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Solutions to a class of linear-quadratic-Gaussian (LQG) stochastic team problems with nonclassical information *

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Abstract: We consider a stochastic dynamic team problem with two controllers and nonclassical information, which can be viewed as the transmission of a garbled version of a Gaussian message over a number of noisy channels under a fidelity criterion. We show that the optimum solution (under a quadratic loss functional) consists of linearly transforming the garbled message to a certain (optimum) power level P^* and then optimally decoding it by using a linear transformation at the receiving end. The power level P^* is determined by the solution of a fifth order algebraic equation. The paper also discusses an extension of this result to the case when the channel noise is correlated with the input random variable, and shows that for the single channel case the optimum solution is again linear.

Keywords: Stochastic control, Stochastic teams, Nonclassical information patterns.

1. Introduction

In stochastic control, the classical information structure requires that the data available at time t still be available at any later time $t' > t$, and actions taken at any time to depend on all the data accumulated up to that time. For systems with linear dynamics, linear measurement equations, Gaussian noise and quadratic criterion, it is known that optimal policies are affine under the classical information structure (Striebel [9], Wonham [14]).

In practice, however, the controller memory may be limited, requiring each action to depend on the most recent datum; or the physical system may be of large size and involve various subsystems which are responsible individually for specific control actions (for example, space missions, air traffic control, etc.), and there may be a delay in the transmission of measurement and control information from one subsystem to another. In such systems the assumptions of classical information cannot be validated. If the delay in the second scenario above is of the one step type, or more generally, if the team information is of the partially nested type, then we call the information structure *quasiclassical* (Witsenhausen [11], Başar and Cruz [2]); for LQG systems of this type affine policies have been shown to be optimum (Ho and Chu [4], Radner [8]).

We refer to an information structure as *nonclassical*, if the action of decision maker j affects the information of i and there is no way in which i can infer the information available to j. Under such information patterns the class of affine functions is not necessarily adequate for the search of an optimum (Witsenhausen [10]). It has been established however, that even for information structures of the nonclassical type, if the cost function does not contain a product term between the decision variables, a basic formulation of the LQG team decision problem admits optimum linear solutions (Bansal and Başar [1]). In this paper, we further extend the class of LQG team problems for which the search for an optimum may be confined to affine policies.

2. Problem statement

Consider the following team decision problem.

Problem P1.

Minimize $J = E\left[k_0 u_0^2 + s_0 u_0 x + (u_1 - x)^2\right]$

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over all Borel measurable maps γ_0 and γ_1, where

$$u_0 = \gamma_0(x + v), \qquad u_1 = \gamma_1(z),$$

and x, v are independent zero mean Gaussian variables with variances 1 and σ_v^2, respectively. The vector z is an n-tuple $(z_1, \dots, z_n)$ where

$$z_i = \lambda_i \gamma_0(x + v) + w_i, \quad \lambda_i \neq 0,$$

with the w_i's being independent, zero mean Gaussian noises with variances $\sigma_{w_i}^2 > 0$; these random variables are further independent of x and v.

The problem consists of encoding and transmitting over a Gaussian channel (Figure 1), where the source output may be distorted prior to encoding and the input to the decoder is a garbled version of the encoder output. Such a model is of use in a practical situation where the initial disturbance is due to uncoded transmission over which the system designer has no control. The problem is related to that of transmission of noisy information with minimum distortion as considered by Wolf and Ziv [13]. Here, however, the channel is multidimensional, the hard power constraint has been replaced by a soft constraint in the form of the $E[k_0 u_0^2]$ term, and in addition there is a correlation cost designated by $s_0 E[u_0 x]$. The encoder–decoder pair is required to cooperate in minimizing the performance index.

We show in the sequel that the search for an optimum solution to P1 may be confined to linear strategies for the encoder as well as for the decoder. Expressions for the optimum linear solution are given and it is further shown that for the special case with $n = 1$ the channel noise may be allowed to be correlated with the channel input while the search for optimum policies may still be confined to linear strategies.

