lambda)!} \cdot \left(1 + \frac{1}{\mu}\right) \cdot (1+o(1)), \text{ as } \lambda\to\infty \tag{8}
$$

$$
\text{(iii)} \quad \lim_{\lambda\to+\infty} \frac{1}{\lambda} e^{\frac{1}{K}\lambda} \cdot Pr\{|Y-\lambda| \geq \mu\lambda\} = 0, \text{ if } K > 1/\log(4/e) \text{ and } \mu \in (\mu^*, 1), \tag{9}
$$

$$
\text{where } \mu^* \text{ is the root of } -\mu^* + (1+\mu^*)\log(1+\mu^*) = \frac{1}{K}.
$$

Proof. (i) By the definition of a Poisson random variable, we have

$$\begin{aligned}
P_1 &= \sum_{0\le k\le(1-\mu)\lambda} \frac{\lambda^k}{k!}e^{-\lambda} \\
&= e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2}+\cdots+\frac{\lambda^{(1-\mu)\lambda}}{((1-\mu)\lambda)!}\right) \\
&= e^{-\lambda}\frac{\lambda^{(1-\mu)\lambda}}{((1-\mu)\lambda)!}\left(1+\frac{(1-\mu)\lambda}{\lambda}+\frac{((1-\mu)\lambda)((1-\mu)\lambda-1)}{\lambda^2}+\cdots+\frac{((1-\mu)\lambda)!}{\lambda^{(1-\mu)\lambda}}\right) \\
&= e^{-\lambda}\frac{\lambda^{(1-\mu)\lambda}}{((1-\mu)\lambda)!}\left(1+(1-\mu)+(1-\mu)\left(1-\mu-\frac{1}{\lambda}\right)+\cdots+\Pi_{j=0}^{(1-\mu)\lambda-1}\left(1-\mu-\frac{j}{\lambda}\right)\right) \\
&\triangleq e^{-\lambda}\frac{\lambda^{(1-\mu)\lambda}}{((1-\mu)\lambda)!}\cdot M_1(\lambda).
\end{aligned}$$

Since $\sum_{k=0}^{\infty}(1-\mu)^k=\frac{1}{\mu}$, for any $\epsilon>0$, there is $N_0=N_0(\epsilon)>0$ such that

$$\left|\sum_{k=0}^{N_0}(1-\mu)^k-\frac{1}{\mu}\right| \le \frac{\epsilon}{3}, \text{ and}$$

$$\sum_{k=N_0+1}^{\infty}(1-\mu)^k \le \frac{\epsilon}{3}.$$

One can also choose $\lambda_0=\lambda_0(N_0,\epsilon)>0$ large enough such that

$$\left|1+\sum_{k=0}^{N_0-1}\Pi_{j=0}^{k}\left(1-\mu-\frac{j}{\lambda}\right)-\sum_{k=0}^{N_0}(1-\mu)^k\right| \le \frac{\epsilon}{3}, \quad \forall\lambda>\lambda_0.$$

We also have for λ satisfying $(1-\mu)\lambda>N_0$,

$$\sum_{k=N_0}^{(1-\mu)\lambda-1}\Pi_{j=0}^{k}\left(1-\mu-\frac{j}{\lambda}\right) \le \sum_{k=N_0+1}^{\infty}(1-\mu)^k\le\frac{\epsilon}{3}, \quad \forall\lambda>\lambda_0.$$

Hence, for any λ satisfying $\lambda>\lambda_0$ and $(1-\mu)\lambda>N_0$,

$$\begin{aligned}
\left|M_1(\lambda)-\frac{1}{\mu}\right| &\le \left|1+\sum_{k=0}^{N_0-1}\Pi_{j=0}^{k}\left(1-\mu-\frac{j}{\lambda}\right)-\sum_{k=0}^{N_0}(1-\mu)^k\right|+ \\
&\quad \left|\sum_{k=0}^{N_0}(1-\mu)^k-\frac{1}{\mu}\right|+\left|\sum_{k=N_0}^{(1-\mu)\lambda-1}\Pi_{j=0}^{k}(1-\mu-\frac{j}{\lambda})\right| \\
&\le \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\le\epsilon.
\end{aligned}$$

So $M_1(\lambda)\to\frac{1}{\mu}$ as $\lambda\to\infty$. Hence

$$P_1 = e^{-\lambda}\frac{\lambda^{(1-\mu)\lambda}}{((1-\mu)\lambda)!}\cdot\frac{1}{\mu}\cdot(1+o(1)), \quad \text{as } \lambda\to\infty.$$

