Incentive-Based Pricing for Network Games with Complete and Incomplete Information*

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Abstract

In this paper, we introduce the concept of dynamic pricing within the context of our previous Stackelberg network game model and view the ISP's policy as an incentive policy, and the underlying game as a reverse Stackelberg game. We study this incentive-design problem under complete information as well as incomplete information. In both cases, we show that the game is not generally *incentive controllable* (that is, there may not exist pricing policies that would lead to attainment of a Pareto-optimal solution), but it is ϵ-incentive controllable (that is, the ISP can come arbitrarily close to a Pareto-optimal solution). The paper also includes a comparative study of the solutions under static and usage-based pricing policies, illustrated by numerical computations.

Index Terms

Dynamic pricing, Incentive policy, Reverse Stackelberg game, Incentive controllability, Pareto-optimal solution, Team solution, Incentive-design problem, Complete information, Incomplete information.

I. Introduction

Recent years have seen a surge of activity in the use of the framework and tools of game theory in studying pricing issues in communication networks, and particularly Internet pricing. Basic knowledge of game theory can be acquired from books such as [5] and [6], while [7] and [8] provide an introduction to this specific research area of network pricing.

The starting point for our work here is the Stackelberg game model formulated in [1]. In this model, the natural players are the Internet Service Providers (ISPs) and the individual users, where the ISPs are the *leaders* and the users are the *followers*. The leaders set the prices for the resources they offer (in this case, bandwidth) and the followers respond by their choice of the amount of bandwidth (or flow) that they are willing to pay for. The payoff for each ISP is the total revenue it collects (minus a fixed cost), and for each user it captures a tradeoff between the desire for high bandwidth (or flow) and despise for high congestion cost due to bottleneck links and high payment for flow. The presence of congestion cost in the net utility functions of users leads to a natural interaction (coupling) between the decisions of all users using a particular link, which in turn necessitates the modeling of the decision making process among the users (for each fixed pricing policy of the ISPs) as a noncooperative game, with Nash equilibrium being a natural candidate for a solution. In the case of multiple ISPs, similarly the Nash equilibrium solution concept can be adopted at the higher level of the Stackelberg game.

Our earlier work on this class of problems has looked at static pricing policies for such network games, and has obtained results on the existence and the uiqueness of an equilibrium, as well as its characterization. This has led to appealing admission policies as well as capacity-expansion schemes when the user population is large. In particular, [1] and [2] have studied the interrelation of the ISP and the users, under uniform pricing and differentiated pricing, respectively. Furthermore, we have extended the model to an environment where the users types may not be known to the ISPs, and studied this network game under incomplete information in [3].

In this paper, we consider an extension of the earlier model in a significant new direction. As in [4], we allow the leader's (let us say there is only one ISP, but still multiple users) policy to be *dynamic*, in the sense that it is allowed to depend on the actions of the users (that is, we have a usage-based policy). Hence, the ISP's policy may also be viewed as an incentive policy, and the underlying game a reverse Stackelberg game ([5], pp. 392–396). Intuitively, by turning to a dynamic policy from a static one, the ISP takes some control over the users' flows, and consequently could expect an improved profit. On the other hand, a user's payoff cannot deteriorate too much because of the fact that he always has the choice of not participating, which imposes a restriction on the incentive policies that the ISP is allowed to choose.

Our goal here is to find a Pareto-optimal solution, what we refer to as the *team solution*, and to obtain the incentive policies to achieve that optimal, what we call the *solution of the incentive-design problem*. We first consider the complete

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information game, where the ISP knows all user types, and hence can deduce their (unique) responses to an announced usage-based pricing policy. Subsequently, we study the incomplete information game where the ISP knows only the probability distribution of the user types, and hence cannot fully deduce the follower responses.

The paper is organized as follows. We first formulate the problem for the single user case, under complete information as well as incomplete information. Then the complete information game and the incomplete information game are dealt with in the next two sections III and IV, respectively. In Section III, which deals with complete information, the team solution is obtained, following by the discussion on incentive controllability, linear incentives, quadratic incentives and general incentives. The team solution is also compared with the Stackelberg game solution obtained in [1]. In Section IV for incomplete information, some properties of the optimal solution are discussed first, which leads to an inductive method to solve for the team solution as well as the solution of the incentive-design problem. Numerical examples are also provided to illustrate the idea. Finally, we provide formulation for the incentive-design problem for the multiple user case and conclude the paper with some discussion.

II. Incentive-Design Problem Formulation

Consider a link of capacity nc accessed by n users, and the Stackelberg game model formulated in [1] and generalized in [2]. Let x_i be the flow of User i and p_i be the price per unit flow charged to him by the ISP. Then, User i's net utility is

$$F_{w_i}(x_i, x_{-i}; p_i) = w_i \log(1 + x_i) - \frac{1}{nc - x_i - x_{-i}} - p_i x_i,$$

where w_i is a positive parameter representing User i's type and $x_{-i} := \sum_{j=1}^{n} x_j - x_i$. Now given the prices announced by the leader (ISP), the n followers (users) play a noncooperative game, which admits a Nash equilibrium $\{x_i^s(p_1, \cdots, p_n) \geq 0\}_{i=1}^n$ satisfying

$$\max_{x_i \in [0, nc - x_{-i}^s(p_1,\cdots,p_n)]} F_{w_i}(x_i, x_{-i}^s(p_1, \cdots, p_n); p_i) = F_{w_i}(x_i^s(p_1, \cdots, p_n), x_{-i}^s(p_1, \cdots, p_n); p_i)$$

for all $i \in \{1, \cdots, n\}$. The existence and the uniqueness of such a Nash equilibrium for a given n-tuple $\{p_i\}$ has been established in [1]. Then the ISP's optimization problem is

$$\max_{\{p_i \geq 0\}_{i=1}^n} \sum_{i=1}^{n} p_i x_i^s(p_1, \cdots, p_n),$$

which also admits a unique solution [2].

In this paper, we allow the ISP's pricing policy to be dynamic and reframe the problem as a reverse Stackelberg game. Instead of setting a fixed unit price p_i for User i, and thus charging User i the amount $p_i x_i$, the ISP announces the total charge to him, $r_i = \gamma_i(x_i)$, as a function of his flow, which is not necessarily linear. We start with a special case with a single user and c taking the value of 1. Then the user's net utility can be expressed as

$$F_w(x; r) := w \log(1 + x) - \frac{1}{1 - x} - r$$

for $0 < x < 1$; for $x = 0$, $r \equiv 0$ and $F_w(0; r) \equiv -1$. To compute the "team solution", which is defined as the action outcome desired by the leader (ISP), we identify two possible cases.

A. Complete Information

In this case, we assume that the user's type, w, is known to the ISP. As stated in [5], pp. 392–396, the team solution is

$$(x^t, r^t) = \arg \max_{0 \leq x < 1, r \geq 0} r, \tag{1}$$

$$\text{s. t.} \quad F_w(x; r) \geq -1, \tag{2}$$

which we assume at this point to exist. The constraint (2) comes from the fact that the user always has the choice of not participating, which guarantees a minimum net utility of -1 for him.

Now the incentive-design problem is to find a $\gamma : [0, 1) \to \mathcal{R}$, such that

$$\arg \max_{0 \leq x < 1} F_w(x; \gamma(x)) = x^t, \tag{3}$$

$$\gamma(x^t) = r^t. \tag{4}$$

Note that $\gamma(0) \equiv 0$.

