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Optimum design of measurement channels and control policies for linear-quadratic stochastic systems *

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Abstract: In the design of optimal controllers for linear-quadratic stochastic systems, a standard assumption is that the measurement channels are fixed and linear, and the measurement noise is Gaussian. In this paper we relax the first part of this restriction and raise the issue of the derivation of optimum measurement structures as a part of the overall design. Toward this end, we take the measuement process as one given by a Wiener integral, and modify the cost function so that it now places some soft constraints on the measurement strategy. Using some results from information theory, we show that the scalar version (for both finite and infinite horizons) of this joint design problem admits an optimum, dictating linear designs for both the controller and the measurement strategy. For the vector version, however, it is possible for a nonlinear design to improve over the best linear one. In both cases, best linear designs involve the solutions of nonlinear (deterministic) optimal control problems.

Keywords: Stochastic control; LQG design; Optimum signal design; Dynamic optimization; Decentralized systems

1. Introduction

In the linear-quadratic-Gaussian (LQG) controller design problem, which is a widely used model in engineering, economics and operations research, the objective is to choose an optimal controller u_t for a stochastic system described by

$$dx_t = Ax_t\,dt + Bu_t\,dt + F\,dv_t, \quad t \geq 0, \tag{1}$$

by minimizing the quadratic performance index

$$\tilde{J} = E\left\{\int_{t_0}^{t_f} e^{-\beta t}\left[x_t^T Q x_t + u_t^T R u_t\right]\,dt + x_{t_f}^T Q_f x_{t_f}\right\} \tag{2}$$

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where $Q \geq 0$, $R > 0$, $Q_f \geq 0$ are known matrices, with the first two possibly depending on the time variable $t \geq t_0$, and $\beta > 0$ is a discount factor.

In (1), x_t is the state vector of dimension n, u_t is the control vector of dimension r, v_t is the additive stochastic disturbance term which is taken as an n-dimensional standard Wiener process, and A, B, F are matrices of appropriate dimensions, which are allowed to depend on the time variable t, $t_0 \leq t \leq t_f$. Perhaps a more familiar form of (1) for the readership of this journal is the white-noise model:

$$\dot{x}_t = Ax_t + Bu_t + \tilde{F}\xi_t, \quad t \geq 0, \tag{3}$$

where ξ_t is a standard Gaussian vector white noise process with zero mean.

Both (1) and (3) are driven by the initial state $x_{t_0} = x_0$, which is a Gaussian random vector with mean zero and covariance Σ_0, i.e., $x_0 \sim N(0, \Sigma_0)$.

The control u_t does not have direct access to the state, but to a noisy version of it, y_t, $t \geq t_0$, which is generated by

$$\mathrm{d}y_t = Hx_t\,\mathrm{d}t + G\,\mathrm{d}w_t, \quad y_{t_0} = 0, \tag{4}$$

where w_t is another standard Wiener process, independent of $\{v_t,\ t_0 \leq t \leq t_f\}$, x_0, and of the same dimension as the measurement process y_t, $t \geq t_0$, say m. In (4), H is an $m \times n$ matrix, possibly depending on t, and G is a nonsingular matrix, that is $GG^{\mathrm{T}} > 0$. Let us denote the causal dependence of u on y by

$$u_t = \gamma_t(y_0^t), \quad y_0^t := \{y_\tau,\ t_0 \leq \tau < t\} \tag{5}$$

where γ_t, $t \geq t_0$, is a control policy.

The well-known LQG theory [1] says that there is a unique controller of the type (5) that minimimes (2) subject to (1), and this controller is linear and exhibits certainty equivalence. It is given by

$$u_t = \gamma_t(y_0^t) = \tilde{\gamma}_t(\hat{x}_t) = -R^{-1}B^{\mathrm{T}}P(t)\hat{x}_t, \quad t \geqslant 0, \tag{6}$$

where $\hat{x}_t$ is generated by the Kalman filter:

$$\mathrm{d}\hat{x}_t = A\hat{x}_t\,\mathrm{d}t + Bu_t\,\mathrm{d}t + K(t)\left[\mathrm{d}y_t - H\hat{x}_t\,\mathrm{d}t\right], \qquad \hat{x}_{t_0} = 0, \tag{7}$$

$$K(t) := \Sigma(t)H^{\mathrm{T}}(t)[GG^{\mathrm{T}}]^{-1}, \tag{8}$$

$$\dot{\Sigma} = A\Sigma + \Sigma A^{\mathrm{T}} + FF^{\mathrm{T}} - \Sigma H^{\mathrm{T}}[GG^{\mathrm{T}}]^{-1}H\Sigma, \qquad \Sigma(t_0) = \Sigma_0, \tag{9}$$

and P is given as the unique solution of the dual (backward propagating) Riccati differential equation

$$\dot{P} + A_\beta^{\mathrm{T}}P + PA_\beta - PBR^{-1}B^{\mathrm{T}}P + Q = 0, \qquad Q(t_f) = Q_f, \tag{10}$$

