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Automatica 00 (2001) 000–000

www.elsevier.com/locate/automatica

Regular paper

Disturbance attenuating output-feedback control of nonlinear systems with local optimality[☆]

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Received 9 October 1999; received in final form 7 November 2000

For a class of nonlinear systems a constructive output-feedback control design achieves local near-optimality and semiglobal inverse optimality with a prescribed $\mathscr{L}_2$-gain.

Abstract

Locally optimal backstepping is extended to *output-feedback systems* with input disturbances and nonlinearities that depend only on the measured output. The constructive design blends worst-case filtering with backstepping, and results in a disturbance attenuating dynamic output-feedback controller that achieves semiglobal inverse optimality and local near-optimality. © 2001 Published by Elsevier Science Ltd.

Keywords: Backstepping; Dynamic output-feedback; Nonlinear systems; Local optimality; Inverse optimality; Robust control; Disturbance attenuation

1. Introduction

While linear $\mathscr{H}_\infty$-design is fully developed (Doyle, Glover, Khargonekar, & Francis, 1989; Başar & Bernhard, 1995), the development of its nonlinear counterpart has proven to be much more difficult. Nonlinear $\mathscr{H}_\infty$ optimality conditions (van der Schaft, 1993; Lu & Doyle, 1995; Battilotti, 1996) yield either local (Isidori, 1994; Isidori & Kang, 1995), or infinite dimensional controllers (Başar & Bernhard, 1995; Didinsky, Başar, & Bernhard, 1993; Ball, Helton, & Walker 1993; Krener, 1995; James & Baras, 1995). Constructive nonlinear designs either do not achieve optimality (Li, 1997), or do not penalize the cost of control (Tezcan & Başar, 1999). To avoid the need to solve the Hamilton–Jacobi–Isaacs (HJI) equation, recent research has concentrated on *inverse optimal* designs that first find a stabilizing feedback control law and then determine a cost functional which is minimized (Freeman & Kokotović, 1996; Krstić & Li, 1998; Ezal, Pan, & Kokotović, 2000). With such designs, even though stability properties similar to those of an optimal design are achieved, a prescribed performance level cannot be guaranteed. For full-state feedback problems, a new backstepping design procedure developed in Ezal et al. (2000) renders the resulting controller $\mathscr{H}_\infty$-optimal for the linearized system, and globally inverse optimal for the nonlinear system.

In this paper we construct finite dimensional dynamic output-feedback control laws which achieve local near-optimality and semiglobal inverse optimality with a prescribed $\mathscr{L}_2$-gain for nonlinear systems in output-feedback form. Such systems have nonlinearities that depend solely on the measured output. The design combines the locally optimal backstepping procedure of Ezal et al. (2000) with the cost-to-come methods for robust estimation (Didinsky et al., 1993; Pan & Başar, 1998; Tezcan & Başar, 1999). The main problem is formulated in Section 2. The design procedure is presented in Section

[☆]This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor H. Nijmeijer under the direction of Editor Roberto Tempo.

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PII: S0005-1098(01)00024-3

AUT 1706 PROD.TYPE: COM
pp. 1–13 (col.fig.: NIL)
ED: CHANAKSHI
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3 and followed by an example in Section 4. The paper ends with the concluding remarks of Section 5 and three appendices. In addition to supporting the developments in the main sections of the paper, the appendices also contain substantial material of independent interest. Appendix A summarizes the cost-to-come method and $\mathscr{H}_\infty$-filtering, while Appendix B provides the solution to a benchmark problem which is the main problem with the controller also having access to the derivative of the output. Because of this additional information, the solution to the benchmark problem sets a lower bound on the cost of the main problem, and plays a key role in proof of the main result given in Appendix C.

In the notation we adopt in this paper, $A_{[i]}$ consists of the first i columns and the first i rows of $A \in \mathbb{R}^{n\times n}$. The same notation is used for $x_{[i]}$, $f_{[i]}$, $G_{1[i]}$, and $a_{[i]} := [a_{i1} \ \cdots \ a_{ii}]$ where a_{ij} is the i, jth element of A. The zero n-vector is denoted by 0_n, and $A_{\{i+1\}}$, $A_{12\{i+1\}}$ and $A_{21\{i+1\}}$ are defined by

$$A \equiv \begin{bmatrix} A_{[i]} & A_{12\{i+1\}} \\ A_{21\{i+1\}} & A_{\{i+1\}} \end{bmatrix}.$$

$\mathbf{\Pi}(t)$ is the vector formed by stacking up the columns of the matrix $\Pi(t)$, $q_x(x) := (\partial q/\partial x)(x)$, $q_{xx}(x) := \frac{1}{2}(\partial^2 q/\partial x^2)(x)$, and $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$ means that there exist constants c_1 and c_2 such that $|\delta_1(\varepsilon)| \leqslant c_2|\delta_2(\varepsilon)|$, whenever $|\varepsilon| < c_1$.

2. Problem formulation

We consider nonlinear systems in *output-feedback form*

$$\dot{x} = Ax + \breve{f}(x_1) + G_1(x_1)w + B_2 u, \tag{1a}$$

$$y = x_1 =: C_1 x, \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^q$ is the disturbance, $u \in \mathbb{R}$ is the control, $y \in \mathbb{R}$ is the output, $B_2 = [0'_{n-1} \ 1]'$, and A is such that its ith row is $[a_{[i]} \ 1 \ 0'_{n-i-1}]$. We assume that $\breve{f}(x_1) := [\breve{f}_1(x_1)' \cdots \breve{f}_n(x_1)']'$ and $G_1(x_1) := [g_1(x_1)' \ \cdots \ g_n(x_1)']'$ are sufficiently smooth and $\breve{f}_x(0) = 0$. In this structure, (A, B_2) is controllable and (A, C_1) is observable. Denoting $B_1 := [b'_1 \ \cdots \ b'_n]' := G_1(0)$, we assume that (A, B_1) is controllable. With $w = 0$, we know that this class of systems is globally stabilizable by output-feedback (Marino & Tomei, 1991; Kanellakopoulos, Kokotović, & Morse, 1991).

2.1. The output-feedback problem

In the region where the linear dynamics

$$\dot{x} = Ax + B_1 w_\ell + B_2 u_\ell,$$

$$y = C_1 x$$

are valid, the $\mathscr{H}_\infty$ disturbance attenuation problem is to design a stabilizing dynamic output-feedback control law which minimizes, for the worst-case disturbance, the cost functional

$$J_\ell(u_\ell, w_\ell) = \int_0^\infty [x'Qx + Ru_\ell^2 - \gamma^2 w'_\ell w_\ell]\,\mathrm{d}t, \tag{2}$$

where $Q > 0$ and $R > 0$ are prespecified. This problem is equivalent to the linear dynamic game

$$U_\ell(x(0)) := \min_{u_\ell} \max_{w_\ell,\, x(0)} J_\ell(u_\ell, w_\ell) < \infty,$$

where $U_\ell(x)$ is the upper value function. We assume that the optimal attenuation level $\gamma^* > 0$ exists and, hence, the desired level of attenuation $\gamma > \gamma^*$ is achievable.