(ii) Similarly we have

$$\begin{aligned}
P_2 &= \sum_{k \geq (1+\mu)\lambda} \frac{\lambda^k}{k!} e^{-\lambda} \\
&= e^{-\lambda} \left(\frac{\lambda^{(1+\mu)\lambda}}{((1+\mu)\lambda)!} + \frac{\lambda^{(1+\mu)\lambda+1}}{((1+\mu)\lambda+1)!} + \cdots \right) \\
&= e^{-\lambda} \frac{\lambda^{(1+\mu)\lambda}}{((1+\mu)\lambda)!} \left(1 + \frac{\lambda}{(1+\mu)\lambda+1} + \frac{\lambda^2}{((1+\mu)\lambda+2)((1+\mu)\lambda+1)} + \cdots \right) \\
&= e^{-\lambda} \frac{\lambda^{(1+\mu)\lambda}}{((1+\mu)\lambda)!} \left(1 + \sum_{k=1}^{\infty} \frac{1}{\Pi_{j=1}^{k} \left((1+\mu) + \frac{j}{\lambda} \right)} \right) \\
&\triangleq e^{-\lambda} \frac{\lambda^{(1+\mu)\lambda}}{((1+\mu)\lambda)!} \cdot M_2(\lambda).
\end{aligned}$$

Since $1 + \sum_{k=1}^{\infty} \frac{1}{(1+\mu)^k} = 1 + \frac{1}{\mu}$, for any $\epsilon > 0$, there is $N_0 = N_0(\epsilon) > 0$ such that

$$\left| 1 + \sum_{k=1}^{N_0} \frac{1}{(1+\mu)^k} - \left(1 + \frac{1}{\mu} \right) \right| \leq \frac{\epsilon}{3}, \text{ and}$$

$$\sum_{k=N_0+1}^{\infty} \frac{1}{(1+\mu)^k} \leq \frac{\epsilon}{3}, \quad \forall \lambda > \lambda_0.$$

Also one can choose $\lambda_0 = \lambda_0(N_0, \epsilon) > 0$ large enough such that

$$\left| 1 + \sum_{k=1}^{N_0} \frac{1}{\Pi_{j=1}^{k} \left((1+\mu) + \frac{j}{\lambda} \right)} - \left(1 + \sum_{k=1}^{N_0} \frac{1}{(1+\mu)^k} \right) \right| \leq \frac{\epsilon}{3}, \quad \forall \lambda > \lambda_0.$$

We also have

$$\sum_{k=N_0+1}^{\infty} \frac{1}{\Pi_{j=1}^{k} \left((1+\mu) + \frac{j}{\lambda} \right)} \leq \sum_{k=N_0+1}^{\infty} \frac{1}{(1+\mu)^k} \leq \frac{\epsilon}{3}, \quad \forall \lambda > \lambda_0.$$

Hence, for any $\lambda > \lambda_0$, we have

$$\begin{aligned}
\left| M_2(\lambda) - \left(1 + \frac{1}{\mu} \right) \right| &\leq \left| 1 + \sum_{k=1}^{N_0} \frac{1}{\Pi_{j=1}^{k} \left((1+\mu) + \frac{j}{\lambda} \right)} - \left(1 + \sum_{k=1}^{N_0} \frac{1}{(1+\mu)^k} \right) \right| \\
&\quad + \left| 1 + \sum_{k=1}^{N_0} \frac{1}{(1+\mu)^k} - \left(1 + \frac{1}{\mu} \right) \right| + \left| \sum_{k=N_0+1}^{\infty} \frac{1}{\Pi_{j=1}^{k} \left((1+\mu) + \frac{j}{\lambda} \right)} \right| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \leq \epsilon.
\end{aligned}$$

This means that $M_2(\lambda) \to 1 + \frac{1}{\mu}$ as $\lambda \to \infty$. So we know

$$P_2 = e^{-\lambda} \frac{\lambda^{(1+\mu)\lambda}}{((1+\mu)\lambda)!} \cdot \left(1 + \frac{1}{\mu} \right) \cdot (1 + o(1)), \quad \text{as } \lambda \to \infty.$$

(iii) By Stirling's formula, we know

$$\begin{aligned}
\frac{1}{\lambda}e^{\frac{1}{K}\lambda}\cdot P_1 &= \frac{1}{\lambda}e^{(\frac{1}{K}-1)\lambda}\cdot\frac{\lambda^{(1-\mu)\lambda}}{((1-\mu)\lambda)!}\cdot\frac{1}{\mu}\cdot(1+o(1))\\
&= \frac{1}{\lambda}e^{(\frac{1}{K}-1)\lambda}\cdot\frac{\lambda^{(1-\mu)\lambda}}{\sqrt{2\pi(1-\mu)\lambda}\left(\frac{(1-\mu)\lambda}{e}\right)^{(1-\mu)\lambda}}\cdot\frac{1}{\mu}\cdot(1+o(1))\\
&= \frac{1}{\lambda\sqrt{2\pi(1-\mu)\lambda}}\cdot\exp\left\{\left(\frac{1}{K}-1+1-\mu\right)\lambda-(1-\mu)\lambda\log(1-\mu)\right\}\cdot\frac{1}{\mu}\cdot(1+o(1))\\
&= \frac{1}{\lambda\sqrt{2\pi(1-\mu)\lambda}}\cdot\exp\left\{\left(\frac{1}{K}-(\mu+(1-\mu)\log(1-\mu))\right)\cdot\lambda\right\}\cdot\frac{1}{\mu}\cdot(1+o(1))\\
&= \frac{1}{\lambda\sqrt{2\pi(1-\mu)\lambda}}\cdot\exp\left\{\left(\frac{1}{K}-\psi_\alpha(\mu)\right)\cdot\lambda\right\}\cdot\frac{1}{\mu}\cdot(1+o(1)),
\end{aligned}$$

where $\psi_\alpha(\mu)$ is as defined in Lemma 3.2.4.