3. Optimum solution to P1

Toward obtaining the solution to P1, we define

$$y = x + v$$

and let

$$m \triangleq E[x \mid y].$$

We now make the following observations.

(a) Whatever the choice of γ_0 be, the optimum choice for γ_1 is

$$\gamma_1(z) = E[x \mid z].$$

(b)

$$E\left[(E(x \mid z) - x)^2\right] = E\left[(E(x \mid z) - m)^2\right] + E\left[(m - x)\right]^2,$$

since

$$E\left[(E(x \mid z) - E(x \mid y)) \cdot (E(x \mid y) - x)\right] = 0.$$

(c)

$$E[\gamma_0(y)x] = E[\gamma_0(y)m].$$

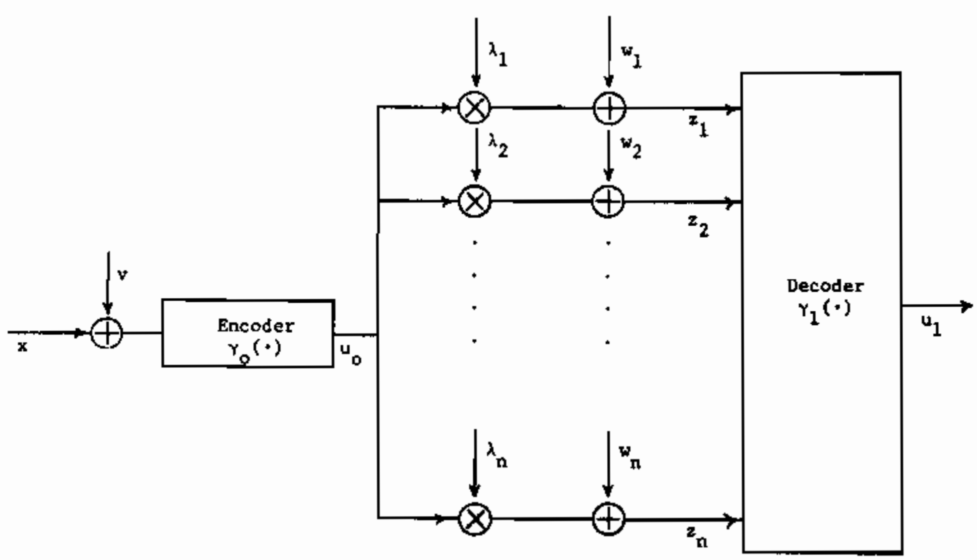

Fig. 1. A schematic diagram for Problem 1.

(d) The variable m $(=y/(1+\sigma_v^2))$ is a zero mean Gaussian random variable with variance

$$\sigma_m^2 = 1/(1+\sigma_v^2).$$

In view of these observations, problem P1 may now be rewritten as P2 below.

Problem P2.

Minimize $\quad J = E\left[k_0 u_0^2 + s_0 u_0 m + (u_1 - m)^2 + c\right]$

over γ_0', γ_1, where

$$u_0 = \gamma_0'(m), \quad u_1 = \gamma_1(z)$$

and

$$c = E\left((m-x)^2\right) = \sigma_v^2 \sigma_m^2.$$

Let J_p denote the infimum of J under the constraint $E[u_0'^2] = P^2$, i.e.,

$$J_p = \inf_{\gamma_0', \gamma_1 : E[\gamma_0'^2] = P^2} J(\gamma_0', \gamma_1). \tag{3.1}$$

We first have

$$J_p \geqslant k_0 P^2 + \inf_{E[\gamma_0'^2] = P^2} s_0 E[u_0 m] + \inf_{E[\gamma_0'^2] = P^2} E(u_1 - m)^2 + c. \tag{3.2}$$

By the Cauchy–Schwartz inequality, we know that

$$\inf_{E[\gamma_0'^2] = P^2} s_0 E[u_0 m] = -|s_0| \sigma_m. \tag{3.3}$$

We now consider the optimization problem

$$\inf_{E[\gamma_0'^2] = P^2} E(u_1 - m)^2.$$

Since m, z and u_1 form a Markov chain, we have

$$I\{m;\, u_1\} \leqslant I\{m;\, z\} \tag{3.4}$$

where $I\{a;\, b\}$ denotes the mutual information of random variables a and b.