The nonlinear optimal disturbance attenuation problem for system (1) is to design a stabilizing dynamic output-feedback control law which minimizes, for the worst-case disturbance, a cost functional of the form

$$J(u, w) = \int_0^\infty [q(x) + r(x)u^2 - \gamma^2 w'w]\,\mathrm{d}t,$$

where $q(x) \geqslant 0$ and $r(x) > 0$ are not specified beforehand, that is, our global objective is *inverse optimality*. For the equivalent dynamic game

$$U(x(0)) := \min_{u} \max_{w, x(0)} J(u, w) < \infty,$$

the function $U(x)$ is the upper value function.

Local optimality: *An inverse optimal feedback control law $u = \mu(y)$ with value function $U(x)$ is also locally $\mathscr{H}_\infty$-optimal if $U_\ell(x) \equiv x'[U_{xx}(0)]x$ where $U_\ell(x)$ is the $\mathscr{H}_\infty$-optimal value function of the linear dynamic game.*

Thus, for simultaneous local optimality and global inverse optimality, we impose the requirements $q_{xx}(0) = Q$ and $r(0) = R$.

2.2. Design methodology

We rely on $\mathscr{H}_\infty$-filtering theory to design a robust observer $\dot{\hat{x}} = \hat{F}(y, \hat{x}, u)$, where $\hat{x} \in \mathbb{R}^n$ is the estimate of x, and the estimation error $x - \hat{x} =: \tilde{x}(t) \to 0$ as $t \to \infty$ for any $w(t) \in \mathscr{L}_2$. However, the cost-to-come methodology (Appendix A) requires that the output y depend on a measurement disturbance in a *non-singular* way, a requirement not satisfied by system (1). To avoid this singularity, we introduce a small measurement noise εv,

$$\dot{x} = Ax + \breve{f}(x_1) + G_1(x_1)w + B_2 u, \tag{3a}$$

$$y = C_1 x + \varepsilon v, \tag{3b}$$

where $\varepsilon > 0$. The optimal solution of the singular noise-free problem (1), if it exists, is the limit, as $\varepsilon \to 0$, of the small-noise problem (3).

As in Pan and Başar (1998), the limit as $\varepsilon \to 0$ is interpreted via a benchmark noise-free problem in which the output derivative $\dot{y}$ is available for feedback. Asymptotically, as $\varepsilon \to 0$, the performance of the $\mathscr{H}_\infty$-filter for the small-noise problem leads to the same performance as the $\mathscr{H}_\infty$-filter for the benchmark problem. Hence, the solution of this benchmark problem provides a tight lower bound on the optimum performance for the small-noise problem. The details of the benchmark problem are in Appendix B. To proceed further, we need to distinguish the notions of near-optimality and suboptimality.

Near-optimality: An inverse optimal feedback control law $u = \mu(y)$ *with value function* $U(x)$ *is also locally near-optimal if* $U_\ell(x) \equiv x'[U_{xx}(0)]x + O(\varepsilon)$ *where* $U_\ell(x)$ *is the* $\mathscr{H}_\infty$*-optimal value function of the linear dynamic game.*

Suboptimality: An inverse optimal feedback control law $u = \mu(y)$ *with value function* $U(x)$ *is also locally suboptimal if* $U_\ell(x) \leqslant x'[U_{xx}(0)]x$ *where* $U_\ell(x)$ *is the* $\mathscr{H}_\infty$*-optimal value function of the linear dynamic game.*

Our approach is to replace (2) with

$$\hat{J}_\ell(u_\ell, w_\ell) = \int_0^\infty [\tilde{x}'\tilde{Q}_1\tilde{x} + \hat{x}'\tilde{Q}_2\hat{x} + Ru_\ell^2 - \gamma^2 w_\ell' w_\ell]\,\mathrm{d}t, \quad (4)$$

where $\tilde{Q}_1$ and $\tilde{Q}_2$ are picked in such a way that, given any $Q > 0$, $\tilde{x}'\tilde{Q}_1\tilde{x} + \hat{x}'\tilde{Q}_2\hat{x} \geqslant x'Qx$. This results in a near-optimal control law which guarantees the same level of disturbance attenuation with respect to any cost functional that is no larger than (4), including (2). We assume that the optimal attenuation level $\gamma_\varepsilon^* > 0$ exists for each ε and, hence, $\gamma > \gamma_\varepsilon^*$ is achievable. We will also achieve semiglobal inverse optimality with respect to a cost functional of the form

$$\hat{J}(u, w) = \int_0^\infty [q(\tilde{x}, \hat{x}) + r(\tilde{x}, \hat{x})u^2 - \gamma^2 w'w]\,\mathrm{d}t, \quad (5)$$

where $q(\tilde{x}, \hat{x}) \geqslant 0$ and $r(\tilde{x}, \hat{x}) > 0$ are determined a posteriori.

We now introduce three assumptions which will be needed in the design to be presented in the next section.

Assumption 1. There exists $c_n > 0$ such that $N(x_1) := g_1(x_1)g_1'(x_1) \geqslant c_n > 0$ for all $x_1 \in \mathbb{R}$. In the linear case $N_0 := b_1 b_1' \geqslant c_n$.

Assumption 2. Given $Q_1 \equiv \tilde{Q}_{1\{2\}} > 0$, the filtering GARE

$$0 = \Pi_s^\infty \hat{A}_0' + \hat{A}_0 \Pi_s^\infty - \Pi_s^\infty[\gamma^2 \hat{C}_1' N_0^{-1} \hat{C}_1 - Q_1]\Pi_s^\infty + \gamma^{-2}\hat{B}_0\hat{B}_0' \quad (6)$$

with $\hat{C}_1 := [1\ 0_{n-2}']$, $\hat{A}_0 := A_{\{2\}} - B_{1\{2\}} b_1' N_0^{-1} \hat{C}_1$, and

$$\hat{B}_0 := (B_{1\{2\}}[I - b_1' N_0^{-1} b_1] B_{1\{2\}}')^{1/2},$$

admits $\Pi_s^\infty > 0$ such that $\hat{A}_0 - \gamma^2 \Pi_s^\infty \hat{C}_1' N_0^{-1} \hat{C}_1$ is Hurwitz.

Assumption 3. Given $Q_2 \equiv \tilde{Q}_2 > 0$ and $R > 0$, the control GARE

$$0 = P_s A + A' P_s + P_s\left(\frac{1}{\gamma^2}\hat{B}_1\hat{B}_1' - B_2 R^{-1} B_2'\right)P_s + Q_2 \quad (7)$$

with $\hat{B}_1 := [N_0^{1/2}\ \ \hat{\Gamma}_{1_0}']'$, and $\hat{\Gamma}_{1_0} = (B_{1\{2\}} b_1' + \gamma^2 \Pi_s^\infty \hat{C}_1') N_0^{-1/2}$ admits $P_s > 0$ such that

$$A - \left(B_2 R^{-1} B_2' - \frac{1}{\gamma^2}\hat{B}_1\hat{B}_1'\right)P_s \quad \text{and} \quad A - B_2 R^{-1} B_2' P_s$$

are Hurwitz.