B. Incomplete Information

In this case, the user's type, w, is not revealed to the ISP; rather, the ISP knows only the distribution of the user's type, say (assuming a discrete distribution), $w = w^i$ w.p. $q_i \in (0,1)$, $i = 1, \cdots, m$, where $\sum_{j=1}^m q_j = 1$. Then the team solution becomes

$$\{(x^{it}, r^{it})\}_{i=1}^m = \quad \arg \max_{\{0 \le x^i < 1, r^i \ge 0\}_{i=1}^m} \{E[r] = \sum_{j=1}^m q_j r^j\}, \tag{5}$$

$$\text{s. t.} \quad F_{w^i}(x^i; r^i) \ge -1, \quad 1 \le i \le m, \tag{6}$$

$$F_{w^i}(x^i; r^i) \ge F_{w^i}(x^j; r^j), \quad 1 \le i \ne j \le m, \tag{7}$$

where we again assume at this point that a solution exists. The constraint (7) is necessary such that a user with the type w^i will choose (x^{it}, r^{it}) which is the flow-charge pair desired for him.

As a result, the incentive-design problem is to find a $\gamma : [0,1) \to \mathcal{R}$, such that for $1 \le i \le m$,

$$\arg \max_{0 \le x < 1} F_{w^i}(x; \gamma(x)) = x^{it}, \tag{8}$$

$$\gamma(x^{it}) = r^{it}, \tag{9}$$

and $\gamma(0) \equiv 0$.

Following the definition in [5], pp. 392–396, the incentive problem defined above is (linearly) incentive controllable if there exists a (linear) γ-function with $\gamma(0) \equiv 0$ such that (3) and (4) for complete information, or (8) and (9) for incomplete information, are satisfied, respectively.

III. Complete Information

A. Team Solution

For convenience, define

$$Q(x; w) := w \log(1 + x) - \frac{1}{1 - x} + 1.$$

Then we can easily see that $F_w(x; r) = Q(x; w) - 1 - r$. Hence, (2) is equivalent to $Q(x; w) \ge r$. If $w \le 1$, $Q(x; w) < 0$ for $0 < x < 1$. Obviously, the only feasible point satisfying (2) is $(x, r) = (0, 0)$. In this case, the user will always choose not to participate. On the other hand, if $w > 1$, then $Q(x; w) > 0$ for $0 < x < x^w_{\max} < 1$, where $x^w_{\max}$ can be obtained by solving $Q(x; w) = 0$. Actually, to maximize r, we must have $r = Q(x; w)$. Since $Q(x; w)$ is strictly concave in x for $x \in [0, 1)$, its maximum is achieved at the point where $\frac{\partial Q(x;w)}{\partial x} = 0$, provided that such a point exists in $[0, 1)$. In fact it does, and we have

$$w = \frac{1 + x}{(1 - x)^2} =: \alpha^{-1}(x),$$

or equivalently,

$$x = \frac{1 + 2w - \sqrt{1 + 8w}}{2w} =: \alpha(w).$$

Note that the function defined above, $\alpha : [1, \infty) \to [0, 1)$, is strictly increasing and thus its inverse is also well-defined. Furthermore, extend the definition of α to $[0, 1)$ so that $\alpha(w) \equiv 0$ for $w \le 1$; note that it is continuous at $w = 1$. Finally, we obtain the unique team solution as

$$x^t = \alpha(w),$$
$$r^t = Q(\alpha(w); w).$$

B. Solution of the Incentive-Design Problem

1) Incentive Controllability: After the team solution is obtained, the incentive controllability of the problem is to be studied next. Suppose there exists a γ-function as stated in the definition. Combining (3) and (4), we know that $F_w(x; \gamma(x))$ for $0 \le x < 1$ achieves its maximum $F_w(x^t; r^t) = -1$ at (x^t, r^t). Note that besides (x^t, r^t), γ also needs to go through $(0, 0)$, for which we have $F_w(0; 0) = -1$. In other words, (x^t, r^t) cannot be the unique maximizing point. Since the user is indifferent at these two points, and there is always a possibility that the user may choose his flow to be zero, which will lead to no profit for the ISP, thus, strictly to say, the problem is not incentive controllable.

However, by applying the same technique as in Example 7.4 of [5], we can show that the problem is ϵ-incentive controllable. A problem is called ϵ-incentive controllable if there exists an incentive design such that the leader can come arbitrarily close to the team solution (x^t, r^t). The method here is to first find a γ-function with $\gamma(0) \equiv 0$ satisfying (3) and (4), and then make a small "dip" in the feasible set near (x^t, r^t) to guarantee the uniqueness of the maximizing point. To illustrate this, we first study linear incentives and then proceed to more general incentives. We assume that $w > 1$, since otherwise x^t is not positive.

2) Linear Incentive: A linear function γ going through $(0,0)$ and (x^t, r^t) must be $\gamma(x) = x \cdot r^t/x^t$. Now we need to see whether this linear function satisfies (3) or not. Note that for $0 < a < 1$, $\gamma(ax^t) = ar^t$. Thus, (ax^t, ar^t) is some point along this line between $(0,0)$ and (x^t, r^t). On the other hand, remember that $0 = Q(0; w)$ and $r^t = Q(x^t; w)$, where $Q(x; w)$ is strictly concave in x. Hence, $Q(ax^t; w) > ar^t$. As a result,

$$F_w(ax^t; \gamma(ax^t)) = Q(ax^t; w) - 1 - ar^t > -1 = F_w(x^t; r^t).$$

Therefore, (3) cannot be satisfied and the problem is not linearly incentive controllable. Note that a linear γ corresponds to the constant unit price scheme in [1] and [2], which shows that the classical Stackelberg version of the pricing problem cannot admit a solution that is team-optimal.

3) Quadratic Incentive: Now suppose that $\gamma(x) = a_1 x + a_2 x^2$, which is a quadratic function satisfying $\gamma(0) = 0$. For (3) and (4) to hold, we need to have

$$a_1 x^t + a_2 (x^t)^2 = r^t, \tag{10}$$

$$\frac{d}{dx} F_w(x; \gamma(x))|_{x=x^t} = \frac{\partial Q(x; w)}{\partial x}|_{x=x^t} - \frac{d\gamma(x)}{dx}|_{x=x^t} = \frac{\partial Q(x; w)}{\partial x}|_{x=x^t} - (a_1 + 2a_2 x^t) = 0. \tag{11}$$

Recall that $\frac{\partial Q(x;w)}{\partial x}|_{x=x^t} = 0$. Thus, from the above two equations, we obtain that

$$a_2 = -\frac{r^t}{(x^t)^2} \quad \text{and} \quad a_1 = -2a_2 x^t = \frac{2r^t}{x^t}.$$

Note that (11) is necessary but not sufficient for (3) to hold. We still need to verify (3) for the above obtained quadratic function.

First, we can prove the following inequality (see Appendix I):

$$\frac{d^2}{dx^2} F_w(x; \gamma(x))|_{x=x^t} = \frac{\partial^2 Q(x; w)}{\partial x^2}|_{x=x^t} - \frac{d^2\gamma(x)}{dx^2}|_{x=x^t} < 0.$$

Thus, $F_w(x; \gamma(x))$ achieves a local maximum at x^t.

Next, we need to show that for $0 \le x < 1$, $F_w(x; \gamma(x)) \le F_w(x^t; \gamma(x^t)) = -1$, or equivalently, $Q(x; w) \le \gamma(x)$. Note that $Q(0; w) = \gamma(0) = 0$ and $Q(x^t; w) = \gamma(x^t) = r^t$. For x other than 0 and x^t, we actually have the following result:

Proposition 1: For $0 < x \ne x^t < 1$, $Q(x; w) < \gamma(x)$.