Also by Stirling's formula, we can compute that

$$\begin{aligned}
\frac{1}{\lambda}e^{\frac{1}{K}\lambda}\cdot P_2 &= \frac{1}{\lambda\sqrt{2\pi(1+\mu)\lambda}}\cdot\exp\left\{\left[\frac{1}{K}-(-\mu+(1+\mu)\log(1+\mu))\right]\cdot\lambda\right\}\cdot(1+\frac{1}{\mu})(1+o(1)),\\
&= \frac{1}{\lambda\sqrt{2\pi(1+\mu)\lambda}}\cdot\exp\left\{\left[\frac{1}{K}-\psi_\beta(\mu)\right]\cdot\lambda\right\}\cdot(1+\frac{1}{\mu})(1+o(1)),
\end{aligned}$$

where $\psi_\beta(\mu)$ is as defined in Lemma 3.2.4.

Now, by Lemma 3.2.4, for K,μ satisfying (9), we know that

$$\frac{1}{K}-\psi_\alpha(\mu)<\frac{1}{K}-\psi_\beta(\mu)<0.$$

Hence we have,

$$\frac{1}{\lambda}e^{\frac{1}{K}\lambda}\cdot\Pr\{|Y-\lambda|\geq\mu\lambda\}=\frac{1}{\lambda}e^{\frac{1}{K}\lambda}\cdot(P_1+P_2)\longrightarrow 0,\ \text{as }\lambda\to\infty. \qquad \Box$$

Proof of Lemma 3.1. According to Lemma 3.2.3, if we can show that for any $K>1/\log(4/e)$,

$$\lim_{n\to\infty}\Pr\left\{\max_i|\tilde{N}_i^n-K\log n|\leq\mu K\log n\right\}=1,\quad\forall\mu\in(\mu^*,1),$$

where $\mu^*\in(0,1)$ is the sole root of the equation $-\mu^*+(1+\mu^*)\log(1+\mu^*)=\frac{1}{K}$, then Lemma 3.1 is true.

Recall $M_n\triangleq\frac{n}{K\log n}$. By invoking the independence property of the Poisson process for the random variables $\tilde{N}_1^n,\tilde{N}_2^n,\ldots,\tilde{N}^n_{\frac{n}{K\log n}}$, we have

$$\begin{aligned}
\Pr\left\{\max_{1\leq i\leq M_n}\left|\tilde{N}_i^n-K\log n\right|\leq\mu K\log n\right\} &= \Pi_{i=1}^{M_n}\Pr\left\{\left|\tilde{N}_i^n-K\log n\right|\leq\mu K\log n\right\}\\
&= \left(\Pr\left\{\left|\tilde{N}_1^n-K\log n\right|\leq\mu K\log n\right\}\right)^{M_n}\\
&= \left(1-\Pr\left\{\left|\tilde{N}_1^n-K\log n\right|>\mu K\log n\right\}\right)^{\frac{n}{K\log n}}\\
&= \exp\left\{\frac{n}{K\log n}\cdot\log\left(1-\Pr\left\{\left|\tilde{N}_1^n-K\log n\right|>\mu K\log n\right\}\right)\right\}.
\end{aligned}$$

If we let $\rho_n \triangleq K \log n$, which is the mean value of $\tilde{N}_1^n$, then

$$\Pr\left\{\max_i \left|\tilde{N}_i^n - K\log n\right| \le \mu K \log n\right\} = \exp\left\{\frac{e^{\frac{\rho_n}{K}}}{\rho_n} \cdot \log\left(1 - \Pr\left\{\left|\tilde{N}_1^n - \rho_n\right| > \mu\rho_n\right\}\right)\right\}.$$

Since by Chebyshev's inequality,

$$\Pr\left\{\left|\tilde{N}_1^n - \rho_n\right| > \mu\rho_n\right\} \le \frac{\mathrm{var}(\tilde{N}_1^n)}{(\mu\rho_n)^2} = \frac{\rho_n}{(\mu\rho_n)^2} = \frac{1}{\mu^2\rho_n} \longrightarrow 0, \quad n \to \infty,$$

we have

$$\Pr\left\{\max_i \left|\tilde{N}_i^n - K\log n\right| \le \mu K \log n\right\} = \exp\left\{-\frac{e^{\frac{\rho_n}{K}}}{\rho_n} \cdot \Pr\left\{\left|\tilde{N}_1^n - \rho_n\right| > \mu\rho_n\right\} \cdot (1 + o(1))\right\}.$$