3. Main problem: locally near-optimal design

As in Pan and Başar (1998) and Tezcan and Başar (1999), we obtain the $\mathscr{H}_\infty$-optimal observer for the small-noise system (3). In the limit as $\varepsilon \to 0$, this observer becomes the observer obtained for the benchmark problem in Appendix B.

3.1. Local design

The $\mathscr{H}_\infty$-optimal filter for the linearization of system (3) is given by

$$0 = \Pi^\infty(\varepsilon)A' + A\Pi^\infty(\varepsilon) - \Pi^\infty(\varepsilon)\left[\frac{\gamma^2}{\varepsilon^2}C_1' C_1 - \tilde{Q}_1\right]\Pi^\infty(\varepsilon) + \gamma^{-2} B_1 B_1', \quad (8a)$$

$$\dot{\hat{x}} = A\hat{x} + \frac{\gamma^2}{\varepsilon}\Pi^\infty(\varepsilon)C_1'\zeta + B_2 u, \quad (8b)$$

where $\zeta := (1/\varepsilon)C_1\tilde{x} \equiv (1/\varepsilon)\tilde{x}_1$. The optimal control law, which minimizes the cost functional (4), is $u = \mu_\ell(\varepsilon; \hat{x}) := -R^{-1}B_2' P(\varepsilon)\hat{x}$, where $P(\varepsilon)$ is the solution of the control GARE

$$0 = P(\varepsilon)A + A'P(\varepsilon) + P(\varepsilon)\left[\frac{\gamma^2}{\varepsilon^2}\Pi^\infty(\varepsilon)C_1' C_1 \Pi^\infty(\varepsilon) - B_2 R^{-1} B_2'\right]P(\varepsilon) + \tilde{Q}_2. \quad (9)$$

Symmetric positive definite solutions $\Pi^\infty(\varepsilon)$ and $P(\varepsilon)$ to the filtering GARE (8a) and the control GARE (9) exist for all $\gamma > \gamma_\varepsilon^*$ (Başar & Bernhard, 1995). Both of these equations are singularly perturbed by ε. As in Kokotović and Yackel (1972) and Pan and Başar (1994), we substitute

$$\Pi^\infty(\varepsilon) = \begin{bmatrix} 0 & 0 \\ 0 & \Pi_s^\infty \end{bmatrix} + \varepsilon \begin{bmatrix} \Pi_{11}^\infty & \Pi_{12}^\infty \\ \Pi_{12}^{\infty\prime} & \Pi_{22}^\infty \end{bmatrix} + O(\varepsilon^2)$$

into the filtering GARE (8a). Then, collecting like powers of ε, we recover the filtering GARE (6) with $\hat{C}_1 \equiv A_{12\{2\}}$, and solve it explicitly for $\Pi_{11}^{\infty} = \gamma^{-2} N_0^{1/2} > 0$ and $\Pi_{12}^{\infty\prime} = \gamma^{-2}(B_{1\{2\}} b_1' + \gamma^2 \Pi_s^{\infty} \hat{C}_1') N_0^{-1/2}$. Moreover, with $\hat{B}_1$ defined by Assumption 3, $\varepsilon^{-1}\Pi^{\infty}(\varepsilon)C_1' \equiv \gamma^{-2}\hat{B}_1 + O(\varepsilon)$. Thus, the control GARE (9) reduces to

$$0 = P(\varepsilon)A + A'P(\varepsilon) + P(\varepsilon)\left[\frac{1}{\gamma^2}\hat{B}_1\hat{B}_1 - B_2 R^{-1} B_2'\right]P(\varepsilon) + \tilde{Q}_2 + O(\varepsilon),$$

which is a regular $O(\varepsilon)$ perturbation of GARE (7) and, hence, $P(\varepsilon) = P_s + O(\varepsilon)$.

Using matrix inversion identities and $\varepsilon\zeta \equiv \tilde{x}_1$, the value function of this filtering problem, $W_\ell(\varepsilon; \tilde{x}) = \tilde{x}'(\Pi^{\infty}(\varepsilon))^{-1}\tilde{x}$, is $W_\ell(\varepsilon;\tilde{x}) = \tilde{x}_{\{2\}}'(\Pi_s^{\infty})^{-1}\tilde{x}_{\{2\}} + O(\varepsilon)$. In view of $\hat{x}_1 = x_1 - \varepsilon\zeta$ and $P(\varepsilon) = P_s + O(\varepsilon)$, the control value function, $V_\ell(\varepsilon;\hat{x}) = \hat{x}'P(\varepsilon)\hat{x}$, is $V_\ell(\varepsilon;\hat{x}) = \mathbf{x}'P_s\mathbf{x} + O(\varepsilon)$, where $\mathbf{x} := [x_1 \ \ \hat{x}_{\{2\}}']'$. These value functions are $O(\varepsilon)$-close to the filter value function $W_\ell(\tilde{x}_{\{2\}})$ and the control value function $V_\ell(\mathbf{x})$ for the benchmark problem of Appendix B. Thus, the value function for the output-feedback problem, $U_\ell(\varepsilon;\tilde{x},\hat{x})$, is $O(\varepsilon)$-close to the value function (B.6) of the benchmark problem, and

$$\lim_{\varepsilon\to 0} U_\ell(\varepsilon;\tilde{x},\hat{x}) = \tilde{x}_{\{2\}}'(\Pi_s^{\infty})^{-1}\tilde{x}_{\{2\}} + \mathbf{x}'P_s\mathbf{x} = U_\ell(\mathbf{x},\tilde{x}_{\{2\}}). \tag{10}$$

The local properties of the linear design are summarized as follows.

Lemma 1. *For all $\gamma > \gamma_\varepsilon^*$, the control law $u = \mu_\ell(\varepsilon;\hat{x}) := -R^{-1}B_2'P(\varepsilon)\hat{x}$, applied to the linearization of system* (3) *with the linear observer* (8), *is $\mathscr{H}_\infty$-optimal with respect to the cost functional* (4), *and is suboptimal with respect to the cost functional* (2). *Furthermore, when $w_\ell \equiv 0$, the equilibrium $(x,\hat{x}) = (0{,}0)$ is exponentially stable, while for all $w_\ell(t) \in \mathscr{L}_2$, all system signals are bounded and converge to zero as $t \to \infty$.*