Proof: To see the proposition, it is convenient to compare $\eta(x; w) := Q(x; w)/x$ with $\xi(x) := \gamma(x)/x = a_1 + a_2 x$ that is a linear function, and equivalently show that $\eta(x; w) < \xi(x)$ for $0 < x \ne x^t < 1$.

First, since $Q(x; w)$ and $\gamma(x)$ coincide at (x^t, r^t) and so do their first-order derivatives, we have

$$\frac{\partial \eta(x; w)}{\partial x}|_{x=x^t} = \left[\frac{1}{x}\frac{\partial Q(x; w)}{\partial x} - \frac{1}{x^2}Q(x; w)\right]|_{x=x^t} = \left[\frac{1}{x}\frac{d\gamma(x)}{dx} - \frac{1}{x^2}\gamma(x)\right]|_{x=x^t} = \frac{d\xi(x)}{dx}|_{x=x^t}.$$

Hence, $\eta(x; w)$ and $\xi(x)$ are tangent at $(x^t, r^t/x^t)$.

Furthermore, since $Q(x; w)$ has a smaller second-order derivative than $\gamma(x)$ does at (x^t, r^t),

$$\begin{aligned}\frac{\partial^2 \eta(x; w)}{\partial x^2}|_{x=x^t} &= \left[\frac{1}{x}\frac{\partial^2 Q(x; w)}{\partial x^2} - \frac{2}{x^2}\frac{\partial Q(x; w)}{\partial x} + \frac{2}{x^3}Q(x; w)\right]|_{x=x^t} \\ &< \left[\frac{1}{x}\frac{d^2\gamma(x)}{dx^2} - \frac{2}{x^2}\frac{d\gamma(x)}{dx} + \frac{2}{x^3}\gamma(x)\right]|_{x=x^t} = \frac{d^2\xi(x)}{dx^2}|_{x=x^t} = 0,\end{aligned}$$

which means that $\eta(x; w)$ is strictly concave at $(x^t, r^t/x^t)$.

We can also prove that $\frac{\partial^3 \eta(x;w)}{\partial x^3} < 0$ for $0 < x < 1$ (see Appendix II). Thus, as x increases from 0 to 1, $\eta(x; w)$ is either concave, or first convex and then concave. For the former case, obviously the proposition holds; for the latter one, to reach the same conclusion, it is sufficient to show that $\lim_{x \to 0^+} \eta(x; w) = w - 1 < \xi(0) = a_1$, which can be verified to be true (see Appendix II). ■

Hence we have proved that the above obtained $\gamma(x) = a_1 x + a_2 x^2$ is the unique quadratic incentive such that $\gamma(0) = 0$ and (3) and (4) hold. Next, we make a small "dip" in the feasible set near the team solution. For example, we can keep all the points along $\gamma(x)$ except for replacing (x^t, r^t) with $(x^t, r^t - \epsilon)$. For this revised incentive, $(x^t, r^t - \epsilon)$ becomes the unique maximizing point for the user with a net utility of $-1 + \epsilon$, which guarantees a profit of $r^t - \epsilon$ for the ISP.

4) General Incentive: To further understand the incentive-design problem, we provide a graphical illustration in Figure 1. Take $w = 100$ as an example. The feasible set, as defined in (1) and (2), is the region below $F_w(x; r) = -1$, or equivalently, $r = Q(x; w)$. Then the team solution can be obtained as $x^t = 0.8635$ and $r^t = 55.9196$. The quadratic incentive obtained previously, $\gamma(x) = a_1 x + a_2 x^2$, where $a_1 = 129.5200$ and $a_2 = -74.9980$, is also depicted in the figure. In fact, any incentive policy which coincides with $F_w(x; r) = -1$ at $(0,0)$ and (x^t, r^t) and falls above $F_w(x; r) = -1$ at all other points satisfy (3) and (4). Then, by making a small "dip" in the feasible set near the team solution, the ISP can come arbitrarily close to the optimal profit r^t, leaving the user with a net utility that is slightly higher than $F_w(x^t; r^t) = -1$.

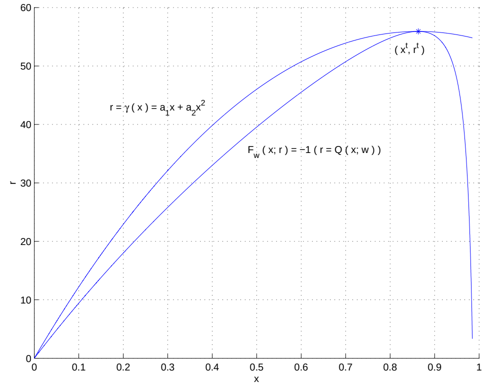

Fig. 1. Graphical illustration of incentives for the complete information case ($w = 100$)

C. *Team Solution vs Stackelberg Game Solution*

Now that the problem is ϵ-incentive controllable, we compare the team solution with the Stackelberg game solution. It has been shown in [1] that under the Stackelberg game model, the user's flow and the optimal profit for the ISP are

$$x^s = \frac{w^{\frac{1}{3}} - 1}{w^{\frac{1}{3}} + 1}, \quad \text{and} \quad r^s = (w^{\frac{1}{3}} - 1)^2(\frac{1}{2}w^{\frac{1}{3}} + \frac{1}{4}),$$

given $w > 1$. Figure 2 shows for different values of w the comparison of the team solution with the Stackelberg game solution. Not surprisingly, adoption of the incentive policy improves the ISP's profit. From the right side, we can see that for w near 1, r^t is almost two times r^s and as w increases to 200, r^t is still around 60 percent more than r^s.

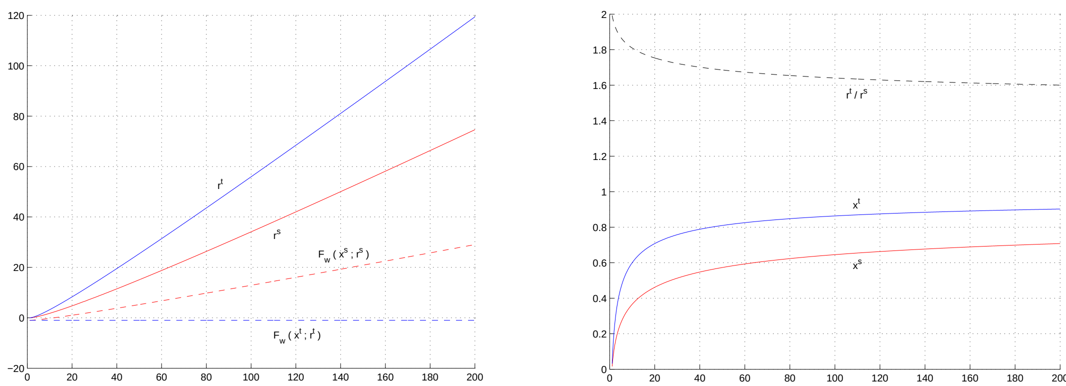

Fig. 2. Comparison of the team solution with the Stackelberg game solution for the complete information case

IV. Incomplete Information

A. *Constraint Reduction*

From the analysis for complete information, we know that if a user's type as captured by the parameter w is less than 1, he will always choose not to participate. Thus, W.L.O.G., we can assume that $w^1 > \cdots > w^m > 1$. To compute the team solution, we first deduce some properties that this solution must satisfy.

Lemma 1: Given $w^i > w^j > 1$, for any $x^i \in [0, 1)$, $x^j \in [0, 1)$, $r^i \geq 0$ and $r^j \geq 0$ such that

$$F_{w^i}(x^i; r^i) \geq F_{w^i}(x^j; r^j) \quad \text{and} \quad F_{w^j}(x^j; r^j) \geq F_{w^j}(x^i; r^i) \tag{12}$$

are satisfied, we must have $x^i \geq x^j$.