Hence, by Lemma 3.2.5 (iii), we deduce that

$$\Pr\left\{\max_i |\tilde{N}_i^n - K\log n| \le \mu K \log n\right\} \longrightarrow 1, \text{ as } n \to \infty. \qquad \square$$

3.3 Proof of the necessity part of Theorem 1

Suppose we connect each node in $\mathcal{G}_n$ to its $\epsilon \log n$ nearest neighbors. Denote the resulting graph by $\mathcal{G}(n, \epsilon\log n)$. Then it suffices to show that for some $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr\{\mathcal{G}(n, \epsilon\log n) \text{ is connected}\} = 0.$$

Let us tessellate $\mathcal{G}(n, \epsilon\log n)$ by $\mathcal{T}_S^n$, with K, μ satisfying Lemma 3.1. Put a trap of size $\sqrt{a}\ l_n$ at the center of each square of $\mathcal{T}_S^n$, where $a \in (0,1)$ is a parameter and l_n is the size of the squares S_i^n. See Figure 4.

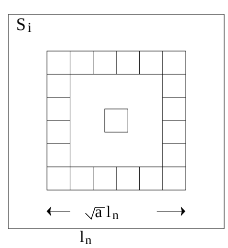

Figure 4: A trap in a small square.

According to Lemma 3.1,

$$\lim_{n\to\infty} \Pr\{\max_i |N_i^n - K\log n| \le \mu K \log n\} = 1. \tag{10}$$

So, if we let

$$A_i^n \triangleq \{\text{ No } (\epsilon\log n + 1)\text{-filling event occurs in the trap of } S_i^n\},$$
$$Q^n \triangleq \{(k_1, k_2, \cdots, k_{M_n}): \quad k_1 + k_2 + \cdots + k_{M_n} = n, \text{ and } k_i \ge 0, \ \forall i\},$$

then we have

$$
\begin{aligned}
&\Pr\{\mathcal{G}(n,\epsilon\log n) \text{ is connected }\}\\
&\quad \le \Pr\{A_i^n,\forall i\}\\
&\quad = \sum_{(k_1,k_2,\cdots,k_{M_n})\in Q^n} \Pr\{A_i^n,\forall i;\ N_i^n=k_i,\ \forall i\}\\
&\quad = \sum_{(k_1,k_2,\cdots,k_{M_n})\in Q^n} \Pr\{A_i^n,\ \forall i|N_i^n=k_i,\ \forall i\}\cdot\Pr\{N_i^n=k_i,\ \forall i\}\\
&\quad = \left(\sum_{\max_i|k_i-K\log n|\le\mu K\log n} + \sum_{Otherwise}\right)\Pr\{A_i^n,\ \forall i|N_i^n=k_i,\ \forall i\}\cdot\Pr\{N_i^n=k_i,\ \forall i\}\\
&\quad \le \sum_{\max_i|k_i-K\log n|\le\mu K\log n}\left(\Pi_{i=1}^{M_n}\Pr\{A_i^n|N_i^n=k_i\}\right)\cdot\Pr\{N_i^n=k_i,\forall i\}\\
&\qquad\quad + \sum_{Otherwise} 1\cdot\Pr\{N_i^n=k_i,\ \forall i\}\\
&\quad = \sum_{\max_i|k_i-K\log n|\le\mu K\log n}\left(\Pi_{i=1}^{M_n}\ \Pr\{A_i^n|N_i^n=k_i\}\right)\cdot\ \Pr\{N_i^n=k_i,\ \forall i\}+\ o(1), \qquad (11)
\end{aligned}
$$

where (11) comes from Lemma 3.1.

Now, suppose $N_i^n = k_i$ where $k_i \in [(1-\mu)K\log n, (1+\mu)K\log n]$. Note that actually k_i i.i.d. nodes are thrown into S_i^n; see Figure 4. For (K,μ) with $(1-\mu)K > 21\epsilon$, we have

$$
\begin{aligned}
&\Pr\{A_i^n|N_i^n=k_i\}\\
&\quad = 1-\ \Pr\ \{\text{An } (\epsilon\log n+1)\text{-filling event occurs in the trap of } S_i^n \mid N_i^n=k_i\}\\
&\quad \le 1-\binom{k_i}{\epsilon\log n+1,\ \cdots,\ \epsilon\log n+1,\ k_i-21(\epsilon\log n+1)}\cdot\left(\frac{a}{36}\right)^{21(\epsilon\log n+1)}\cdot(1-a)^{k_i-21(\epsilon\log n+1)}\\
&\quad \le 1-\binom{(1-\mu)K\log n}{\epsilon\log n+1,\ \cdots,\ \epsilon\log n+1,\ (1-\mu)K\log n-21(\epsilon\log n+1)}\cdot\left(\frac{a}{36}\right)^{21(\epsilon\log n+1)}\cdot(1-a)^{k_i-21(\epsilon\log n+1)}.
\end{aligned}
$$