Proof. The time derivative of the output-feedback value function is

$$\dot{U}_\ell = -\tilde{x}'\tilde{Q}_1\tilde{x} - \hat{x}'\tilde{Q}_2\hat{x} - Ru_\ell^2 + \gamma^2 w_\ell' w_\ell - \gamma^2|w_\ell - v_{f_\ell}|^2 - \gamma^2(\zeta - v_{c_\ell})^2 + R(u - \mu_\ell)^2,$$

where the worst-case filtering disturbance is $v_{f_\ell} := (1/\gamma^2) B_1'(\Pi^{\infty}(\varepsilon))^{-1}\tilde{x}$, and the worst-case control disturbance is $v_{c_\ell} := (1/\varepsilon)C_1\Pi^{\infty}(\varepsilon)P(\varepsilon)\hat{x}$. Hence, the control law $u_\ell = \mu_\ell(\varepsilon;\hat{x})$ minimizes the cost functional (4) for all $\gamma > \gamma_\varepsilon^*$, and the equilibrium $(x,\hat{x}) = (0{,}0)$ is exponentially stable when $w_\ell \equiv 0$. Moreover, when $w(t) \in \mathscr{L}_2$ all system signals converge to zero as $t \to \infty$ (Teel, 1999). If $\tilde{Q}_1$ and $\tilde{Q}_2$ satisfy $\tilde{x}'\tilde{Q}_1\tilde{x} + \hat{x}'\tilde{Q}_2\hat{x} \geqslant x'Qx$, then

$$\dot{U}_\ell \leqslant -x'Qx - Ru_\ell^2 + \gamma^2 w_\ell' w_\ell - \gamma^2|w_\ell - v_{f_\ell}|^2 - \gamma^2(\zeta - v_{c_\ell})^2 + R(u - \mu_\ell)^2,$$

which implies that $u_\ell = \mu_\ell(\varepsilon;\hat{x})$ is suboptimal with respect to cost functional (2), that is, $U_\ell^*(x(0)) \leqslant U_\ell(\varepsilon;\tilde{x}(0),\hat{x}(0))$, where

$$U_\ell^*(x(0)) := \min_{u_\ell}\max_{w_\ell} \int_0^{\infty} [x'Qx + Ru_\ell^2 - \gamma^2 w_\ell' w_\ell]\,dt \tag{11}$$

is the optimal value function associated with the standard linear $\mathscr{H}_\infty$-optimal control problem. □

3.2. Nonlinear design

The worst-case observer for the nonlinear system (3) is

$$\dot{\Pi}(\varepsilon) = \Pi(\varepsilon)A' + A\Pi(\varepsilon) - \Pi(\varepsilon)\left[\frac{\gamma^2}{\varepsilon^2}C_1'C_1 - \Theta\right] \times \Pi(\varepsilon) + \gamma^{-2}G_1(x_1)G_1'(x_1), \tag{12}$$

$$\dot{\hat{x}} = A\hat{x} + \breve{f}(x_1) + \frac{\gamma^2}{\varepsilon^2}\Pi(\varepsilon)C_1'(x_1 - \hat{x}_1) + B_2 u, \tag{13}$$

where $\Theta = \gamma^{-2}\Pi^{-1}\hat{M}\Pi^{-1}$ with some design parameter $\hat{M} > 0$. Because this observer is not in strict-feedback form, we consider a reduced-order observer obtained in the limit as $\varepsilon \to 0$. Substituting

$$\Pi(\varepsilon;t) = \begin{bmatrix} 0 & 0 \\ 0 & \Pi_s(t)\end{bmatrix} + \varepsilon\begin{bmatrix}\Pi_{11}(t) & \Pi_{12}(t) \\ \Pi_{12}'(t) & \Pi_{22}(t)\end{bmatrix} + O(\varepsilon^2)$$

into DRE (12) and letting $\varepsilon \to 0$, we obtain $\Pi_{11} = \gamma^{-2}N^{1/2}(x_1)$ and

$$\dot{\Pi}_s = \Pi_s\hat{A}' + \hat{A}\Pi_s - \Pi_s[\gamma^2\hat{C}_1'N^{-1}(x_1)\hat{C}_1 - \Theta_s]\Pi_s + \gamma^{-2}\hat{B}\hat{B}', \tag{14}$$

where $\hat{C}_1 \equiv A_{12\{2\}}, \Theta_s(\Pi_s) := \gamma^{-2}\Pi_s^{-1}\hat{M}_{\{2\}}\Pi_s^{-1}$, $\Pi'_{12} = \gamma^{-2}\,\hat{\Gamma}_1(x_1,\Pi_s)$, and the explicit expressions for $\hat{A}(x_1)$, $\hat{B}(x_1)$, and $\hat{\Gamma}_1(x_1,\Pi_s)$ are defined by (A.3). Note that the reduced-order DRE (14) is identical to the DRE (B.7a) for the benchmark problem, and is time invariant whenever $G_1(x_1) \equiv B_1$. As in Appendix B, we select the initial condition of the filter to be $\Pi_s(0) = \Pi_s^{\infty} > 0$, and $\hat{M}_{\{2\}} := \gamma^2\Pi_s^{\infty}Q_1\Pi_s^{\infty}$ where $Q_1 := \hat{Q}_{1\{2\}} > 0$ and Π_s^{∞} is the steady state solution of the filtering GARE (6).

The approximation in the filter dynamics we have made thus far is that of reducing the order of the DRE. Two further approximations $a_{11}\hat{x}_1 \approx a_{11}x_1$ and $A_{21\{2\}}\hat{x}_1 \approx A_{21\{2\}}x_1$ are not essential but are now made to simplify the stability analysis of the approximate filter

$$\dot{\hat{x}}_1 = a_{11}x_1 + \hat{x}_2 + \breve{f}_1(x_1) + \frac{1}{\varepsilon}N^{1/2}(x_1 - \hat{x}_1),$$

$$\hat{x}_1(0) = \hat{x}_{1_o}, \tag{15a}$$

$$\dot{\hat{x}}_{\{2\}} = A_{\{2\}}\hat{x}_{\{2\}} + \varphi(x_1) + \frac{1}{\varepsilon}\hat{\Gamma}_1(x_1,\Pi_s)(x_1 - \hat{x}_1) + B_{2\{2\}}u, \tag{15b}$$

where $\varphi(x_1) := A_{21\{2\}}x_1 + \breve{f}_{\{2\}}(x_1)$, $\hat{x}_{\{2\}}(0) := 0$, and $\hat{x}_{1_0}$ is the initial condition for the estimate of the measurement. Since $v \equiv 0$, we can always set $\hat{x}_{1_0} = x_1(0)$. System (3), the reduced-order DRE (14), filter (15), and the fast dynamics of the error $\zeta = (1/\varepsilon)(x_1 - \hat{x}_1)$ are written together as

$$\dot{\xi} = \mathscr{A}_{11}(\xi) + \mathscr{A}_{12}(\xi)\zeta + \mathscr{B}_{11}(\xi)w + \mathscr{B}_{12}(\xi)u,$$
$$\varepsilon\dot{\zeta} = \mathscr{A}_{21}(\xi) - \mathscr{A}_{22}(\xi)\zeta + \mathscr{B}_{21}(\xi)w, \tag{16}$$

where $\xi := [x_1 \;\; \hat{x}'_{\{2\}} \;\; \tilde{x}'_{\{2\}} \;\; \mathbf{\Pi}'_s]'$ and $\mathscr{A}_{22} \equiv (\mathscr{B}_{21}\mathscr{B}'_{21})^{1/2} \geqslant c_n^{1/2} > 0$ for all ξ. Using singular perturbation arguments (Kokotović, Khalid, & O'Reilly, 1986), we let $\varepsilon \to 0$, compute the slow manifold $\zeta_s := \mathscr{A}_{22}^{-1}(\mathscr{A}_{21} + \mathscr{B}_{21}w_s)$ and, substituting in (16), arrive at