Also, we have the inequality

$$I\{m;\, u_1\} \geqslant \frac{1}{2} \log \frac{\sigma_m^2}{E\left\{(u_1 - m)^2\right\}} \tag{3.5}$$

(Wyner [14]), and the equality

$$I\{m;\, z\} = H\{z\} - H\{z \mid m\} \tag{3.6}$$

where $H\{a\}$ is the entropy of the random variable a and $H\{a \mid b\}$ is the conditional entropy of the random variable a given b.

The correlation matrix for vector z, with $E[u_0^2] = P^2$, is

$$C_z = \begin{bmatrix} \lambda_1^2 P^2 + \sigma_{w_1}^2 & \dots & \lambda_1 \lambda_n P^2 \\ \lambda_n \lambda_1 P^2 & \dots & \lambda_n^2 P^2 + \sigma_{w_n}^2 \end{bmatrix}$$

which has the determinant

$$|C_z| = \prod_{i=1}^{n} \sigma_{w_i}^2 \left(1 + P^2 \sum_i \frac{\lambda_i^2}{\sigma_{w_i}^2}\right). \tag{3.7}$$

Since for any n-variate random vector with fixed mean and covariance matrix the maximum entropy is attained by a normal distribution with an appropriate density function (see, for example, Kagan et al. [5]), we have

$$\begin{aligned} H\{z\} &\leqslant \tfrac{1}{2} \log(2\pi e)^n |C_z| \\ &= \tfrac{1}{2} \log(2\pi e)^n \prod_{i=1}^{n} \sigma_{w_i}^2 (1 + P^2 \lambda) \end{aligned} \tag{3.8}$$

where $\lambda \triangleq \sum_i \lambda_i^2 / \sigma_{w_i}^2$.

Using (3.8) in (3.6) we get

$$I\{m;\, z\} \leqslant \frac{1}{2} \log\left(1 + P^2 \sum_i \frac{\lambda_i^2}{\sigma_{w_i}^2}\right) \tag{3.9}$$

where we have made use of the expression

$$H\{z \mid m\} = \tfrac{1}{2} \log(2\pi e)^n \prod_{i=1}^{n} \sigma_{w_i}^2,$$

since for fixed γ_0', the vector z given m consists of n independent Gaussian random variables.

Using (3.5) and (3.9) in (3.4) we get

$$E_{E[\gamma_0'^2] = P^2}\left\{(u_1 - m)^2\right\} \geqslant \frac{\sigma_m^2}{1 + P^2 \lambda}. \tag{3.10}$$

It follows from (3.2) (using (3.3) and (3.10)) that

$$\begin{aligned} J_p &\geqslant k_0 P^2 - |s_0| P \sigma_m + \frac{\sigma_m^2}{1 + P^2 \lambda} + 1 - \sigma_m^2 \\ &\geqslant \min_{P \geqslant 0} \left[k_0 P^2 - |s_0| P \sigma_m + \frac{\sigma_m^2}{1 + P^2 \lambda} + 1 - \sigma_m^2\right] \\ &= k_0 P^{*2} - |s_0| P^* \sigma_m + \frac{\sigma_m^2}{1 + P^{*2} \lambda} + 1 - \sigma_m^2 \end{aligned} \tag{3.11}$$

where

$$P^* = \text{Arg} \min_P \left[k_0 P^2 - |s_0| P\sigma_m + \frac{\sigma_m^2}{1+P^2\lambda} \right] > 0.$$