where

$$A_\beta := A - \tfrac{1}{2}\beta I. \tag{11}$$

For the infinite horizon version (i.e., $t_f = \infty$) the two Riccati equations are replaced by their algebraic counterparts:

$$A\Sigma + \Sigma A^{\mathrm{T}} + FF^{\mathrm{T}} - \Sigma H^{\mathrm{T}}[GG^{\mathrm{T}}]^{-1}H\Sigma = 0, \tag{12}$$

$$A_\beta^{\mathrm{T}}P + PA_\beta + Q - PBR^{-1}B^{\mathrm{T}}P = 0. \tag{13}$$

These equations are assured of unique nonnegative definite solutions, under which the controller (6) leads to a stable feedback system and a stable filter, if

$$(A, B) \text{ and } (A, F) \text{ are controllable; } (A, Q) \text{ and } (A, H) \text{ are observable.} \tag{14}$$

Note that one of the essential modeling assumptions of the LQG theory is that the measurement scheme (4) is fixed as given. This may not always be a reasonable assumption, however, especially in large scale decentralized systems with more than one decision maker, where the control decisions are often taken on the basis of information generated by other members of the same system and garbled by noisy communication channels. As also indicated in [2], we may identify agents in large decentralized systems with one of two kinds of roles: (i) agents who perform the communication tasks of generating information bearing signals, and (ii) agents who perform the control functions of forming estimates, minimizing erros and reducing costs. This flexibility opens the possibility (and necessity) of simultaneously designing measurement and control strategies, and implementing them in a decentralized fashion.

This paper addresses such a stochastic decision problem, which can be viewed as an extension of the LQG model briefly described above, where a new design element is included in the measurement equation (4). Specifically, we replace (4) by

$$\mathrm{d}y_t = h_t(x_t,\ y_0^t)\ \mathrm{d}t + G\ \mathrm{d}w_t,\quad y_{t_0} = 0, \tag{15}$$

where h_t, $t \geq t_0$, is a general (possibly nonlinear) function, which allows the measurement at time t to depend on the current value of the state as well as the past values of the measurement. The driving measurement noise w_t, $t \geq t_0$, is again as defined earlier, following (4).

Our interest lies in the derivation of an optimal measurement strategy (within the class described above by (15)) along with the corresponding optimal control, both chosen under the new performance index

$$J = \bar{J} + E\left\{\int_{t_0}^{t_f} \mathrm{e}^{-\beta t} h_t^{\mathrm{T}}(x_t,\ y_0^t) N h_t(x_t,\ y_0^t)\ \mathrm{d}t\right\},\quad N > 0, \tag{16}$$

which places some soft constraints on the measurement strategy. Hence, we seek a pair γ_t^*, h_t^*, $t \geq t_0$, from the classes of functions identified above, such that, for all permissible γ_t, h_t, $t \geq t_0$,

$$J(\gamma_t^*,\ h_t^*;\ t \geq t_0) \leq J(\gamma_t,\ h_t;\ t \geq t_0). \tag{17}$$

The discrete-time version of this problem was earlier discussed in [2], where it was shown that for the scalar model the best measurement strategy is to amplify the *innovation* at each stage to a certain power threshold level, with this threshold obtained from the solution of a discrete-time nonlinear optimal control problem using dynamic programming. For the infinite-horizon version, these threshold levels converge to a fixed constant, leading to the existence of optimal linear stationary policies. For higher order problems, however, [2] has established the possibility of nonlinear optimum designs, and also obtained the best linear designs, through the solutions of nonlinear optimal control problems.

The continuous-time problem studied in this paper requires somewhat different mathematical tools than those used in [2], but it will turn out that the results to be obtained are qualitatively similar to those of [2]. We again first study the scalar problem (in the next section), and obtain the optimum joint design, which has a linear structure. We then present, in Section 3, several numerical examples to illustrate the results of Section 2. Following this, in Section 4 we solve the problem with higher-order dynamics, when the function h_t in (15) is restricted to be affine. Section 5 concludes the paper.

2. Solution to the scalar problem

We consider here the one-dimensional version of the general problem, rewritten as

$$\mathrm{d}x_t = ax_t\ \mathrm{d}t + bu_t\ \mathrm{d}t + f\ \mathrm{d}v_t,\qquad x_{t_0} = x_0 \sim N(0,\ \sigma_0) \tag{18a}$$

$$\mathrm{d}y_t = h_t(x_t,\ y_0^t)\ \mathrm{d}t + g\ \mathrm{d}w_t,\qquad y_{t_0} = 0,\quad t \geq t_0. \tag{18b}$$

$$u_t = \gamma_t(y_0^t), \tag{19}$$

where $\{v_t\}$, $\{w_t\}$ are standard independent Wiener processes, which are also independent of x_0. The control and measurement functions γ_t and h_t are taken to be Borel measurable, and the cost function is

$$J = E\left\{ x_{t_f}^2 q_f + \int_{t_0}^{t_f} e^{-\beta t}\left[x_t^2 q + u_t^2 r \right] \mathrm{d}t \right\} + E\left\{ \int_{t_0}^{t_f} e^{-\beta t} n h_t^2 \, \mathrm{d}t \right\}, \tag{20}$$

where $q \geq 0$, $q_f \geq 0$, with at least one of them positive, $r > 0$, $n > 0$.