$$\dot{\xi}_s = \mathscr{A}_1(\xi_s) + \mathscr{B}_1(\xi_s)w_s + \mathscr{B}_2u_s, \tag{17}$$

where $\mathscr{A}_1 := \mathscr{A}_{11} + \mathscr{A}_{12}\mathscr{A}_{22}^{-1}\mathscr{A}_{21}$, $\mathscr{B}_1 := \mathscr{B}_{11} + \mathscr{A}_{12}\mathscr{A}_{22}^{-1}\mathscr{B}_{21}$, and $\mathscr{B}_2 := \mathscr{B}_{12}$. System (17) is identical to (B.17), and its subsystem

$$\dot{\Pi}_s = \pi_s(x_{s_1}, \Pi_s), \tag{18a}$$

$$\dot{\hat{x}}_{s\{2\}} = A_{\{2\}}\hat{x}_{s\{2\}} + \varphi(x_{s_1}) + \hat{\Gamma}_1(x_{s_1}, \Pi_s)\varpi + B_{2\{2\}}u_s \tag{18b}$$

is identical to observer (B.7) obtained for the benchmark problem. Hence, as $\varepsilon \to 0$, the reduced-order observer (18) constitutes the limit for the observer given by Eq. (13).

It was also demonstrated for the benchmark problem in Appendix B that the closed-loop system (17) with

$$u = \mu(\mathbf{x}, \Pi_s) := -\tfrac{1}{2}r^{-1}(\xi)\mathscr{B}'_2(\xi)U'_\xi \tag{19}$$

and $w = v(\mathbf{x}, \tilde{x}_{\{2\}}, \Pi_s) := (1/2\gamma^2)\mathscr{B}'_1(\xi)U'_\xi$ satisfies the HJI equation (B.19) with the value function (B.18), that is, $U(\xi) := V(\mathbf{x}, \Pi_s) + W(\tilde{x}_{\{2\}}, \Pi_s)$. Therefore, the feedback control law (B.13) obtained for the benchmark problem will be utilized without any changes, except that the states $(\hat{x}_{\{2\}}, \Pi_s)$ will now be provided by observer (15) rather than by (B.7). Next, we introduce the composite state $\xi_c := [\xi' \;\; \zeta]'$, and write system (16) as

$$\dot{\xi}_c = \mathscr{F}(\xi_c) + \mathscr{G}_1(\xi_c)w + \mathscr{G}_2u. \tag{20}$$

A Lyapunov function for this system is

$$\mathscr{V}(\xi_c) = U(\xi) + \varepsilon\mathscr{U}(\xi, \zeta),$$

where $\mathscr{U}(\xi,\zeta) := \gamma^2|\zeta - \mathscr{A}_{22}^{-1}\mathscr{A}_{21} - (1/2\gamma^2)\mathscr{A}_{22}^{-1}\mathscr{B}_{21}\mathscr{B}'_{11}U'_\xi|^2_{\mathscr{A}_{22}^{-1}}$. Given $\beta > 0$ and $\varepsilon > 0$, we define the set $\Omega(\beta, \varepsilon) := \Omega_\xi(\beta) \times \Omega_\zeta(\beta, \varepsilon)$, where

$$\Omega_\xi(\beta) := \{\xi \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n\times n} > 0 \,|\, U(\xi) \leqslant \beta\},$$

$$\Omega_\zeta(\beta, \varepsilon) := \{\zeta \in \mathbb{R} \,|\, \varepsilon\mathscr{U}(\xi, \zeta) \leqslant \beta, \forall \xi \in \Omega_\xi(\beta)\}.$$

To state the main theorem of this section, we now take the disturbance $w(t)$ to belong to the set $\mathscr{W}_2(c_{w_2}) := \{w(t) \in \mathscr{L}_2 \,|\, \|w(t)\|_2 \leqslant c_{w_2}\}$, where c_{w_2} is a prespecified known positive constant.

Main Theorem. *If Assumptions* 1–3 *are satisfied, then for each pair* (β, c_{w_2}) *there exists* $\varepsilon_2^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_2^*)$, $w(t) \in \mathscr{W}_2(c_{w_2})$, *and*

$$\xi_c(0) \in \Omega(\beta, \varepsilon) \cap \{\Pi_s(0) = \Pi_s^\infty > 0, \hat{x}_{\{2\}}(0) = 0_{n-1}\}, \tag{21}$$

the closed-loop system (19), (20) *achieves*:

(1) *The boundedness of all signals, and their convergence to the equilibrium, i.e.,* $\xi_c(t) \to \xi_{c_0} = [0'_{2n-1} \;\; \mathbf{\Pi}_s^{\infty\prime} \;\; 0]'$ *as* $t \to \infty$.

(2) *Semiglobal inverse* $\mathscr{H}_\infty$*-optimality with respect to a cost functional of the form*

$$\hat{J}(u, w) = \int_0^\infty [\mathscr{Q}(\xi_c) + r(\xi_c)u^2 - \gamma^2 w'w]\,\mathrm{d}t, \tag{22}$$

where $\mathscr{Q}(\xi_c) \geqslant 0$, $r(\xi_c) > 0$, *and* $\gamma > \gamma_\varepsilon^* > 0$.

(3) *Local near-optimality with respect to the cost functional* (4) *where* $\tilde{Q}_1$, $\tilde{Q}_2$ *and R are prespecified positive definite matrices, and* $\gamma > \gamma_\varepsilon^* > 0$. *Moreover, local near-suboptimality is achieved with respect to the cost functional* (2) *where Q and R are prespecified positive definite matrices.*

Proof. The proof of the Main Theorem is found in Appendix C.[1] □

4. Example

Our task is to design an output-feedback control law for the nonlinear system

$$\dot{x}_1 = x_1^2 + x_2 + w, \tag{23a}$$

$$\dot{x}_2 = -w + u, \tag{23b}$$

$$y = x_1, \tag{23c}$$

that is locally optimal for the cost functional

$$J_\ell(u, w) = \int_0^\infty [x_1^2 + x_2^2 + u^2 - \gamma^2 w^2]\,\mathrm{d}t. \tag{24}$$

As we did with (4), we replace (24) with

$$\hat{J}_\ell(u, w) = \int_0^\infty [\tilde{x}_1^2 + 2\tilde{x}_2^2 + \hat{x}_1^2 + 2\hat{x}_2^2 + u^2 - \gamma^2 w^2]\,\mathrm{d}t, \tag{25}$$

where $\tilde{x} = x - \hat{x}$, $\hat{x}$ is the estimate of x, and $\tilde{x}_1^2 + 2\tilde{x}_2^2 + \hat{x}_1^2 + 2\hat{x}_2^2 \geqslant x_1^2 + x_2^2$.