Note that P^* necessarily exists, since at $P = 0$ the function to be minimized is decreasing and as $P \to \infty$, $J_p \to \infty$, implying that the search can be confined to a closed bounded region of $\mathbf{R}^1$ over which a continuous function always admits a minimum. Taking derivatives, we find that the value $P = P^*$ which attains this minimum satisfies

$$(2k_0P^* - |s_0|\sigma_m)\left(P^{*2} + \frac{1}{\lambda}\right)^2 = \frac{2P^*\sigma_m^2}{\lambda}. \tag{3.12}$$

We now have

$$J_{\text{opt}} \triangleq \inf_{\gamma_0,\gamma_1} J(\gamma_0, \gamma_1) = \inf_{P \geqslant 0} J_p$$

$$\geqslant k_0 P^{*2} - (|s_0|\sigma_m)P^* + \frac{\sigma_m^2}{1+P^{*2}\lambda} + 1 - \sigma_m^2 \tag{3.13}$$

which gives us a lower bound for the infimum of J.

We now finally show that this lower bound is tight and can be achieved by some linear policies γ_0 and γ_1. Starting with $u_0 = \gamma_0(y) = \beta y$ and $u_1 = \gamma_1(z) = E[x\,|\,z]$, we first find the corresponding cost to be

$$J(\beta) = k_0\beta^2(1+\sigma_v^2) + s_0\beta + \frac{\sigma_m^2}{1+\beta^2(1+\sigma_v^2)\lambda} + 1 - \sigma_m^2,$$

and then observe that this becomes identical to (3.13) if we choose

$$\beta = \beta^* = -(\text{sgn } s_0)P^*\sigma_m. \tag{3.14}$$

This then leads to Theorem 1 below.

Theorem 1. (i) *Problem* P1 *admits an optimum solution which is linear in the measurements available to the decision makers. The corresponding gain coefficients may be found by solving for the roots of a fifth order polynomial which is given by equation* (3.12).

(ii) *If P^* denotes the root of* (3.12) *rendering a minimum value of J_p, the optimum solution to* P1 *is given by*

$$\gamma_0^*(y) = \beta^* y$$

and

$$\gamma_1^*(z) = E[x\,|\,z] = \frac{1}{1+\sigma_v^2}\sum_{i=1}^{n} \frac{\lambda_i\beta}{\sigma_{w_i}^2\left[\left(1/(1+\sigma_v^2)\right)+\beta^2\lambda\right]} z_i$$

where β^* and P^* are related by (3.14).

(iii) *The minimum achievable value of the performance index is*

$$J(\beta^*) = k_0\beta^{*2}(1+\sigma_v^2) + s_0\beta^* + \frac{1}{(1+\sigma_v^2)\left[1+\beta^{*2}\lambda(1+\sigma_v^2)\right]} + \frac{\sigma_v^2}{1+\sigma_v^2}.$$

4. An extension

We now restrict ourselves to the case with $n = 1$ and show that in this case the channel noise may be allowed to be correlated with the channel input while the search for an optimum solution may still, without any loss of generality, be confined to linear strategies. Towards this end we consider the scalar team decision problem P3 below.

Problem P3.

Minimize $\quad J = E\left[k_0u_0^2 + s_0u_0x + (u_1 - x)^2\right]$

over Borel measurable maps γ_0 and γ_1, where

$$u_0 = \gamma_0(x+v)$$

and

$$u_1 = \gamma_1(u_0 + \alpha(x+v) + w).$$

Here x, v and w are independent Gaussian random variables with variances 1, σ_v^2 and σ_w^2, respectively.