Proof: Since $w^i > w^j$, (12) implies that

$$F_{w^i}(x^i; r^i) \geq F_{w^i}(x^j; r^j) \geq F_{w^j}(x^j; r^j) \geq F_{w^j}(x^i; r^i).$$

Hence, we have

$$F_{w^i}(x^i; r^i) - F_{w^j}(x^i; r^i) = (w^i - w^j)\log(1 + x^i) \geq F_{w^i}(x^j; r^j) - F_{w^j}(x^j; r^j) = (w^i - w^j)\log(1 + x^j),$$

which means $x^i \geq x^j$. ∎

A direct result of Lemma 1 is as follows:

Corollary 1: For $w^1 > \cdots > w^m > 1$, $x^{1t} \geq \cdots \geq x^{mt} \geq 0$. Futhermore,

$$F_{w^i}(x^{it}; r^{it}) \geq F_{w^i}(x^{jt}; r^{jt}) \geq F_{w^j}(x^{jt}; r^{jt}) \geq F_{w^j}(x^{it}; r^{it}), \quad 1 \leq i < j \leq m.$$

Now we are ready to prove the following theorem, which plays an important role in the derivation of $\{(x^{it}, r^{it})\}_{i=1}^m$.

Theorem 1: For $w^1 > \cdots > w^m > 1$, to compute the action outcome desired by the ISP, $\{(x^{it}, r^{it})\}_{i=1}^m$, the constraints (6) and (7) can be simplified as

$$x^1 \geq \cdots \geq x^m \geq 0; \quad F_{w^m}(x^m; r^m) = -1 \text{ and } F_{w^i}(x^i; r^i) = F_{w^i}(x^{i+1}; r^{i+1}), \quad 1 \leq i \leq m-1. \tag{13}$$

Proof: By Corollary 1, $\{(x^{it}, r^{it})\}_{i=1}^m$ must satisfy $x^{1t} \geq \cdots \geq x^{mt} \geq 0$. We prove the second half of the theorem in several steps.

Step 1: From Corollary 1, we know that for any $i < m$, the solution must satisfy

$$F_{w^i}(x^{it}; r^{it}) \geq F_{w^i}(x^{mt}; r^{mt}) \geq F_{w^m}(x^{mt}; r^{mt}).$$

Therefore, the constraint (6) can be reduced to $F_{w^m}(x^m; r^m) \geq -1$. Furthermore, from the definition of $F_w(x; r)$, to maximize the objective function in (5), we should make r^i's as large as possible, or equivalently, $F_{w^i}(x^i; r^i)$'s as small as possible. Therefore, the equality actually holds for (6), which can be written as

$$F_{w^m}(x^m; r^m) = -1.$$

The constraint (7) will not affect this equality since it is a "soft" constraint instead of being a "hard" one. By "hard" constraint, we mean that some given constant bounds are involved; otherwise, a constraint is called a "soft" one.

Step 2: Fix (x^m, r^m) such that $x^m \geq 0$ and $F_{w^m}(x^m; r^m) = -1$. Then (7) can be rewritten as

$$F_{w^i}(x^i; r^i) \geq F_{w^i}(x^m; r^m) \text{ and } F_{w^m}(x^i; r^i) \leq F_{w^m}(x^m; r^m), \quad 1 \leq i \leq m-1; \tag{14}$$

$$F_{w^i}(x^i; r^i) \geq F_{w^i}(x^j; r^j), \quad 1 \leq i \neq j \leq m-1. \tag{15}$$

In fact, for $1 \leq i < j \leq m-1$,

$$F_{w^i}(x^i; r^i) - F_{w^j}(x^j; r^j) \geq F_{w^i}(x^j; r^j) - F_{w^j}(x^j; r^j) \geq F_{w^i}(x^m; r^m) - F_{w^j}(x^m; r^m).$$

The first inequality comes from (15) and the second inequality holds since $x^j \geq x^m \geq 0$. Therefore, $F_{w^j}(x^j; r^j) \geq F_{w^j}(x^m; r^m)$ implies that $F_{w^i}(x^i; r^i) \geq F_{w^i}(x^m; r^m)$. On the other hand,

$$F_{w^m}(x^i; r^i) = F_{w^j}(x^i; r^i) - (w^j - w^m)\log(1 + x^i) \leq F_{w^m}(x^j; r^j) = F_{w^j}(x^j; r^j) - (w^j - w^m)\log(1 + x^j),$$

because of (15) and the fact that $x^i \geq x^j \geq 0$. So, $F_{w^m}(x^j; r^j) \leq F_{w^m}(x^m; r^m)$ implies that $F_{w^m}(x^i; r^i) \leq F_{w^m}(x^m; r^m)$. As a conclusion, (14) can be reduced to

$$F_{w^{m-1}}(x^{m-1}; r^{m-1}) \geq F_{w^{m-1}}(x^m; r^m) \text{ and } F_{w^m}(x^{m-1}; r^{m-1}) \leq F_{w^m}(x^m; r^m).$$

Note that (15) is a soft constraint. Thus, to maximize the objective function in (5), (14) can be further reduced to

$$F_{w^{m-1}}(x^{m-1}; r^{m-1}) = F_{w^{m-1}}(x^m; r^m),$$

and then the second half of the constraint is in fact equivalent to $x^{m-1} \geq x^m$.

Step 3: For $2 \leq k \leq m-1$, fix $\{(x^i, r^i)\}_{i=k}^m$ such that $x^k \geq \cdots \geq x^m \geq 0$, $F_{w^m}(x^m; r^m) = -1$ and $F_{w^i}(x^i; r^i) = F_{w^i}(x^{i+1}; r^{i+1})$ for $k \leq i \leq m-1$. Rewrite (15) as

$$F_{w^i}(x^i; r^i) \geq F_{w^i}(x^k; r^k) \text{ and } F_{w^k}(x^i; r^i) \leq F_{w^k}(x^k; r^k), \quad 1 \leq i \leq k-1; \tag{16}$$

$$F_{w^i}(x^i; r^i) \geq F_{w^i}(x^j; r^j), \quad 1 \leq i \neq j \leq k-1. \tag{17}$$

(17) is a soft constraint. For the hard constraint (16), following the similar reasoning as in Step 2, it can be reduced to

$$F_{w^{k-1}}(x^{k-1}; r^{k-1}) = F_{w^{k-1}}(x^k; r^k)$$

and $x^{k-1} \geq x^k$.

Step 4: By repeating Step 3 for k from $m-1$ to 2, we inductively prove the theorem.

■

Corollary 2: For $w^1 > \cdots > w^m > 1$, $\{(x^{it}, r^{it})\}_{i=1}^m$ must satisfy: $x^{1t} = \alpha(w^1)$ and $x^{it} \leq \alpha(w^i)$ for $2 \leq i \leq m$.

Proof: We prove this corollary by induction.

First, we must have $x^{mt} \leq \alpha(w^m)$. If this does not hold, i.e., $x^{mt} > \alpha(w^m)$, then we can compare r^{mt} with $r^{m'}$ defined as r^m achieved at $x^m = \alpha(w^m)$. In fact, from $F_{w^m}(x^m; r^m) = -1$, we know that $r^{m'} = Q(\alpha(w^m); w^m)$ and $r^{mt} = Q(x^{mt}; w^m)$. Since $Q(x^m; w^m)$ is strictly concave in x^m and reaches the maximum at $x^m = \alpha(w^m)$, we have that $r^{m'} > r^{mt}$, Thus, when x^m is reduced to $\alpha(w^m)$, r^m is actually improved. Furthermore, since x^m decreases and as a result $F_{w^{m-1}}(x^m; r^m) = F_{w^m}(x^m; r^m) + (w^{m-1} - w^m)\log(1 + x^m)$ decreases as well, the constraints in (13) are actually relaxed for all the other flow-charge pairs. Therefore, a higher value for $E[r]$ can be achieved, which contradicts to the assumption that x^{mt} is optimal.