[1] One can also show that if the input disturbance is bounded, i.e., $w(t) \in \mathscr{L}_\infty$, then all system signals remain bounded in a semiglobal fashion (Ezal, 1998).

The nonlinear reduced-order observer (15) for this problem is

$$\dot{\hat{x}}_1 = \hat{x}_2 + x_1^2 + \frac{1}{\varepsilon}(x_1 - \hat{x}_1), \tag{26a}$$

$$\dot{\hat{x}}_2 = \frac{1}{\varepsilon}(\gamma^2 \Pi_s - 1)(x_1 - \hat{x}_1) + u, \tag{26b}$$

$$\dot{\Pi}_s = 2\Pi_s - (\gamma^2 - \Theta_s)\Pi_s^2, \tag{26c}$$

where $\hat{x}(0) = 0$, $\Theta_s(\Pi_s(t)) = \gamma^{-2}\Pi_s^{-1}(t)M\Pi_s^{-1}(t) > 0$ and $M := 2\gamma^2\Pi_s^{\infty}\Pi_s^{\infty} > 0$. We select $\Pi_s(0) = \Pi_s^{\infty}$, where Π_s^{∞} is the steady-state solution of the DRE (26c). This yields $\Pi_s(t) \equiv \Pi_s^{\infty} = 2(\gamma^2 - 2)^{-1}$ and $\Theta_s(\Pi_s^{\infty}) = 2$, and we select $\gamma = 3 > \sqrt{2}$, so that $\Pi_s(0) = \Pi_s^{\infty} = 0.29 > 0$.

In the benchmark problem, where $\dot{y} = \dot{x}_1$ is available for feedback, the extended output is defined by $\bar{y} = \dot{x}_1 - x_1^2$, and the reduced-order cost functional is obtained by setting $\tilde{x}_1 \equiv 0$ in (25), that is

$$\tilde{J}_\ell(u, w) = \int_0^\infty [2\tilde{x}_2^2 + x_1^2 + 2\hat{x}_2^2 + u^2 - \gamma^2 w^2]\,\mathrm{d}t. \tag{27}$$

The worst-case observer (B.7) for the benchmark problem with $\varpi = \bar{y} - \hat{x}_2$ is $\dot{\hat{x}}_2 = (\gamma^2\Pi_s^{\infty} - 1)\varpi + u$, and the DRE is (26c). Since $\varpi \equiv \dot{x}_1 - x_1^2 - \hat{x}_2$, we rewrite the observer dynamics as

$$\dot{x}_1 = x_1^2 + \hat{x}_2 + \varpi, \tag{28a}$$

$$\dot{\hat{x}}_2 = 1.57\varpi + u. \tag{28b}$$

For the linearization of system (28) and the cost functional

$$\mathbf{J}_\ell(u, w) = \int_0^\infty [x_1^2 + 2\hat{x}_2^2 + u^2 - \gamma^2\varpi^2]\,\mathrm{d}t,$$

the full-state feedback $\mathscr{H}_\infty$ problem has a solution $P_s > 0$ to (7) for all $\gamma > \gamma^* = 2.622 > \sqrt{2}$, resulting in $u_\ell = \mu_\ell(\mathbf{x}) = -B_2' P_s \mathbf{x} = -2.91x_1 - 4.24\hat{x}_2$. Using the Cholesky factorization $P_s = L'\Delta L$, and following the design procedure of Appendix B, we compute the locally optimal nonlinear control law as

$$u = \bar{\mu}(z) := -4.24\bar{r}^{-1}(z)z_2, \tag{29}$$

where $\bar{r}(z)$ is defined by (B.16), $R = 1$, $\bar{\sigma}(z_1) = 1.61z_1 + 0.97z_1^2$, and $z = \Phi(\mathbf{x})$ is $z_1 = x_1$ and $z_2 = \hat{x}_2 + 0.69x_1 + x_1^2$.

The control law (29), derived for the benchmark problem, together with the observer derived for the output-feedback problem (26), achieves semiglobal inverse optimality with respect to (22) and local near-optimality with respect to (25). Moreover, this dynamic output-feedback control law is locally near-suboptimal with respect to (24).

5. Conclusions

For a class of nonlinear systems with additive uncertainty, we have obtained in this paper an output-feedback control law that achieves local near-optimality and semiglobal inverse $\mathscr{H}_\infty$-optimality. If the derivative of the output measurement is also available for feedback, then the same control law with a reduced-order observer is shown (in the benchmark problem of Appendix B) to achieve local $\mathscr{H}_\infty$-optimality and global inverse $\mathscr{H}_\infty$-optimality. Unlike standard linear $\mathscr{H}_\infty$-designs which penalize the actual state of the system, the performance index we consider penalizes the estimation error and the estimate of the state. When the penalty matrices are chosen carefully, the optimal controller which corresponds to the modified performance index is shown to be suboptimal with respect to the standard $\mathscr{H}_\infty$-performance index.

The local optimality property obtained in this paper is for the linearized system with respect to a quadratic performance index. For the full-state feedback problem a higher-order locally optimal design was developed in Pan, Ezal, Krener, and Kokotović (2000) for nonquadratic cost functionals using a nonlinear Cholesky factorization. With considerable effort that approach can be extended to situations when the linearization of the system is not controllable, but the system is stabilizable via some nonlinear terms. A simple example of such a system is $\dot{x} = xu$ with the cost functional $J = \int_0^\infty (x^4 + u^2)\,\mathrm{d}t$. Similar extensions seem possible for the output-feedback case, and are open to future research.

Acknowledgements

This research was supported in part by the National Science Foundation under Grants ECS-9812346 and ECS-9896140, the Air Force Office of Scientific Research under Grants F49620-95-1-0409 and F49620-98-1-0087, and the U.S. Department of Energy under Grant DE-FG-02-97-ER-13939.

Appendix A. Nonlinear $\mathscr{H}_\infty$-filtering

The *cost-to-come* methodology and $\mathscr{H}_\infty$-filtering theory (Didinsky et al., 1993, Didinsky, Pan, & Başar, 1995; Pan & Başar, 1998; Tezcan & Başar, 1999) is summarized here in the context of the nonlinear system

$$\dot{\chi} = \hat{A}\chi + \varphi(\bar{y}) + \Gamma_1(\bar{y})w + \Gamma_2(\bar{y})u, \quad \chi(0) = \chi_0,$$
$$\bar{y} = \hat{C}_1\chi + \hat{H}_{12}(\bar{y})w, \tag{A.1}$$

where $\chi \in \mathbb{R}^{\hat{n}}$ is the state, $u \in \mathbb{R}$ is the control, $w \in \mathbb{R}^{\hat{q}}$ is the disturbance, $\bar{y} \in \mathbb{R}^{\hat{m}}$ is the output, χ_0 is unknown, while $\bar{y}(0) = \bar{y}_0$ is known. It is assumed that $N(\bar{y}) := \hat{H}_{12}(\bar{y})\hat{H}'_{12}(\bar{y}) > 0$ for all $\bar{y} \in \mathbb{R}^{\hat{m}}$. For system (A.1), we use the nonlinear observer