2.1. Derivation of the optimum solution

We seek a pair (h^*, γ^*) such that the cost $J(\gamma, h)$ is minimized. Toward this end, we first invoke, for the measurement process, the linear structure

$$h_t(x_t, y_0^t) = H_t \cdot (x_t - \hat{x}_{t|t}), \qquad \hat{x}_{t|t} := E\left[x_t \middle| y_0^t\right], \tag{21}$$

where H_t is a function of t, yet to be determined. For each H_t, $\hat{x}_{t|t}$ is given by the Kalman filter:

$$\mathrm{d}\hat{x}_{t|t} = (a\hat{x}_{t|t} + bu_t)\,\mathrm{d}t + K_t\,\mathrm{d}y_t, \qquad x_{t_0|t_0} = 0, \tag{22}$$

$$K_t = H_t \sigma_t / g^2, \tag{23}$$

$$\dot{\sigma}_t = 2a\sigma_t + f^2 - H_t^2 \sigma_t^2 / g^2, \qquad \sigma_{t_0} = \sigma_0. \tag{24}$$

Here, the separation principle applies, and the unique optimal control law is given by

$$u_t = \gamma_t(y_0^t) = -(1/r) b p_t \hat{x}_{t|t}, \tag{25}$$

$$\dot{p}_t = -(2a - \beta) p_t + p_t^2 (b^2/r) - q, \qquad p_{t_f} = q_f, \tag{26}$$

and the corresponding value of the cost is (as a function of H_t)

$$J = \int_{t_0}^{t_f} e^{-\beta t}\left[\sigma_t b^2/r \cdot p_t^2 + p_t f^2\right] \mathrm{d}t + p_0\sigma_0 + \int_{t_0}^{t_f} e^{-\beta t} H_t^2 \sigma_t n \, \mathrm{d}t$$

$$\equiv \underbrace{ng^2 \int_{t_0}^{t_f} e^{-\beta t}[m_t\sigma_t + c_t'\sigma_t]\,\mathrm{d}t}_{L(c')} + \int_{t_0}^{t_f} e^{-\beta t} p_t f^2 \,\mathrm{d}t + p_0\sigma_0 \tag{27}$$

with

$$c_t' := H_t^2/g^2 \text{ and } m_t := b^2 p_t^2 / rng^2.$$

Now let $c_t = c_t'\sigma_t$. Then, to obtain the best measurement strategy in the class specified, we have to solve the nonlinear dynamic optimization problem

$$\min_{\{c_t \geq 0\}} \quad L(c) = \int_{t_0}^{t_f} e^{-\beta t}[m_t\sigma_t + c_t]\,\mathrm{d}t \tag{28a}$$

$$\text{s.t.} \quad \dot{\sigma}_t = 2a\sigma_t + f^2 - c_t\sigma_t, \qquad \sigma_{t_0} = \sigma_0. \tag{28b}$$

Note that the function $L(c)$ is bounded from below (by zero), and as $c \to \infty$, $L(c) \to \infty$, implying that there exists a constant $K > 0$ (could be sufficiently large) such that $\min_{\{c_t \geq 0\}} L(c) = \min_{\{0 \leq c_t \leq K\}} L(c)$.

This dynamic optimization problem can be viewed as a nonlinear optimal control problem, with c being the control variable (constrained to be nonnegative), and σ the state. As such, it can be solved using either dynamic programming or the minimum principle, with the former being applicable if there exists a continuously differentiable value function. We now first explore this possibility, and write down the associated Hamilton–Jacobi–Bellman (HJB) equation:

$$-\frac{\partial V}{\partial t}=\min_{c_t\geq 0}\left\{c_t\ \mathrm{e}^{-\beta t}+m_t\sigma_t\ \mathrm{e}^{-\beta t}+\frac{\partial V}{\partial \sigma}\cdot\left(2a\sigma_t+f^2-c_t\sigma_t\right)\right\},\qquad V(t_\mathrm{f},\sigma)\equiv 0. \tag{29}$$

Clearly a solution to the pointwise minimization exists if, and only if, $\mathrm{e}^{-\beta t}+(\partial V/\partial\sigma)\sigma_t\leq 0$, under which

$$-\frac{\partial V}{\partial t}=m_t\sigma_t\ \mathrm{e}^{-\beta t}+\frac{\partial V}{\partial\sigma}\left(2a\sigma_t+f^2\right),\qquad V(t_f,\sigma)\equiv 0. \tag{30}$$