By Stirling's formula, we know

$$
\begin{aligned}
&\binom{(1-\mu)K\log n}{\epsilon\log n+1,\ \cdots,\ \epsilon\log n+1,\ (1-\mu)K\log n-21(\epsilon\log n+1)}\\
&\quad = \ e^{O(\log\log n)}\cdot\frac{((1-\mu)K\log n)!}{(\epsilon\log n!)^{21}\ (((1-\mu)K-21\epsilon)\log n)!}\\
&\quad = \ e^{O(\log\log n)}\frac{\left(\frac{(1-\mu)K\log n}{e}\right)^{(1-\mu)K\log n}}{\left(\frac{\epsilon\log n}{e}\right)^{21\epsilon\log n}\left(\frac{((1-\mu)K-21\epsilon)\log n}{e}\right)^{((1-\mu)K-21\epsilon)\log n}}\\
&\quad = \ \exp\{O(\log\log n)+((1-\mu)K\log((1-\mu)K)-21\epsilon\log\epsilon-((1-\mu)K-21\epsilon)\log((1-\mu)K-21\epsilon))\cdot\log n\}\\
&\quad \triangleq \ \exp\left\{O(\log\log n)+\widehat{\phi}(\epsilon,\mu,K)\cdot\log n\right\}.
\end{aligned}
$$

Hence, we have,

$$
\Pr\{A_i^n|N_i^n=k_i\} \quad\le\quad 1-\exp\left\{O(\log\log n)+\widehat{\phi}(\epsilon,\mu,K)\cdot\log n\right\}\cdot(\frac{a}{36})^{21(\epsilon\log n+1)}\cdot(1-a)^{k_i-21(\epsilon\log n+1)}
$$

$$\begin{aligned}
&= 1-\exp\left\{o(\log n)+\widehat{\phi}(\epsilon,\mu,K)\cdot\log n+21\epsilon\log n\cdot\log\frac{a}{36}+(k_i-21\epsilon\log n)\cdot\log(1-a)\right\}\\
&= 1-\exp\left\{o(\log n)+\widehat{\phi}(\epsilon,\mu,K)\cdot\log n+21\epsilon\log\frac{a}{36}\log n+((1+\mu)K-21\epsilon)\log n\cdot\log(1-a)\right\}\\
&= 1-\exp\left\{\psi(\epsilon,a,K,\mu)\cdot\log n\cdot(1+o(1))\right\},
\end{aligned}$$

where we define

$$\begin{aligned}
\psi(\epsilon,a,K,\mu) \triangleq\ & (1-\mu)K\log((1-\mu)K)-21\epsilon\log\epsilon-((1-\mu)K-21\epsilon)\log((1-\mu)K-21\epsilon)\\
&+21\epsilon\log\frac{a}{36}+((1+\mu)K-21\epsilon)\log(1-a).
\end{aligned}$$

Let us define the set

$$\begin{aligned}
\mathcal{D}_{\epsilon,a,K,\mu} \triangleq\ & \{(\epsilon,a,K,\mu)\in\mathcal{R}^4:\ -1<\psi(\epsilon,a,K,\mu)<0,(1-\mu)K>21\epsilon>0,a\in(0,1),\\
& \text{and } (K,\mu) \text{ satisfy Lemma 3.1.}\}
\end{aligned}$$

It is easy to verify that $\mathcal{D}_{\epsilon,a,K,\mu}\neq\phi$. Note that one set of feasible choices is $\epsilon=0.074$, $K=1973.9$, $\mu=0.032$ and $a=0.001$, whence we obtain $\psi=-0.9996$.

For any $(\epsilon,a,K,\mu)\in\mathcal{D}_{\epsilon,a,K,\mu}$, there are constants $N_{\epsilon,a,K,\mu}>0$, and $\delta\in(0,1)$, such that

$$\Pr\{A_i^n|N_i^n=k_i\} \leq 1-n^{-\delta}, \quad \forall(k_i,n):|k_i-K\log n|\leq\mu K\log n, n\geq N_{\epsilon,a,K,\mu}.$$

Hence, by Remark 2, (11) and Lemma 3.1, for any $(\epsilon,a,K,\mu)\in\mathcal{D}_{\epsilon,a,K,\mu}$,

$$\begin{aligned}
\Pr\{\mathcal{G}(n,\epsilon\log n)\text{is connected}\} &\leq \sum_{\max_i|k_i-K\log n|\leq\mu K\log n}(1-n^{-\delta})^{M_n}\Pr\{N_i^n=k_i,\ \forall i.\}+o(1)\\
&= (1-n^{-\delta})^{\frac{n}{K\log n}(1+o(1))}\cdot\Pr\{\max_i|N_i^n-K\log n|\leq\mu K\log n\}+o(1)\\
&\leq (1-n^{-\delta})^{\frac{n}{K\log n}(1+o(1))}\cdot 1+o(1)\\
&= \left(1-\frac{1}{n^\delta}\right)^{-n^\delta\cdot\frac{-n^{1-\delta}}{K\log n}(1+o(1))}+o(1)\\
&= o(1), \quad \text{as } n\to\infty.
\end{aligned}$$

□

4 $\Theta(\log n)$ neighbors are sufficient for connectivity

In this section we prove the sufficiency part of Theorem 1.