$$\dot{\hat{\chi}} = \hat{A}\hat{\chi} + \varphi(\bar{y}) + \hat{\Gamma}_1(\bar{y},\Pi)\varpi + \Gamma_2(\bar{y})u, \tag{A.2a}$$

$$\dot{\Pi} = \Pi\hat{A}_r' + \hat{A}_r\Pi - \Pi[\gamma^2\hat{C}_1'N^{-1}\hat{C}_1 - \Theta]\Pi + \gamma^{-2}\Gamma_1[I - \hat{H}_{12}'N^{-1}\hat{H}_{12}]\Gamma_1', \tag{A.2b}$$

where $\hat{\chi}(0) = \hat{\chi}_0$, $\Pi(0) = \gamma^{-2}\Theta_0^{-1}$, and

$$\hat{\Gamma}_1(\bar{y},\Pi) = (\Gamma_1\hat{H}_{12}' + \gamma^2\Pi\hat{C}_1')N^{-1/2}, \tag{A.3a}$$

$$\hat{A}_r(\bar{y}) = \hat{A} - \hat{K}(\bar{y})\hat{C}_1, \tag{A.3b}$$

$$\varpi(\bar{y}) = N^{-1/2}(\bar{y} - \hat{C}_1\hat{\chi}), \tag{A.3c}$$

$$\hat{K}(\bar{y}) = \Gamma_1(\bar{y})\hat{H}_{12}'(\bar{y})N^{-1}(\bar{y}) \in \mathbb{R}^{\hat{n}\times\hat{m}}. \tag{A.3d}$$

The cost functional associated with system (A.1), (A.2) is

$$\hat{J}(u,w) = \int_0^\infty [(\chi - \hat{\chi})'\Theta(\bar{y}(t))(\chi - \hat{\chi}) + l(\bar{y},\hat{\chi},u) - \gamma^2 w'w]\,dt, \tag{A.4}$$

where $l(\bar{y},\hat{\chi},u) \geqslant 0$ and $\Theta(\bar{y}(t)) > 0$. Our objective is to find an output-feedback control law which solves the differential game $\inf_u \sup_{\chi_0,w} \hat{J}(u,w) < \infty$. We assume that the optimal disturbance attenuation level $\gamma^* > 0$ exists and, hence, any desired level of attenuation $\gamma > \gamma^*$ is achievable. The cost-to-come analysis establishes that this differential game for (A.1), (A.2) is equivalent to a differential game with respect to the equivalent disturbance ϖ in the $\hat{\chi}$-coordinates:

$$\inf_u \sup_{\chi_0,w} \hat{J}(u,w) \equiv \inf_u \sup_{\varpi} \mathbf{J}(u,\varpi) < \infty,$$

where $\mathbf{J}(u,\varpi) := \int_0^\infty [l(\bar{y},\hat{\chi},u) - \gamma^2\varpi'(\bar{y})\varpi(\bar{y})]\,dt$. This new differential game constitutes the control design and the function $l(\bar{y},\hat{\chi},u)$ is constructed during the controller design procedure.

The worst-case observer (A.2) has some desirable properties. For example, the time derivative of the candidate value function $W(\tilde{\chi},\Pi) := \tilde{\chi}'\Pi^{-1}(t)\tilde{\chi} \geqslant 0$ along the error dynamics is

$$\dot{W} = -q_1(\tilde{\chi},t) - \gamma^2\varpi'\varpi - \gamma^2|w - v_f|^2 + \gamma^2 w'w,$$

where $\tilde{\chi} := \chi - \hat{\chi}$, $q_1(\tilde{\chi},t) := |\tilde{\chi}|^2_\Theta$, and the worst-case disturbance is

$$v_f := \hat{H}_{12}'N^{-1/2}\varpi - \hat{H}_{12}'N^{-1}\hat{C}_1\tilde{\chi} + \frac{1}{\gamma^2}[I - \hat{H}_{12}'N^{-1}\hat{H}_{12}]\Gamma_1'\Pi^{-1}\tilde{\chi}. \tag{A.5}$$

The solution of (A.2b) plays an important role in the stability of the closed-loop system. When $\bar{y} = 0$ this differential equation becomes

$$\dot{\Pi} = \Pi(\hat{A} - \hat{K}_0\hat{C}_1)' + (\hat{A} - \hat{K}_0\hat{C}_1)\Pi - \Pi[\gamma^2\hat{C}_1'N_0^{-1}\hat{C}_1 - \hat{Q}_1]\Pi + \gamma^{-2}\Gamma_{1_0}[I - \hat{D}_1'N_0^{-1}\hat{D}_1]\Gamma_{1_0}', \tag{A.6}$$

where $\Gamma_{1_0} := \Gamma_1(0)$, $\hat{D}_1 := \hat{H}_{12}(0)$, $N_0 := N(0)$, $\hat{K}_0 := \hat{K}(0)$, and $\hat{Q}_1 > 0$ is some prespecified matrix.

Lemma A.1. *Given $\gamma > 0$ and $\hat{Q}_1 > 0$, suppose that (A.6) with $\dot{\Pi} \equiv 0$ has a solution $\Pi_\infty = \Pi_\infty' > 0$ such that $\hat{A} - \hat{K}_0\hat{C}_1 - \gamma^2\Pi_\infty\hat{C}_1'N_0^{-1}\hat{C}_1$ is Hurwitz. Let $\Pi_0 = \Pi_\infty$ and $\Theta(\Pi) := \gamma^{-2}\Pi^{-1}(t)M\Pi^{-1}(t) > 0$ for some $M > 0$, and suppose there exists $c_y > 0$ such that $\|\bar{y}(t)\|_\infty \leqslant c_y$. Then, there exist continuous decreasing positive functions $\beta_\pi^-(c_y)$ and $\beta_\Theta^-(c_y)$, and increasing positive functions $\beta_\pi^+(c_y)$ and $\beta_\Theta^+(c_y)$, such that $\forall t \geqslant 0$,*

$$0 < \beta_\pi^-(c_y)I \leqslant \Pi^{-1}(t) \leqslant \beta_\pi^+(c_y)I < \infty,$$

$$0 < \beta_\Theta^-(c_y)I \leqslant \Theta(\Pi(t)) \leqslant \beta_\Theta^+(c_y)I < \infty. \tag{A.7}$$

Moreover, if $M := \gamma^2\Pi_\infty\hat{Q}_1\Pi_\infty$ and $\lim_{t\to\infty}\bar{y}(t) = 0$, then $\lim_{t\to\infty}\Pi(t) = \Pi_\infty$ and $\lim_{t\to\infty}\Theta(t) = \hat{Q}_1$.

The proof of Lemma A.1, given in Ezal, (1998), relies on the continuity of solutions of algebraic Riccati equations (Lancaster & Rodman, 1995, Theorem 11.2.1) and on comparison theorems for differential Riccati equations (Freiling & Jank, 1996).

Appendix B. Benchmark problem: limiting performance

Here we formulate and solve a benchmark problem which is obtained from the main problem of Section 2 by making one additional assumption:

Assumption B.1. The derivative of the output, $\dot{y}$, is available for feedback.