Towards obtaining the solution of P3, we define $u_0' = u_0 + \alpha(x+v)$ and rewrite J as

$$J = E\left[k_0(u_0' - \alpha y)^2 + s_0(u_0' - \alpha y)x + (u_1 - x)^2\right]$$

with

$$u_0' = \gamma_0'(x+v), \qquad u_1 = \gamma_1(u_0' + w)$$

and

$$y = x + v.$$

As done earlier for the n-dimensional case, we introduce $m = E[x \mid y]$, and rewrite the above optimization problem as

$$\text{Minimize} \quad J = E\Big[k_0 u_0'^2 + s_0' u_0' m + (u_1 - m)^2 + c\Big]$$

over γ_0', γ_1 where

$$u_0' = \gamma_0'(m) \quad \text{and} \quad u_1 = \gamma_1(u_0' + w),$$

and c is a constant that does not depend on either u_0' or u_1. Now, in this new setting, an analysis identical to the one for the multichannel case yields optimum linear strategies for this problem, and we have Theorem 2 below.

Theorem 2. *The stochastic team decision Problem* P3 *admits a solution which is linear in the observation variables, with the corresponding gain coefficients obtained by solving for the roots of a fifth order polynomial, i.e., the optimum decision rules* $(\hat{\gamma}_0, \hat{\gamma}_1)$ *are given by*

$$\hat{\gamma}_0(y) = \hat{\beta}^* y$$

and

$$\hat{\gamma}_1(z) = \frac{\hat{\beta}^* + \alpha}{(\hat{\beta}^* + \alpha)^2(1 + \sigma_v^2) + \sigma_w^2} z.$$

Here

$$\hat{\beta}^* = \operatorname{Arg} \min_{\beta} J_\beta$$

with

$$J_\beta \triangleq k_0 \beta^2 (1 + \sigma_v^2) + s_0 \beta + \frac{(\beta + \alpha)^2 \sigma_v^2 + \sigma_w^2}{(\beta + \alpha)^2 (1 + \sigma_v^2) + \sigma_w^2},$$

i.e., $\hat{\beta}^*$ *is the solution to*

$$\left[2k_0 \hat{\beta}^* (1 + \sigma_v^2) + s_0\right]\left[(\hat{\beta}^* + \alpha)^2 (1 + \sigma_v^2) + \sigma_w^2\right]^2 = 2\sigma_w^2 (\hat{\beta}^* + \alpha)$$

which renders minimum value to J_β.

5. Conclusion

The classical information transmission problem, viewed as a team problem, generally assumes that one can directly encode the variable to be recovered at the receiving end. A model which allows distortion prior to transmission was first considered by Dobrushin and Tsybakov [3]. If we suppose that the message to be transmitted is a temperature read by a digital thermometer or the pixel levels of an image, then the message would not be an exact copy of the object of interest but a noise bearing variant of it, and the analysis in this paper would then be applicable.

A special case, however, is when an ungarbled version of the random variable of interest is indeed available at the encoder. It is notable that in order to establish the optimality of linear solutions for LQG team problems in the absence of cross terms between the decision variables, Bansal and Başar [1] have used this special case with the further restriction that $n = 1$.

We also find that for the case where the channel noise is uncorrelated with the encoder input, the nature of the solution is such that it may be considered as first extracting the message from its noisy version under a mean square criterion and then transmitting this extracted message. This is in accordance with the scheme reported for the hard power constraint version with $n = 1$ by Wolf and Ziv [13], as well as with the 'disconnection principle' introduced for finite alphabets by Witsenhausen [12].

We should also note that channels of the form considered in this paper have been called multipath channels in the literature (Ovseevich and Pinsker [6]). In such channels, even though there is a single transmitter, the reception is as though a number of channels are operative in parallel. Pinsker [7] also gives expressions and estimates for the quantity of information contained in observations with respect to an estimated parameter for a fixed and random number of observations. For the special case where each of the λ_i's are assumed to be unity, our result on maximum information between m and z (eqn. (3.9)) corresponds to the expression derived in Pinsker [7] when the number of observations is fixed, and the estimated parameter and the noises are all Gaussian. Pinsker's approach, however, is based on sufficient statistics whereas here we employed known results on entropy maximization.

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