Definition 4.1. *Disk tessellation $\mathcal{T}_D^n(a,b)$. Suppose the unit square is located with corner at the origin, as in Figure 5. Let r be such that $\pi r^2=\frac{K\log n}{n}$, where $K>0$ is tunable parameter. We consider a grid of squares of size $2r$, with corners at $(a \bmod 2r, b \bmod 2r)$ as in Figure 5. Inside each square, we inscribe a disk of area $\frac{K\log n}{n}$. We call the set of all the disks intersecting the unit square as the Disk Tessellation $\mathcal{T}_D^n(a,b)$ (with a minor abuse of the term "tessellation").*

Similarly to the square tessellation, we name the small disks intersecting the unit square as D_i^n, for $i\leq M_n$. Let the number of the n nodes that fall into disk S_i^n be denoted as N_i^n, which is a random variable.

The proof of the following lemma is very similar to the proof of Lemma 3.1, in the sense that we can just replace the small squares there by small disks here. So we omit the proof.

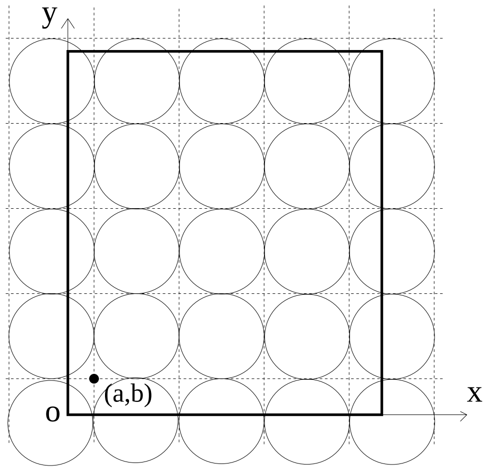

Figure 5: The disk tessellation $\mathcal{T}_D^n(a,b)$.

Lemma 4.1. *For any $K > 1/\log(4/e)$ and any point sequence $\{(a_n, b_n) \in \mathcal{R}^2,\ n = 1, 2, \cdots\}$,*

$$\lim_{n\to\infty} Pr\{N_i^n \le (1+\mu)K\log n, \text{for any disk } D_i^n \text{ in tessellation } \mathcal{T}_D^n(a_n,b_n)\} = 1, \quad \forall \mu \in (\mu^*, 1),$$

where μ^* is the sole root of

$$-\mu^* + (1+\mu^*)\log(1+\mu^*) \quad = \quad \frac{1}{K}. \tag{12}$$

Proof of the sufficiency part of Theorem 1. We want to prove that for any $\delta > 0$, $\mathcal{G}(n, (2/\log(4/e) + \delta)\log n)$ will be connected in the sense of Theorem 1.

According to (12), $\mu^* \to 1$ as $K \to (1/\log(4/e))^+$. So for any $\delta > 0$, there is a constant $\delta' > 0$ such that

$$K = 1/\log(4/e) + \delta' \quad \Longrightarrow \quad (1+\mu^*)K < 2/\log(4/e) + \delta. \tag{13}$$

From now on, we fix the parameter K in the Disk tessellation to be the one in (13), and fix μ such that

$$1 > \mu > \mu^* \quad \text{and} \quad (1+\mu)K < 2/\log(4/e) + \delta.$$

Let $r_n \triangleq \sqrt{\frac{K\log n}{\pi n}}$, the radius of the disks in the Disk tessellation. Then choose two positive constants $\epsilon, \eta \in (0,1)$ such that

$$\pi(r_n - \epsilon r_n)^2 \quad > \quad \frac{(1+\eta)\log n}{n}. \tag{14}$$

Now let us tessellate the unit square by a collection of several disk tessellations:

$$\mathcal{T}_\epsilon^n \quad \triangleq \quad \left\{\mathcal{T}_D^n(i\cdot\epsilon r_n, j\cdot\epsilon r_n), i,j = 0,1,2,\cdots,2\cdot\left[\frac{1}{\epsilon}\right]+1\right\}.$$

This collection of tessellations has the following property:

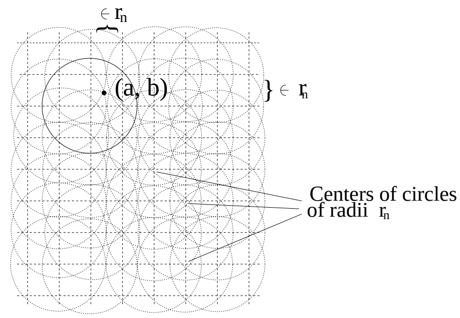

Figure 6: Tessellation collection $\mathcal{T}_\epsilon^n$.

- For any point (a, b) in the unit square, there is a disk in $\mathcal{T}_\epsilon^n$ whose center is within a distance of ϵr_n from the point; see Figure 6.