A candidate solution to this PDE is

$$V(t,\sigma)=\xi(t)\sigma+\eta(t), \tag{31a}$$

$$-\dot{\xi}(t)=2a\xi(t)+m_t\ \mathrm{e}^{-\beta t},\qquad \xi(t_f)=0, \tag{31b}$$

$$-\dot{\eta}(t)=f^2\xi(t),\qquad \eta(t_\mathrm{f})=0. \tag{31c}$$

under which the earlier condition becomes:

$$\xi(t)\sigma(t)\leq \mathrm{e}^{-\beta t}. \tag{32}$$

Hence, under the structural assumption (31), a necessary and sufficient condition for the existence of a solution to the optimal control problem (28) is the existence of a $\{c_t\geq 0\}_{t=t_0}^{t_\mathrm{f}}$ such that (32)–(35) are satisfied:

$$c_t\left(\mathrm{e}^{-\beta t}-\xi(t)\sigma_t\right)=0, \tag{33}$$

$$\dot{\sigma}_t=2a\sigma_t+f^2-c_t\sigma_t,\qquad \sigma_{t_0}=\sigma_0, \tag{34}$$

$$-\dot{\xi}(t)=2a\xi(t)+m_t\ \mathrm{e}^{-\beta t},\qquad \xi(t_\mathrm{f})=0, \tag{35}$$

where

$$m_t:=\frac{b^2}{rng^2}p_t^2,\qquad \dot{p}_t=-(2a-\beta)p_t+\frac{b^2}{r}p_t^2-q,\qquad p_{t_\mathrm{f}}=q_\mathrm{f}. \tag{36}$$

Hence, $\min_{\{c_t\}}L(c)=V(t_0,\sigma_0)\equiv\xi(t_0)\sigma_0+\eta(t_0)$, where

$$\xi(t_0)=-\mathrm{e}^{-2at_0}\int_{t_\mathrm{f}}^{t_0}\mathrm{e}^{2a\tau}\ \mathrm{e}^{-\beta\tau}m_\tau\ \mathrm{d}\tau=\mathrm{e}^{-2at_0}\int_{t_0}^{t_\mathrm{f}}\mathrm{e}^{(2a-\beta)\tau}m_\tau\ \mathrm{d}\tau, \tag{37}$$

$$\eta(t_0)=f^2\int_{t_0}^{t_\mathrm{f}}\xi(\tau)\ \mathrm{d}\tau=f^2\int_{t_0}^{t_\mathrm{f}}\mathrm{e}^{-2at}\ \mathrm{d}t\int_{t_0}^{t}\mathrm{e}^{(2a-\beta)\tau}m_\tau\ \mathrm{d}\tau=f_2\int_{t_0}^{t_\mathrm{f}}\mathrm{d}\tau m_\tau\ \mathrm{e}^{(2a-\beta)\tau}\int_\tau^{t_\mathrm{f}}\mathrm{e}^{-2at}\ \mathrm{d}t$$

$$\Rightarrow\quad \eta(t_0)=f^2\int_{t_0}^{t_\mathrm{f}}\frac{1}{2a}(\mathrm{e}^{-2a\tau}-\mathrm{e}^{-2at_\mathrm{f}})m_\tau\ \mathrm{e}^{(2a-\beta)\tau}\ \mathrm{d}\tau$$

$$=\frac{f^2}{2a}\int_{t_0}^{t_\mathrm{f}}\mathrm{e}^{-\beta\tau}m_\tau\ \mathrm{d}\tau-\frac{f^2}{2a}\int_{t_0}^{t_\mathrm{f}}\mathrm{e}^{-2at_\mathrm{f}}\ \mathrm{e}^{(2a-\beta)\tau}m_\tau\ \mathrm{d}\tau. \tag{38}$$

Thus completing the discussion of the dynamic programming approach to the optimization problem (28), we now apply the *minimum principle* to the same problem. We first define the Hamiltonian associated with this problem as

$$H=\mathrm{e}^{-\beta t}(m_t\sigma_t+c_t\sigma_t)+\lambda_t\left(2a\sigma_t+f^2-c_t\sigma_t^2\right) \tag{39}$$

where λ_t, $t \ge t_0$, is the co-state variable. Any optimal solution c_t, $t \ge t_0$, should minimize H (pointwise), and lead to satisfaction of the two-point boundary value problem

$$-\dot{\lambda}_t = m\,\mathrm{e}^{-\beta t} + 2a\lambda_t - c_t\lambda_t, \qquad \lambda_{t_f} = 0, \tag{40}$$

$$-\dot{\sigma}_t = 2a\sigma_t + f^2 - c_t\sigma_t, \qquad \sigma_{t_0} = \sigma_0. \tag{41}$$

Note that since the Hamiltonian is linear in c_t, the minimizing solution in (39) is obtained from

$$c_t\left[\mathrm{e}^{-\beta t} - \sigma_t\lambda_t\right] = 0, \qquad \mathrm{e}^{-\beta t} - \sigma_t\lambda_t \ge 0. \tag{42}$$

Note that the solutions to (42), for any fixed t, are either

$$c_t = 0, \qquad \mathrm{e}^{-\beta t} - \sigma_t\lambda_t \ge 0, \tag{43}$$

or a c_t that satisfies

$$\mathrm{e}^{-\beta t} = \sigma_t\lambda_t. \tag{44}$$

In the former case, σ_t and λ_t can easily be solved from (41) and (40) (with $c_t = 0$), and the condition in (43) be checked. Because of the boundary condition on λ, this condition will always be satisfied in a neighborhood of the terminal time, implying that there exists an interval $(t_2, t_f]$ on which $c_t = 0$, which says that beyond a time point t_2 no measurement should be taken. If $\sigma_0 = 0$, then the same scenario repeats in a neighborhood of $t = t_0$, which implies the existence of an interval $[t_0, t_1)$ on which again no measurement should be taken.