Now, suppose for some k such that $2 \leq k \leq m$, $x^{it} \leq \alpha(w^i)$ holds for $k \leq i \leq m$. Note that $\alpha(w^{k-1}) > \alpha(w^k) \geq x^{it}$ for $k \leq i \leq m$. Then if $x^{k-1} > \alpha(w^{k-1})$, by reducing x^{k-1} to $\alpha(w^{k-1})$, the expected profit $E[r]$ can be further improved, following the similar reasoning as above. Therefore, $x^{(k-1)t} \leq \alpha(w^{k-1})$.

Finally, it is obvious that $x^{1t} = \alpha(w^1)$.

■

B. Team Solution

For the two equations in (13), $F_{w^m}(x^m; r^m) = -1$ implies $r^m = Q(x^m; w^m)$, while $F_{w^i}(x^i; r^i) = F_{w^i}(x^{i+1}; r^{i+1})$ implies $r^i = r^{i+1} + Q(x^i; w^i) - Q(x^{i+1}; w^i)$, for $1 \leq i \leq m-1$. Now we can compute the team solution by induction on m.

1) Two-Type Incomplete Information: For $w^1 > w^2 > 1$ and $q_1 + q_2 = 1$, $r^2 = Q(x^2; w^2)$ and $r^1 = r^2 + Q(x^1; w^1) - Q(x^2; w^1)$. Therefore, the goal is to maximize

$$E[r] = \sum_{j=1}^{2} q_j r^j = q_1 Q(x^1; w^1) - q_1 Q(x^2; w^1) + Q(x^2; w^2) = q_1 Q(x^1; w^1) + q_2 Q(x^2; \frac{w^2 - q_1 w^1}{q_2}).$$

Note that $v^{2/2} := \frac{w^2 - q_1 w^1}{q_2} < w^2$. We can easily see that $x^{1t} = \alpha(w^1)$ and $x^{2t} = \alpha(v^{2/2})$, which is 0 if $v^{2/2} \leq 1$. Then the maximum of $E[r]$ is $E[r^t] = q_1 Q(x^{1t}; \alpha^{-1}(x^{1t})) + q_2 Q(x^{2t}; \alpha^{-1}(x^{2t}))$. When $v^{2/2} \leq 1$, $x^{2t} = 0$ and $\alpha^{-1}(x^{2t})$ is not clearly defined; but this does not matter since $Q(0; w) \equiv 0$ for any w.

2) Three-Type Incomplete Information: For $w^1 > w^2 > w^3 > 1$ and $\sum_{j=1}^{3} q_j = 1$,

$$\begin{aligned} E[r] = \sum_{j=1}^{3} q_j r^j &= q_1 Q(x^1; w^1) - q_1 Q(x^2; w^1) + (q_1 + q_2) Q(x^2; w^2) - (q_1 + q_2) Q(x^3; w^2) + Q(x^3; w^3) \\ &= q_1 Q(x^1; w^1) + q_2 Q(x^2; \frac{(q_1 + q_2) w^2 - q_1 w^1}{q_2}) + q_3 Q(x^3; \frac{w^3 - (q_1 + q_2) w^2}{q_3}). \end{aligned}$$

Clearly, $v^{2/3} := \frac{(q_1+q_2)w^2 - q_1 w^1}{q_2} < w^2$ and $v^{3/3} := \frac{w^3 - (q_1+q_2)w^2}{q_3} < w^3$. We always have $x^{1t} = \alpha(w^1)$, while the values of x^{2t} and x^{3t} depend on how $v^{2/3}$ is compared with $v^{3/3}$. We discuss several possibilities.

(i) If $v^{2/3} \leq v^{3/3}$, then $x^{2t} = x^{3t} \leq \alpha(v^{3/3})$. So the expected profit can be expressed as

$$E[r] = q_1 Q(x^1; w^1) + (q^2 + q^3) Q(x^2; \frac{w^3 - q_1 w^1}{q_2 + q_3}).$$

Note that

$$v^{2/3} \leq v^{2,3/3} := \frac{w^3 - q_1 w^1}{q_2 + q_3} = \frac{q_2 v^{2/3} + q_3 v^{3/3}}{q_2 + q_3} \leq v^{3/3}.$$

From the discussion for the two-type case, we know that $x^{2t} = x^{3t} = \alpha(v^{2,3/3})$, which is 0 if $v^{2,3/3} \leq 1$.

(ii) If $1 \geq v^{2/3} > v^{3/3}$, $x^{2t} = x^{3t} = 0$.

(iii) If $v^{2/3} > 1 \geq v^{3/3}$, $x^{2t} = \alpha(v^{2/3})$ and $x^{3t} = 0$.

(iv) If $v^{2/3} > v^{3/3} > 1$, $x^{2t} = \alpha(v^{2/3})$ and $x^{3t} = \alpha(v^{3/3})$.

In fact, (ii), (iii) and (iv) can be summarized as: if $v^{2/3} > v^{3/3}$, $x^{2t} = \alpha(v^{2/3})$ and $x^{3t} = \alpha(v^{3/3})$. Finally, the optimal expected profit is $E[r^t] = \sum_{j=1}^{3} q_j Q(x^{jt}; \alpha^{-1}(x^{jt}))$.

3) Multiple-Type Incomplete Information: Suppose that $m \geq 4$, $w^1 > \cdots > w^m > 1$ and $\sum_{j=1}^{m} q_j = 1$. Then,

$$\begin{aligned} E[r] = \sum_{j=1}^{m} q_j r^j &= \sum_{k=1}^{m-1} \sum_{l=1}^{k} q_l [Q(x^k; w^k) - Q(x^{k+1}; w^k)] + Q(x^m; w^m) \\ &= q_1 Q(x^1; w^1) + \sum_{k=2}^{m} q_k Q(x^k; \frac{\sum_{l=1}^{k} q_l w^k - \sum_{l=1}^{k-1} q_l w^{k-1}}{q_k}). \end{aligned}$$

For $2 \leq k \leq m$, $v^{k/m} := \frac{\sum_{l=1}^{k} q_l w^k - \sum_{l=1}^{k-1} q_l w^{k-1}}{q_k} < w^k$. Then we have the following corollary as an extension of Corollary 2. The proof is similar, and hence is omitted here.

Corollary 3: For $w^1 > \cdots > w^m > 1$, $x^{it} \leq \max_{k=i}^{m} \alpha(v^{k/m})$ for $2 \leq i \leq m$.

Now we can discuss the optimal solution based on $v^{k/m}$'s.

(i) If $v^{m-1/m} \leq v^{m/m}$, then $x^{(m-1)t} = x^{mt} \leq \alpha(v^{m/m})$. So,

$$E[r] = q_1 Q(x^1; w^1) + \sum_{k=2}^{m-2} q_k Q(x^k; v^{k/m}) + (q_{m-1} + q_m) Q(x^{m-1}; \frac{w^m - \sum_{j=1}^{m-2} q_j w^{m-2}}{q_{m-1} + q_m}),$$

where

$$v^{m-1/m} \leq v^{m-1,m/m} := \frac{w^m - \sum_{j=1}^{m-2} q_j w^{m-2}}{q_{m-1} + q_m} \leq v^{m/m}.$$

Therefore, we can continue the discussion as for the $(m-1)$-type case and the solution can be obtained inductively.