Since the number of tessellations in $\mathcal{T}_\epsilon^n$ is finite, by Lemma 4.1, we know that

$$\Pr \{ \text{ Every disk of } \mathcal{T}_\epsilon^n \text{ contains no more than } (2/\log(4/e)+\delta)\log n \text{ nodes}\} \to 1, \text{ as } n \to \infty.$$

By the choice of r_n, ϵ and T_ϵ^n, any disk with radius $(1-\epsilon)r_n$ and with its center in the unit square, will be contained in a disk in the collection of tessellations $\mathcal{T}_\epsilon^n$; see Figure 6. So if any of the disks of the tessellation collection $\mathcal{T}_\epsilon^n$ contains no more than $(2/\log(4/e)+\delta)\log n$ nodes of $\mathcal{G}_n$, then each node of $\mathcal{G}(n, (2/\log(4/e)+\delta)\log n)$ will be connected to every node that is within distance of $(1-\epsilon)r_n$. To complete the proof, we need the following theorem on connectivity based on distance.

Theorem 4.1 (Theorem 3.2 in [10]). *(Let $G(n, r(n))$ be the graph formed by connecting every node to its neighbors that are within distance $r(n)$.) Then, graph $G(n, r(n))$, with $\pi r(n)^2 = \frac{\log n + c(n)}{n}$, is connected with probability one as $n \to \infty$ if and only if $c(n) \to \infty$.*

Now, by (14), and the above Theorem 4.1, we know $\mathcal{G}(n, (2/\log(4/e)+\delta)\log n)$ will be connected with high probability.

So if we define $B_n \triangleq \{$ Every disk of $\mathcal{T}_\epsilon^n$ contains no more than $(2/\log(4/e)+\delta)\log n$ nodes $\}$, then

$$\Pr\{\mathcal{G}(n, (2/\log(4/e)+\delta)\log n) \text{ is connected } | B_n\} \longrightarrow 1, \quad \text{as } n \to \infty.$$

Then,

$$\begin{aligned} & \Pr\ \{\mathcal{G}(n, (2/\log(4/e)+\delta)\log n) \text{ is connected }\} \\ = \ & \Pr\{B_n\} \cdot \Pr\{\mathcal{G}(n, (2/\log(4/e)+\delta)\log n) \text{ is connected } |B_n\} \\ & + \Pr\{B_n^c\} \cdot \Pr\{\mathcal{G}(n, (2/\log(4/e)+\delta)\log n) \text{is connected } |B_n^c\} \\ = \ & (1+o(1)) \cdot (1+o(1)) + o(1) \to 1, \quad \text{as } n \to \infty. \end{aligned}$$

Hence we have proved the result. □

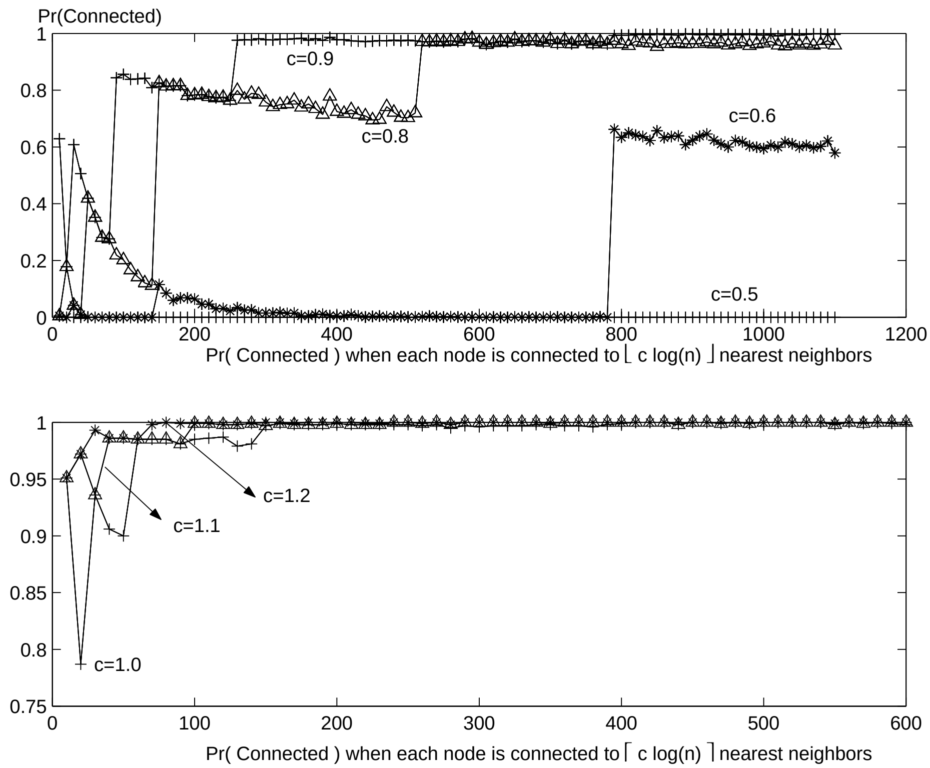

Figure 7: Simulation results for connection based on $c \log n$ nearest neighbors.