Now, if $t_1 < t_2$, in the interval $[t_1, t_2]$ or in any subinterval of it, there will be a *singular control*, obtained by differentiation of (44). Carrying out the manipulations, we arrive at the expression

$$c_t = \frac{\beta^2 + 2f^2m + \sigma_t m(4a - \beta) + \sigma_t\dot{m} - 2a\beta}{2\sigma_t m - \beta}, \tag{45}$$

with the corresponding values of σ and λ being

$$\sigma_t = \frac{\beta + \sqrt{\beta^2 + 4f^2m}}{2m}, \qquad \lambda_t = \frac{\mathrm{e}^{-\beta t}}{\sigma_t}. \tag{46}$$

The condition for (45)–(46) to constitute a solution is for c_t to be nonnegative, which determines the length of the interval over which it is valid. This can be done numerically, as will be discussed in the next section, in the context of some examples. But before doing this, we first verify optimality of the solution (25), along with (21), provided that (42) or (32)–(35) admit solutions.

2.2. *Overall optimality*

We now prove the overall optimality of the pair given by (21) and the solution of the optimal control problem (28). Toward this end, we first rewrite J given by (20) as

$$J = E\left\{\int_{t_0}^{t_f}\mathrm{e}^{-\beta t}\left[u_t - \frac{b}{r}p_t x_t\right]^2 r\,\mathrm{d}t\right\} + x_0' p_0 x_0 + \int_{t_0}^{t_f}\left(\mathrm{e}^{-\beta t}p_t f^2\right)\mathrm{d}t + E\left\{\int_{t_0}^{t_f}\mathrm{e}^{-\beta t}nh_t^2\,\mathrm{d}t\right\} \tag{47}$$

where $p_t > 0$, $t \ge t_0$, is defined by (26).

Fix $E[h_t^2] \le \ell_t$, where ℓ_t is known. Then,

$$\min_{\gamma,h} J = \min_{\{\ell_t\}} \min_{\gamma,\,yh,\,E[h_t^2]\le\ell_t} J$$

where for the inner minimization problem the equivalent cost function is

$$L = E\left\{\int_{t_0}^{t_1} \mathrm{e}^{-\beta t}\frac{b^2}{r^2}p_t^2\left[\frac{r}{bp_t}u_t - x_t\right]^2 r \,\mathrm{d}t\right\} \tag{48}$$

which is to be minimized with respect to γ, h, subject to $E[h_t^2] \le \ell_t$ and

$$\mathrm{d}x_t = ax_t + bu_t + f\,\mathrm{d}v_t, \qquad \mathrm{d}y_t = h_t(x_t, y_0^t)\,\mathrm{d}t + g\,\mathrm{d}w_t. \tag{49}$$

Let $u_t' := (r/bp_t)u_t$. Then we have

$$L = E\left\{\int_{t_0}^{t_1} \mathrm{e}^{-\beta t}\frac{b^2}{r^2}p_t^2[u_t' - x_t]^2 r \,\mathrm{d}t\right\}, \tag{50}$$

$$\mathrm{d}x_t = ax_t + \frac{b^2 p_t}{r}u_t' + f\,\mathrm{d}v_t, \qquad x_{t_0} = x_0, \tag{51}$$

$$\mathrm{d}y_t = h_t(x_t, y_0^t)\,\mathrm{d}t + g\,\mathrm{d}w_t. \tag{52}$$

Now decompose x_t into two components:

$$x_t = x_t^1 + x_t^2, \tag{53}$$

$$\mathrm{d}x_t^1 = ax_t^1 + f\,\mathrm{d}v_t, \qquad x_{t_0}^1 = x_0, \tag{54}$$

$$\mathrm{d}x_t^2 = ax_t^2 + \frac{b^2}{r}p_t u_t', \qquad x_{t_0}^2 = x_0 \tag{55}$$

$$\Rightarrow \quad L = E\left\{\int_{t_0}^{t_1} \mathrm{e}^{-\beta t}\frac{b^2}{r^2}p_t^2\left[u_t'' - x_t^1\right]^2 r \,\mathrm{d}t\right\} \tag{56}$$

where

$$u_t'' := u_t' - x_t^2 \tag{57}$$

and the minimization over u_t' is equivalent to minimization over u_t'', since x_t^2 depends only on the past values of u_t'.