(ii) If $1 \geq v^{m-1/m} > v^{m/m}$, $x^{(m-1)t} = x^{mt} = 0$. Then, $E[r] = q_1 Q(x^1; w^1) + \sum_{k=2}^{m-2} q_k Q(x^k; v^{k/m})$, for which we can proceed as in the $(m-2)$-type case.

(iii) If $v^{m-1/m} > 1 \geq v^{m/m}$, then $x^{mt} = 0$ and $E[r] = q_1 Q(x^1; w^1) + \sum_{k=2}^{m-1} q_k Q(x^k; v^{k/m})$. Proceed as in the $(m-1)$-type case.

(iv) If $v^{m-1/m} > v^{m/m} > 1$, then we need to further look at $v^{m-2/m}$.

(iv a) If $v^{m-2/m} \leq v^{m-1/m}$, then $x^{(m-2)t} = x^{(m-1)t} \leq \alpha(v^{m-1/m})$. Thus,

$$\begin{aligned} E[r] = {} & q_1 Q(x^1; w^1) + \sum_{k=2}^{m-3} q_k Q(x^k; v^{k/m}) \\ & + (q_{m-2} + q_{m-1}) Q(x^{m-2}; \frac{\sum_{j=1}^{m-1} q_j w^{m-1} - \sum_{j=1}^{m-3} q_j w^{m-3}}{q_{m-2} + q_{m-1}}) + q_m Q(x^m; v^m/m), \end{aligned}$$

where

$$v^{m-2/m} \leq v^{m-2,m-1/m} := \frac{\sum_{j=1}^{m-1} q_j w^{m-1} - \sum_{j=1}^{m-3} q_j w^{m-3}}{q_{m-2} + q_{m-1}} \leq v^{m-1/m}.$$

Then proceed as in the $(m-1)$-type case.

(iv b) If $v^{m-2/m} > v^{m-1/m}$, then we proceed to $v^{m-3/m}$, $v^{m-4/m}$ and so on if necessary as follows: if at some point we have $v^{i-1/m} \leq v^{i/m}$ for some i, then $x^{(i-1)t} = x^{it} \leq \alpha(v^{i/m})$ and the m-type case can be reduced to the $(m-1)$-type case similarly as discussed in (iv a); otherwise, $w^1 > v^{2/m} > \cdots > v^{m/m} > 1$ and as a result, $x^{1t} = \alpha(w^1)$ and $x^{it} = \alpha(v^{i/m})$ for $2 \leq i \leq m$.

In conclusion, $\{x^{it}\}_{i=1}^m$ can be obtained inductively and $E[r^t] = \sum_{j=1}^m q_j Q(x^{jt}; \alpha^{-1}(x^{jt}))$. $\{r^{it}\}_{i=1}^m$ are computed accordingly as $r^{mt} = Q(x^{mt}; w^m)$ and $r^{it} = r^{(i+1)t} + Q(x^{it}; w^i) - Q(x^{(i+1)t}; w^i)$ for $1 \leq i \leq m-1$.

C. Solution of the Incentive-Design Problem

Now that the team solution $\{x^{it}, r^{it}\}_{i=1}^m$ has been computed, we turn to finding an incentive policy γ satisfying (8) and (9). Note that $\gamma(0) \equiv 0$ and $\gamma(x^{it}) = r^{it}$ for $1 \leq i \leq m$. Yet from Theorem 1, we know that $F_{w^m}(x^{mt}; r^{mt}) = F_{w^m}(0; 0) = -1$ and $F_{w^i}(x^{it}; r^{it}) = F_{w^i}(x^{(i+1)t}; r^{(i+1)t})$ for $1 \leq i \leq m-1$. For the same reason as in the complete information case, the incentive-design problem for the incomplete information case is not incentive controllable but rather ϵ-incentive controllable. An incentive policy β such that the ISP comes arbitrarily close to the team solution can be achieved by making a small “dip” of γ near the team solution.

First, given the team solution, γ can be determined inductively according to Theorem 1.

Step 1: Start with $i = m$. Remember that $x^{1t} \geq \cdots \geq x^{mt} \geq 0$. If $x^{mt} = 0$, decrease i to the next smaller integer $m-1$. Repeat this until for some i, $x^{it} > 0$. Note that $F_{w^i}(x; r) = F_{w^i}(0, 0) = -1$ is equivalent to $r = Q(x; w^i)$, while $r^{it} = Q(x^{it}; w^i)$. Then we can choose $\gamma(0) = 0$, $\gamma(x) \geq Q(x; w^i)$ for $0 < x < x^{it}$ and $\gamma(x^{it}) = r^{it}$.

Step 2: Now start with $j = i - 1$ and repeatedly decrease j by 1 until for some j, $x^{jt} > x^{it}$. Note that $F_{w^j}(x; r) = F_{w^j}(x^{it}; r^{it})$ is equivalent to $r = Q(x; w^j) - 1 - F_{w^j}(x^{it}; r^{it})$, while $F_{w^j}(x^{jt}; r^{jt}) = F_{w^j}(x^{it}; r^{it})$ from Theorem 1 indicates that $r^{jt} = Q(x^{jt}; w^j) - 1 - F_{w^j}(x^{it}; r^{it})$. Therefore, we can choose $\gamma(x) \geq Q(x; w^j) - 1 - F_{w^j}(x^{it}; r^{it})$ for $x^{it} < x < x^{jt}$ and $\gamma(x^{jt}) = r^{jt}$.

Step 3: Let i take the value of j and repeat Step 2 until j reaches 1. Finally, for $x^{1t} < x < 1$, let $\gamma(x) \geq Q(x; w^1) - 1 - F_{w^1}(x^{1t}; r^{1t})$ and we obtain γ that satisfies (8) and (9).

Next, to guarantee the uniqueness of the optimal solution, we may revise γ in the following way to obtain β. Remember that in the above process to determine γ, we have a number i after Step 1 such that $x^{kt} = 0$ for $i < k \leq m$ and $x^{it} > 0$. Correspondingly, here we let $\beta(x^{kt}) = r^{kt} - \epsilon^k$, where $\epsilon^k = 0$ for $i < k \leq m$ and $\epsilon^i > 0$ for $k = i$. Then, $F_{w^i}(x^{it}; \beta(x^{it})) = -1 + \epsilon^i > -1$. Note that ϵ^i should be small enough such that $F_{w^{i+1}}(x^{it}; \beta(x^{it})) < -1$, which can be achieved since $F_{w^{i+1}}(x^{it},\beta(x^{it})) < F_{w^i}(x^{it},\beta(x^{it}))$. The counterpart of the above Step 2 here is to choose $\beta(x^{kt}) = r^{kt} - \epsilon^k$, where $\epsilon^k = \epsilon^i$ for $j < k < i$ and $\epsilon^j > \epsilon^i$ for $k = j$. Then, $F_{w^j}(x^{jt};\beta(x^{jt})) = F_{w^j}(x^{it};r^{it}) + \epsilon^j > F_{w^j}(x^{it};\beta(x^{it}))$. On the other hand, ϵ^j should be small enough such that $F_{w^{j+1}}(x^{jt};\beta(x^{jt})) < F_{w^{j+1}}(x^{it};\beta(x^{it}))$. Again, repeat this as repeating Step 2 until j reaches 1.