5 A simulation study

The theoretical results of this paper are asymptotic in nature. The question naturally arises as to how rapidly the probability of connectedness approaches one as n is increased. Our current theory is inadequate for this task.

To investigate this we have conducted a simulation study. In each run, n nodes were placed randomly (uniformly iid) in a unit square. For various values of c, it was determined whether the network was connected when each node was connected to its $c \log n$ nearest neighbors. By repeating the simulation 1000 times, the empirical probability that the network is connected is obtained. These empirical probabilities were plotted for various values of n; see Figure 7.

The results suggest that the critical value of c is near one. The jumps in the plots arise because $c \log n$ is generally not an integer. What we display is the probability of connectedness for $\lfloor c \log n \rfloor$ nearest neighbors when $c < 1$, and for $\lceil c \log n \rceil$ nearest neighbors when $c > 1$.

It is an interesting question whether the critical value of c is one. In any case we see that for c larger than about 1.5, the probability of connectedness increases rapidly to one for a modest number of nodes (e.g., $n \approx 30$).

We also present for comparison purposes simulation results for connectivity based on distance; see Figure 8. Shown are the plots when nodes are connected to all others within a distance $\sqrt{\frac{c \log n}{\pi n}}$. In this case the critical c is known to be one (see [10]), and the plots clearly illustrate this.

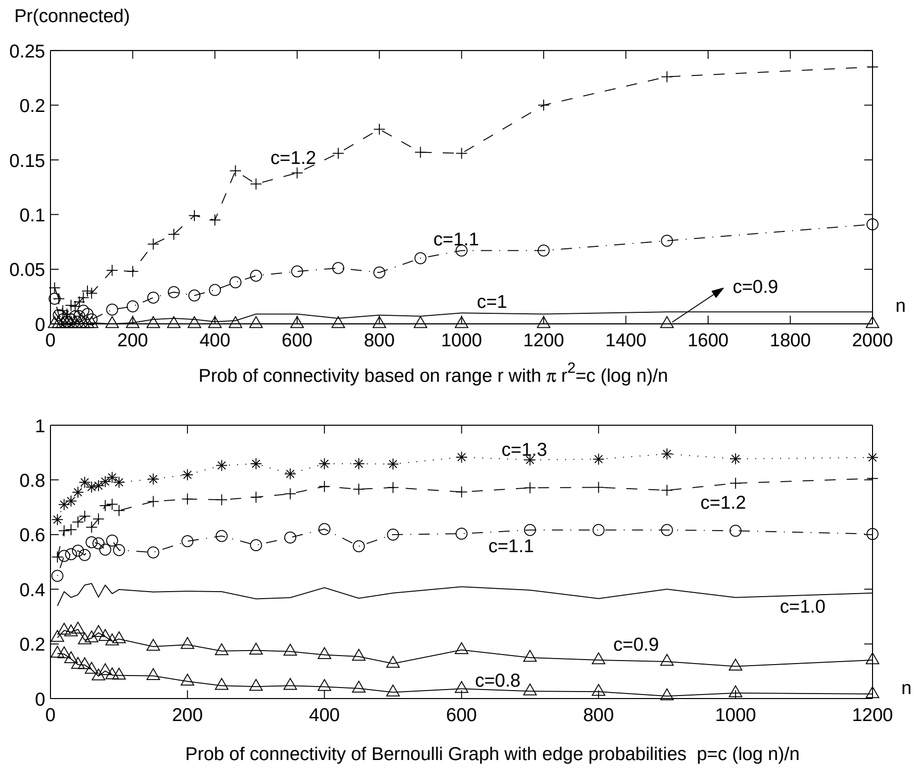

Figure 8: Simulation results for connection based on distance and for Bernoulli graphs.

To illustrate another instance of a problem with a critical value of the preconstant determining connectivity, we exhibit the results for Bernoulli random graphs; see Figure 8. Here an edge is drawn between a pair of nodes with probability $\frac{c \log n}{n}$. Again it is known that the critical value of c is one (see [12]), and the plots illustrate this too. Note however that Bernoulli random graphs do not model wireless networks.

6 Concluding remarks

As the number of nodes participating in a wireless network increases, the number of nearest neighbors each is connected to should not remain constant. Otherwise one obtains a disconnected network. In fact the number of nearest neighbors needs to grow like $\Theta(\log n)$. This contrasts with some previous studies in the 1970s and 1980s which recommended various “magic numbers” of nearest neighbors (three, six, seven, eight etc.). However that will result in network disconnectivity with probability approaching one as the number of nodes increases. We have shown that asymptotic connectivity results when every node is connected to its nearest $5.1774 \log n$ neighbors, while asymptotic disconnectivity results when each node is connected to less than $0.074 \log n$ nearest neighbors. Simulations suggest that there may be a critical value of c, and that it is close to one. However our theory is inadequate to resolve this, and it remains an open problem. In any case when $c \geq 1.5$, the probability of connectedness increases to near one even for modest n. These results should guide schemes for power control, media access control and routing in *ad hoc* networks; see [19].

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