Hence the optimization problem is

$$\text{Min } L(\gamma'', h) \text{ over } u_t'' = \gamma_t''(y_0^t),\ h_t,\ E\left[h_t^2\right] \le \ell_t$$

$$\text{such that} \quad \mathrm{d}x_t^1 = ax_t^1 + f\,\mathrm{d}v_t, \qquad x_{t_0}^2 = x_0,$$

$$\mathrm{d}y_t = h_t(x_t^1, x_t^2, y_0^t)\,\mathrm{d}t + g\,\mathrm{d}w_t,$$

where the differential equation for y_t can equivalently be written as

$$\mathrm{d}y_t = h_t''(x_t^1, y_0^t)\,\mathrm{d}t + g\,\mathrm{d}w_t,$$

this being true because x_t^2 is $\sigma(y_0^t)$-measurable.

This is a problem of the type that arises in the transmission of information over Gaussian channels: The Gaussian message process x_t^2, $t \ge\, > t_0$, is to be transmitted over a continuous-time channel corrupted by additive noise, modeled by a Wiener process $(w_t, t \ge t_0)$, and it is desired to design an encoder (in our case h'') under a given power constraint (ℓ) such that the quadratic error at the receiver (in our case $L(\gamma'', h)$) is minimized after optimum decoding (which, in our case, is γ''). This information transmission problem has been studied before in ([3], pp. 177–195), where it has been shown that the

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optimal solution for h_t'' (encoder) is linear in the innovations, in which case the optimum choice for the decoder γ_t'' is the Kalman filter. Now, given that h'' is in form

$$h_t''(x_t^1, y_0^t) = \left(x_t^1 - E\left[x_t^1 \middle| y_0^t\right]\right)H_t'',$$

we can invert the transformation used in this subsection, to arrive at the structural form (21) as the optimum one for the original problem.

We are now in a position to summarize the main result of this section in the following theorem:

Theorem 1. *Let there exist a solution to the nonlinear optimal control problem* (28), *to be denoted by* H_t^*, $t \geq t_0$, *after the transformation introduced by* (27). *Then, the scalar joint control/measurement design problem admits an optimal solution, given by*

$$h_t^*(x_t, y_0^t) = (x_t - \hat{x}_{t|t}^*)H_t^*, \qquad \gamma_t^*(y_0^t) = -(1/r)bp_t\hat{x}_{t|t}^*,$$

where $\hat{x}_{t|t}^*$ *is generated by* (22), *with* $H_t = H_t^*$, *and* p_t *is obtained from* (26).

3. Numerical examples

We present in this section two numerical examples to illustrate the results presented in the previous section, and especially the *switching* nature of the optimal measurement scheme.

Example 1. *Parametric values*: $t_f = 10$; $t_0 = 0$; $a = \beta = q_f = \sigma_0 = 0$; $b = f = g = q = r = n = 1$.

Working with (32)–(35), we have

$$\xi(t) = 11 - t - \left(2/(1 + e^{2(t-10)}\right),$$

$$c^*(t) = \begin{cases} \xi(t) - \dfrac{\tanh^2(10-t)}{\xi(t)}, & 0.1125 < t < 8.085, \\ 0 & \text{else,} \end{cases}$$

$$\sigma(t) = \begin{cases} t, & 0 \leq t \leq 0.1125, \\ 1/\xi(t), & 0.115 < t < 8.085, \\ t - 7.972, & t > 8.085. \end{cases}$$

Hence, the optimum policy is to make measurements only in the interval (0.115, 8085) (see Figure 1). The corresponding optimal controller, from (25), is

$$u_t = \gamma_t^*(y_0^t) = -\tanh(t-10)\hat{x}_{t|t},$$

$$d\hat{x}_{t|t} = -\tanh(t-10)\hat{x}_{t|t}\,dt + K_t\,dy_t, \quad \hat{x}_{0|0} = 0, \qquad K_t = \sqrt{(c(t)/\xi(t))}\,.$$

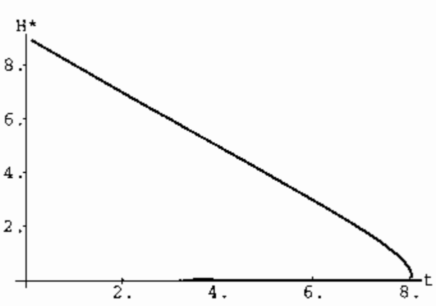

Figure 1. A plot of optimum measurement gain $H_t^* = \sqrt{c^*(t)}$ for $t \in (0.115, 8.085)$

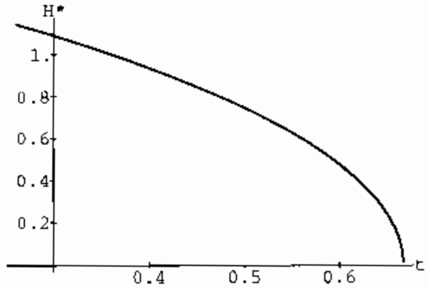

Figure 2. A plot of optimum measurement gain H_t^* for $t \in (0.25158, 0.6666)$

Example 2. *Parameter values*: $a = \beta = q = t_0 = \sigma_0 = 0$; $b = r = n = g = q_f = t_f = 1$.