In conclusion, for such an incentive policy β, where $\beta(x^{it}) = r^{it} - \epsilon^i$ for $1 \le i \le m$ and $\beta = \gamma$ at all other points, a user of the type w^i, $1 \le i \le m$, will choose $(x^{it}, r^{it} - \epsilon^i)$ and as a result the ISP achieves an expected profit of $\sum_{j=1}^m q_j(r^{jt} - \epsilon^j) = E[r^t] - \sum_{j=1}^m \epsilon^j$. By making ϵ^i's small enough, the ISP can come arbitrarily close to the team solution.

D. Numerical Examples

We next illustrate the team solution and the solution of the incentive-design problem by numerical examples. Suppose that there are $m = 5$ types with equal probabilities for all types. Then $E[r] = \sum_{j=1}^5 Q(x^j; v^{j/5})/5$, where $v^{1/5} := w^1$ and $v^{i/5} = i * w^i - (i-1) * w^{i-1}$ for $2 \le i \le 5$.

Example 1: Take w^1 to w^5 to be 60, 40, 30, 24 and 20, respectively. Then,

$$v^{1/5} = 60 > v^{2/5} = 20 > v^{3/5} = 10 > v^{4/5} = 6 > v^{5/5} = 4 > 1.$$

From the case (iv b) in the discussion on team solution, we know immediately that $x^{it} = \alpha(v^{i/5})$ for $1 \le i \le 5$. The numerical results for the team solution are shown in Table I.

TABLE I
TEAM SOLUTION FOR EXAMPLE 1

i	1	2	3	4	5
w^i	60	40	30	24	20
$\alpha^{-1}(x^{it})$	60	20	10	6	4
x^{it}	0.8256	0.7078	0.6	0.5	0.4069
r^{it}	12.1782	10.4873	8.8017	7.3655	6.1421
$F_{w^i}(x^{it};r^{it})$	18.2024	7.4985	2.7984	0.3656	-1
$E[r^t]$	8.9949				

Now that the team solution is known, the first graph in Figure 3 depicts the derivation of γ. The five dashed curves, from the least steep one to the most steep one, stand for $F_{w^5}(x;r) = -1$ ($r = Q(x;w^5)$) and $F_{w^i}(x;r) = F_{w^i}(x^{(i+1)t};r^{(i+1)t})$ ($r = Q(x;w^i) - 1 - F_{w^i}(x^{(i+1)t};r^{(i+1)t})$) for $i = 4$, 3, 2, 1, respectively. Then γ can be any function that coincides with the solid curve at $(0,0)$ and (x^{it}, r^{it})'s but is above it (including along the solid curve) at all other points. Note that the feasible set is the region below the solid curve. From our previous analysis, we know that an incentive β making the ISP come arbitrarily close to optimal would be $\beta = \gamma$ except that $\beta(x^{it}) = r^{it} - \epsilon^i$, where $\epsilon^i = 0$ for $x^{it} = 0$, $\epsilon^i = \epsilon^j$ for $x^{it} = x^{jt}$, $\epsilon^i > \epsilon^j$ for $x^{it} > x^{jt}$, and ϵ^i's are small enough.

Example 2: Now take w^1 to w^5 to be 70, 40, 33, 25 and 20, respectively. Then, $v^{1/5}$ to $v^{5/5}$ become 70, 10, 19, 1 and 0, respectively. First, $1 \ge v^{4/5} > v^{5/5}$. Thus, from the case (ii) in the discussion on team solution, $x^{4t} = x^{5t} = 0$ and the problem is reduced to 3 types. Next, observe that $v^{2/5} < v^{3/5}$. As a result, $x^{2t} = x^{3t}$ from (i), so that we can further convert it to 2 types. Compute $v^{2,3/5} = (3w^3 - w^1)/2 = 14.5$. Since $v^{1/5} > v^{2,3/5} > 1$, finally from (iv b) we obtain that $x^{1t} = \alpha(v^{1/5}) > x^{2t} = x^{3t} = \alpha(v^{2,3/5})$. The results are listed in Table II.

The solution of the incentive-design problem for this example is illustrated in the second graph in Figure 3. Similarly as in Example 1, γ can be any function above the solid curve that is the upper boundary of the feasible set. Then, β can be obtained by making a small "dip" below $(x^{2t}, r^{2t}) = (x^{3t}, r^{3t})$ and (x^{1t}, r^{1t}) ($\epsilon^1 > \epsilon^2 = \epsilon^3 > 0$).

TABLE II
TEAM SOLUTION FOR EXAMPLE 2

i	1	2	3	4	5
w^i	70	40	33	25	20
$\alpha^{-1}(x^{it})$	70	14.5	14.5	1	0
x^{it}	0.8380	0.6615	0.6615	0	0
r^{it}	18.6491	14.8005	14.8005	0	0
$F_{w^i}(x^{it};r^{it})$	17.7856	2.5540	-1	-1	-1
$E[r^t]$	9.6500				

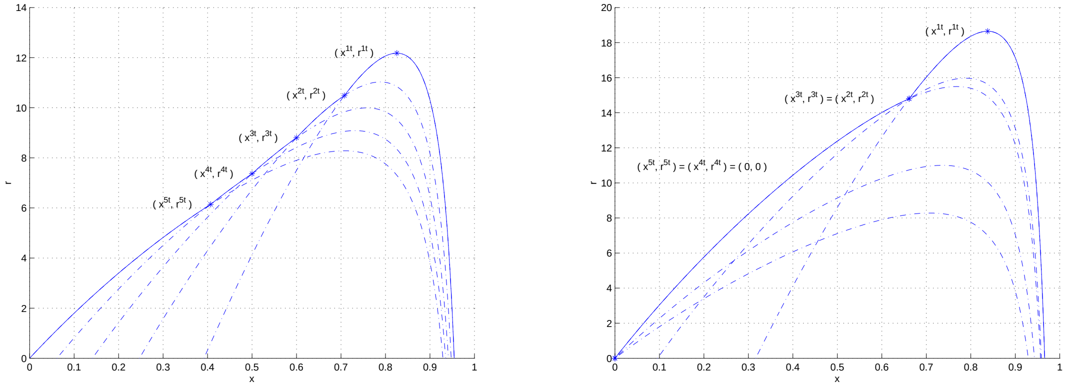

Fig. 3. Graphical illustration of incentives for the incomplete information case

V. Discussion and Extensions

A. *Incentive-Design Problem Formulation for Multiple Users*

Now consider the case with a single ISP and n users. User i's net utility, $1 \le i \le n$, is

$$F_{w_i}(x_i, x_{-i}; r_i) := w_i \log(1 + x_i) - \frac{1}{n - x_i - x_{-i}} - r_i,$$

where $x_{-i} := \sum_{j=1}^{n} x_j - x_i$.

1) Complete Information: Suppose that w_i's are known to the ISP. Then the team solution is

$$\begin{aligned} \{(x_i^t, r_i^t)\}_{i=1}^{n} = \quad & \arg \max_{\{x_i \ge 0, \sum_{j=1}^{n} x^j < n, r_i \ge 0\ \}_{i=1}^{n}} \sum_{j=1}^{n} r_j, \\ \text{s. t.} \quad & F_{w_i}(x_i, x_{-i}; r_i) \ge -\frac{1}{n - x_{-i}}, \quad 1 \le i \le n. \end{aligned}$$

The incentive-design problem is to find $\{\gamma_i\}_{i=1}^{n}$ such that for $1 \le i \le n$,

$$\begin{aligned} \arg \max_{0 \le x_i < n - x_{-i}^t} F_{w_i}(x_i, x_{-i}^t; \gamma_i(x_i)) &= x_i^t, \\ \gamma_i(x_i^t) &= r_i^t, \end{aligned}$$

and it is restricted by $\gamma_i(0) \equiv 0$.