The solutions to (26) and (31) are, respectively,

$$p(t) = 1/(2-t) \quad \text{and} \quad \xi(t) = (1-t)/(2-t), \quad 0 \le t \le 1.$$

Let us consider two different values for f:

(i) $f = 1$. Then, with $H_t \equiv 0$, σ_t obtained from (34) is $\sigma_t = t$, $0 \le t \le 1$, which leads to satisfaction of (32) as a strict inequality. Hence, the optimum policy here is not to use any measurement throughout the interval. (Here the measurement is too costly!)

(ii) $f = 3$, which corresponds to a more noisy system. Then we have two switches in the measurement policy, at time instants

$$t_1 = (10 - \sqrt{28})/18 \simeq 0.25158 \quad \text{and} \quad t_2 = 2/3 \simeq 0.6667.$$

Outside the interval (t_1, t_2) the optimum value for H_t is *zero*, and inside the interval it is (see Figure 2)

$$H_t^* = \sqrt{9(1-t)^2 - 1/(2-t)}, \quad t_1 < t < t_2.$$

The corresponding filter error variance σ is

$$\sigma_t = \begin{cases} 9t, & 0 \le t < t_1, \\ (2-t)/(1-t), & t_1 < t < t_2, \\ 9t - 2, & t_2 \le t \le 1. \end{cases}$$

Our numerical experimentation with other examples has shown that the number of switches is not necessarily upperbounded by two. One can in fact have an arbitrary number of switches in the measurement strategy, depending on the parameters of the problem at hand. Also, depending on the values of the system parameters, it is possible for an optimal solution to (28) not to exist within the class of piecewise continuous controls; in this case one has to look for a solution in an extended class that includes *impulsive* controls.

4. Solution to the vector version

We now return to the original problem formulated in Section 1, and require h in (15) to be a linear function of its arguments. First, using an argument similar to that of Theorem 6 of [2], it can be shown that within the general linear class there is no loss of generality in restricting the measurement strategies to be in a form that is the vector version of (21):

$$h_t(x_t, y_0^t) = H_t \cdot (x_t - \hat{x}_{t|t}), \qquad \hat{x}_{t|t} = E[x_t | y_0^t], \tag{58}$$

where H_t, $t \geq t_0$, is a matrix valued function, of dimensions $r \times n$. For each such H_t, the conditional mean $\hat{x}_{t|t}$ is again given by the Kalman filter:

$$\mathrm{d}\hat{x}_{t|t} = \left(A\hat{x}_{t|t} + Bu_t\right)\,\mathrm{d}t + K_t\,\mathrm{d}y_t, \qquad \hat{x}_{t_0|t_0} = 0, \tag{59}$$

$$K_t = H_t\Sigma_t(GG^{\mathrm{T}})^{-1}, \tag{60}$$

$$\dot{\Sigma}_t = A\Sigma_t + \Sigma_t A^{\mathrm{T}} + FF^{\mathrm{T}} - H_t\Sigma_t(GG^{\mathrm{T}})^{-1}\Sigma_t H_t^{\mathrm{T}}, \qquad \Sigma_{t_0} = \Sigma_0. \tag{61}$$

For each $\{H_t\}$, the unique optimal control law is given by

$$u_t = \gamma_t^*\left(y_0^t\right) = -R^{-1}B^{\mathrm{T}}P\hat{x}_{t|t}, \quad t \geq t_0, \tag{62}$$

where $P(t)$, $t \geq t_0$, is the unique nonnegative definite solution of (10).

Now, to obtain the optimum measurement gain matrix, we have to substitute (62), along with (59), into the performance index (16), to arrive at a new cost function which will have to be optimized with respect to $\{H_t\}$:

$$\min_{\{H_t\}} L(H): \quad L(H) = \int_{t_0}^{t_f} \mathrm{e}^{-\beta t}\,\mathrm{Tr}\left[\Sigma_t\left(M_t + H_t^{\mathrm{T}}NH_t\right)\right]\,\mathrm{d}t + k \tag{63}$$

where

$$M_t := P(t)BR^{-1}B^{\mathrm{T}}P(t), \tag{64}$$

$$k := \mathrm{Tr}\left[P(0)\Sigma_0\right] + \int_{t_0}^{t_f}\mathrm{Tr}\left[P(t)GG^{\mathrm{T}}\right]\,\mathrm{d}t. \tag{65}$$