2) Incomplete Information: For this case, suppose that the ISP only knows that the users' types are independent of each other and for User i, $1 \le i \le n$, $w_i = w_i^{j_i}$ w.p. $q_i^{j_i} \in (0, 1)$, $j_i = 1, \cdots, m_i$, where $\sum_{j_i=1}^{m_i} q_i^{j_i} = 1$. Thus, w.p. $q_1^{j_1} \times \cdots \times q_n^{j_n}$, $(w_1, \cdots, w_n) = (w_1^{j_1}, \cdots, w_n^{j_n})$. Let $\vec{J} := (j_1, \cdots, j_n)^T$, which can take the values from $\vec{J}_f := (1, \cdots, 1)^T$ to $\vec{J}_l := (m_1, \cdots, m_n)^T$. Then the team solution is

$$\begin{aligned} \left\{\{(x_i^{\vec{J}t}, r_i^{\vec{J}t})\}_{i=1}^{n}\right\}_{\vec{J}=\vec{J}_f}^{\vec{J}_l} = \quad & \arg \max_{\left\{\{x_i^{\vec{J}} \ge 0, \sum_{j=1}^{n} x_j^{\vec{J}} < n, r_i^{\vec{J}} \ge 0\}_{i=1}^{n}\right\}_{\vec{J}=\vec{J}_f}^{\vec{J}_l}} \left\{ E\left[\sum_{j=1}^{n} r_j\right] = \sum_{\vec{J}=\vec{J}_f}^{\vec{J}_l} q_1^{j_1} \times \cdots \times q_n^{j_n} \sum_{j=1}^{n} r_j^{\vec{J}} \right\}, \\ \text{s. t.} \quad & F_{w_i^{j_i}}(x_i^{\vec{J}}, x_{-i}^{\vec{J}}; r_i^{\vec{J}}) \ge -\frac{1}{n - x_{-i}^{\vec{J}}}, \quad 1 \le i \le n,\ \vec{J} = \vec{J}_f, \cdots, \vec{J}_l, \\ & F_{w_i^{j_i}}(x_i^{\vec{J}}, x_{-i}^{\vec{J}}; r_i^{\vec{J}}) \ge F_{w_i^{j_i}}(x_i^{\vec{J}'}, x_{-i}^{\vec{J}}; r_i^{\vec{J}'}), \quad 1 \le i \le n,\ \vec{J}, \vec{J}' = \vec{J}_f, \cdots, \vec{J}_l,\ \vec{J} \ne \vec{J}'. \end{aligned}$$

The incentive-design problem is to find $\{\gamma_i\}_{i=1}^{n}$ such that for $1 \le i \le n$ and $\vec{J} = \vec{J}_f, \cdots, \vec{J}_l$,

$$\begin{aligned} \arg \max_{0 \le x_i < n - x_{-i}^{\vec{J}t}} F_{w_i^{j_i}}(x_i, x_{-i}^{\vec{J}t}; \gamma_i(x_i)) &= x_i^{\vec{J}t}, \\ \gamma_i(x_i^{\vec{J}t}) &= r_i^{\vec{J}t}, \end{aligned}$$

and it is restricted by $\gamma_i(0) \equiv 0$.

Incentive controllability is similarly defined for the multiple user case as for the single user case. We are currently extending our previous analysis to solve for the solutions of the incentive-design problems formulated above for multiple users.

B. Concluding Remarks

This paper has studied Internet pricing from the perspective of the ISP by introducing dynamic pricing policies. We have verified the ϵ-incentive controllability for the single ISP, single user case and obtained ϵ-optimal incentive policies. Current work involves extensions to the multiple user case, and further to the multiple ISP case. The results would provide helpful intuitions for the ISPs on the deployment of their pricing policies.

Appendix I
Proof of local maximality of quadratic incentive under complete information

Recall that $w = (1+x^t)/(1-x^t)^2$ and $a_2 = -r^t/(x^t)^2$, where $r^t = Q(x^t; w)$. Therefore,

$$\frac{d^2}{dx^2}F_w(x;\gamma(x))|_{x=x^t} = \frac{\partial^2 Q(x;w)}{\partial x^2}|_{x=x^t} - \frac{d^2\gamma(x)}{dx^2}|_{x=x^t} = -\frac{w}{(1+x^t)^2} - \frac{2}{(1-x^t)^3} - 2a_2$$
$$= -\frac{2x^t + (x^t)^2 - (x^t)^3 + 2(x^t)^4 - 2(1+x^t)^2(1-x^t)\log(1+x^t)}{(x^t)^2(1+x^t)(1-x^t)^3}.$$

Since $0 < x^t < 1$ for $w > 1$, showing that the above quantity is negative is equivalent to proving

$$2x^t + (x^t)^2 - (x^t)^3 + 2(x^t)^4 > 2(1+x^t)^2(1-x^t)\log(1+x^t).$$

This is true because (i) if $x^t = 0$, both sides attain 0 and have zero first-order derivatives, and (ii) for $0 < x^t < 1$, the first-order derivative of the left-hand side is larger than that of the right-hand side, or equivalently,

$$2x^t - (x^t)^2 + 8(x^t)^3 > [2 - 4x^t - 6(x^t)^2]\log(1+x^t),$$

which holds since the left-hand side is always positive, while for $\frac{1}{3} \le x^t < 1$, the right-hand side is nonpositive, and for $0 < x^t < \frac{1}{3}$,

$$2x^t - (x^t)^2 + 8(x^t)^3 > [2 - 4x^t - 6(x^t)^2]x^t > [2 - 4x^t - 6(x^t)^2]\log(1+x^t).$$

Appendix II
Additional proof for Proposition 1

A. Proof of $\frac{\partial^3\eta(x;w)}{\partial x^3} < 0$ for $0 < x < 1$

The second-order derivative of $\eta(x;w)$ can be written as

$$\frac{\partial^2\eta(x;w)}{\partial x^2} = w\left[-\frac{1}{x^2} + \frac{1}{(1+x)^2} - \frac{1}{x^2(1+x)} + \frac{2\log(1+x)}{x^3}\right] - \frac{2}{(1-x)^3}.$$

Note that $-2/(1-x)^3$ is strictly decreasing. Thus, it is sufficient to show that the quantity in the brackets has a negative first-order derivative, i.e.,

$$\frac{2x(1+x)^3 + 4x(1+x)^2 + x^2(1+x) - 2x^4 - 6(1+x)^3\log(1+x)}{x^4(1+x)^3} < 0.$$

Now we compare $2x(1+x)^3 + 4x(1+x)^2 + x^2(1+x)$ with $2x^4 + 6(1+x)^3\log(1+x)$. It can easily be verified that the third-order derivative of the former function is always less than that of the latter one. Hence, the second-order derivatives, which assume the same value at $x = 0$, follow the same order. So do the first-order derivatives. Finally, we have $2x(1+x)^3 + 4x(1+x)^2 + x^2(1+x) < 2x^4 + 6(1+x)^3\log(1+x)$, and it is concluded that $\frac{\partial^3\eta(x;w)}{\partial x^3} < 0$ for $0 < x < 1$.

B. Proof of $w - 1 < a_1$

Recall that

$$w - 1 = \frac{1+x^t}{(1-x^t)^2} - 1 \quad \text{and} \quad a_1 = \frac{2r^t}{x^t} = \frac{2}{x^t}Q(x^t; \frac{1+x^t}{(1-x^t)^2}).$$

Consider them as functions of x^t. Then they assume the same value at 0^+, and the first-order derivative of $w-1$ is less than that of a_1. Since $0 < x^t < 1$, so we must have $w - 1 < a_1$.

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