Note that here the 'control' variable H_t is matrix valued, and the dynamic constraint is the matrix valued state equation (59). Assuming the existence of a continuously differentiable value function for this problem, the associated HJB equation can be written as (by ignoring the constant bias term k in (63))

$$-\frac{\partial V}{\partial t} = \min_{H_t}\left\{\mathrm{e}^{-\beta t}\,\mathrm{Tr}\left[\Sigma_t M_t + \Sigma_t H_t^{\mathrm{T}}NH_t\right]\right.$$
$$\left. + \mathrm{Tr}\left[\frac{\partial V}{\partial \Sigma_t}\left(A\Sigma_t + \Sigma_t A^{\mathrm{T}} + FF^{\mathrm{T}} - H_t\Sigma_t(GG^{\mathrm{T}})^{-1}\Sigma_t H_t^{\mathrm{T}}\right)\right]\right\}, \qquad V(t_f, \Sigma) \equiv 0. \tag{66}$$

Invoking an affine structure for V:

$$V_t(t, \Sigma) = \mathrm{Tr}[\Xi\Sigma] + \eta(t) \tag{67}$$

and using this in (66), we arrive at

$$-\mathrm{Tr}[\dot{\Xi}\Sigma] = \min_{H_t}\left\{\mathrm{e}^{-\beta t}\,\mathrm{Tr}\left[\Sigma_t M_t + \Sigma_t H_t^{\mathrm{T}}NH_t\right]\right.$$
$$\left. + \mathrm{Tr}\left[\Xi\left(A\Sigma_t + \Sigma_t A^{\mathrm{T}} + FF^{\mathrm{T}} - H_t\Sigma_t(GG^{\mathrm{T}})^{-1}\Sigma_t H_t^{\mathrm{T}}\right]\right\} + \dot{\eta}(t)$$
$$= e^{-\beta t}\,\mathrm{Tr}[\Sigma_t M_t] + \mathrm{Tr}\left[\Xi\left(A\Sigma_t + \Sigma_t A^{\mathrm{T}} + FF^{\mathrm{T}}\right)\right] + \dot{\eta}(t)$$

which is satisfied if Ξ and η are chosen according to

$$-\dot{\Xi}(t) = \Xi(t)A + A^{\mathrm{T}}\Xi(t) + \mathrm{e}^{-\beta t}M_t, \qquad \Xi(t_f) = 0, \tag{68}$$

$$-\dot{\eta}(t) = \mathrm{Tr}\left[\Xi(t)FF^{\mathrm{T}}\right], \qquad \eta(t_f) = 0, \tag{69}$$

with the optimal H_t satisfying the equation

$$\varphi(H_t) := e^{-\beta t}\ \mathrm{Tr}\left[\Sigma_t H_t^{\mathrm{T}} N H_t\right] - \mathrm{Tr}\left[\Xi(t) H_t \Sigma_t (GG^{\mathrm{T}})^{-1} \Sigma_t H_t^{\mathrm{T}}\right] = 0. \tag{70}$$

The condition of existence, that replaces (32) in this case, is

$$\min_{H_t} \varphi(H_t) = 0. \tag{71}$$

We can now summarize the main result of this section in the following theorem, which is the counterpart (but a weaker version) of Theorem 1 in the vector case:

Theorem 2. *Let there exist a solution to the nonlinear optimal control problem* (63), *to be denoted by* H_t^*, $t \geq t_0$. *Then, with the measurement strategies restricted to the linear class, the general joint control/measurement design problem admits an optimal solution, given by*

$$h_t^*(x_t, y_0^t) = H_t^* \cdot (x_t - \hat{x}_{t|t}^*), \qquad \gamma_t^*(y_0^t) = -R^{-1}B^{\mathrm{T}}P(t)\hat{x}_{t|t}^*,$$

where $\hat{x}_{t|t}^*$ *is generated by* (59), *with* $H_t = H_t^*$, *and* P *is obtained from* (10).

5. Concluding remarks

We have formulated in this paper a class of joint control/measurement design problems, which extends the framework of the standard LQG models. For the scalar version the globally optimal measurement strategy turns out to be a linear one, with the structure of a time-varying gain multiplying the innovations in the state measurements. This gain is determined from the solution of a nonlinear optimal control problem. For the vector version, the best linear measurement has the same structure, but the possibility of improving upon this design by a nonlinear one is not ruled out in this case. This qualitative difference between the scalar and vector versions is due to the fact that the result from information theory that was used in the proof of Theorem 1 does not extend to the vector case. In both cases, with linear measurements, the optimal controller is linear in the best (minimum-mean-square) estimate of the state.

Some challenges for the future might be a thorough investigation of the numerical aspects of the solutions to the optimal control problems (28) and (63), and the derivation of implementable nonlinear measurement schemes, as well as the corresponding control policies, for the problem of Section 4.

References

[1] Fleming, W.H., and Rishel, R.W., *Deterministic and Stochastic Optimal Control*, Springer-Verlag, Berlin, 1975.

[2] Bansal, R., and Başar, T., "Simultaneous design of measurement and control strategies for stochastic systems with feedback", *Automatica* 25/5 (1989) 679–694.

[3] Liptser, R.S., and Shiryayev, A.N., *Statistics of Random Process, II. Applications*, Springer-Verlag, New York, 1978.