The solution to this problem, in addition to being of independent interest, helps to gauge the inverse-optimality and local near-optimality of the solution of the main problem.

With $\varphi(x_1) := A_{21\{2\}}x_1 + \breve{f}_{\{2\}}(x_1)$, system (1) is equivalently written as

$$\bar{y} := \dot{x}_1 - a_{11}x_1 - \breve{f}_1(x_1) = \hat{C}_1 x_{\{2\}} + g_1(x_1)w, \tag{B.1a}$$

$$\dot{x}_{\{2\}} = A_{\{2\}}x_{\{2\}} + \varphi(x_1) + G_{1\{2\}}(x_1)w + B_{2\{2\}}u, \tag{B.1b}$$

$$y = C_1 x, \tag{B.1c}$$

where $\bar{y}$ is the extended output. For the linearization of system (B.1) our objective is to find a stabilizing control law $u_\ell = \mu_l(y_\ell, \hat{x}_{\{2\}})$ which minimizes the cost functional (4) when $x_1 \equiv \hat{x}_1$, that is

$$\tilde{J}_l(u_\ell, w_\ell) = \int_0^\infty [\tilde{x}_{\{2\}}'Q_1\tilde{x}_{\{2\}} + \mathbf{x}'Q_2\mathbf{x} + Ru^2 - \gamma^2 w_\ell' w_\ell]\,dt, \tag{B.2}$$

where $\mathbf{x} := [x_1\hat{x}_{\{2\}}]'$, and $Q_1 := \tilde{Q}_{1\{2\}} > 0$ and $Q_2 := \tilde{Q}_2 > 0$ satisfy $\tilde{x}'\tilde{Q}_1\tilde{x} + \hat{x}'\tilde{Q}_2\hat{x} \geqslant x'Qx$. For the nonlinear system (B.1), we consider a cost functional of the form

$$\tilde{J}(u,w)=\int_0^\infty [q_1(\tilde{x}_{\{2\}},t)+q_2(\mathbf{x},t)+r(\mathbf{x},t)u^2-\gamma^2 w'w]\,\mathrm{d}t, \tag{B.3}$$

where $q_1(\tilde{x}_{\{2\}},t):=\tilde{x}'_{\{2\}}\Theta_s(t)\tilde{x}_{\{2\}}\geqslant 0$, $q_2(\mathbf{x},t)\geqslant 0$, and $r(\mathbf{x},t)>0$ are allowed to depend on the output waveform $x_1([0,t])$ in a causal manner. Our objective is inverse optimality, and hence we do not specify $\Theta_s(t)$, $q_2(\mathbf{x},t)$, and $r(\mathbf{x},t)$ beforehand. For simultaneous local optimality and global inverse optimality, we require that $\Theta_s(0)=Q_1$, $q_{2_{\mathbf{xx}}}(0,0)=Q_2$, and $r(0,0)=R$.

B.1. Local design

Following the cost-to-come procedure with $\varpi_\ell(\bar{y}_\ell,\hat{x}_{\{2\}})=N_0^{-1/2}(\bar{y}_\ell-\hat{C}_1\hat{x}_{\{2\}})$, the linear observer is

$$\dot{\hat{x}}_{\{2\}}=A_{21\{2\}}x_1+A_{\{2\}}\hat{x}_{\{2\}}+\hat{\Gamma}_{1_o}\varpi_\ell+B_{2\{2\}}u_\ell, \tag{B.4}$$

where $\hat{\Gamma}_{1_o}$ is defined by Assumption 3. The observer dynamics (B.4) and the equivalent disturbance $\varpi_\ell(\bar{y}_\ell,\hat{x}_{\{2\}})$ can be written as

$$0=\pi_s(0,\Pi_s^\infty), \tag{B.5a}$$

$$\dot{\mathbf{x}}=A\mathbf{x}+\hat{B}_1\varpi_\ell+B_2u_\ell, \tag{B.5b}$$

where $\hat{B}_1$ is defined by Assumption 3 and $\Pi_s^\infty>0$ is the solution of the filtering GARE (6) which is denoted by (B.5a).

Lemma B.1. *Under Assumptions* 1–3, *and* B.1, *the control law* $u_\ell=\mu_\ell(\mathbf{x}):=-R^{-1}B_2'P_s\mathbf{x}$, *applied to the linearization of system* (B.1), *in conjunction with the linear observer* (B.4), *is* $\mathscr{H}_\infty$*-optimal with respect to the cost functional* (B.2), *and is suboptimal with respect to the cost functional* (2). *When* $w_\ell\equiv 0$, *the control law enders the equilibrium* $(x,\hat{x}_{\{2\}})=(0,0)$ *exponentially stable. Moreover, for all* $w_\ell(t)\in\mathscr{L}_2$, *all system signals are bounded and converge to zero as* $t\to\infty$.

Proof. The time derivatives of the filtering $W_\ell(\tilde{x}_{\{2\}})$ and the control $V_\ell(\mathbf{x})$ value functions are

$$\dot{W}_\ell=-|\tilde{x}_{\{2\}}|^2_{Q_1}-\gamma^2\varpi_\ell'\varpi_\ell-\gamma^2|w_\ell-v_{f_\ell}|^2+\gamma^2w_\ell'w_\ell,$$

$$\dot{V}_\ell=-\mathbf{x}'Q_2\mathbf{x}-Ru_\ell^2+\gamma^2\varpi_\ell'\varpi_\ell-\gamma^2[\varpi_\ell-v_{c_\ell}]^2+R[u_\ell-\mu_\ell]^2,$$

where

$$v_{f_\ell}:=b_1'N_0^{-1/2}\varpi_\ell-b_1'N_0^{-1}\hat{C}_1\tilde{x}_{\{2\}}+\gamma^{-2}[I-b_1'N_0^{-1}b_1]B_{1\{2\}}(\Pi_s^\infty)^{-1}\tilde{x}_{\{2\}}$$

is the linear worst-case filtering disturbance, and $v_{c_\ell}(\mathbf{x}):=\gamma^{-2}\hat{B}_1'P_s\mathbf{x}$ is the linear worst-case control disturbance. Hence, the time derivative of the composite value function

$$U_\ell(\mathbf{x},\tilde{x}_{\{2\}}):=V_\ell(\mathbf{x})+W_\ell(\tilde{x}_{\{2\}})\equiv\tilde{x}'_{\{2\}}(\Pi_s^\infty)^{-1}\tilde{x}_{\{2\}}+\mathbf{x}'P_s\mathbf{x} \tag{B.6}$$

is

$$\dot{U}_\ell=-\tilde{x}'_{\{2\}}Q_1\tilde{x}_{\{2\}}-\mathbf{x}'Q_2\mathbf{x}-Ru_\ell^2+\gamma^2w_\ell'w_\ell-\gamma^2|w_\ell-v_{f_\ell}|^2-\gamma^2(\varpi_\ell-v_{c_\ell})^2+R[u_\ell-\mu_\ell]^2.$$

Therefore, the control law $u_\ell=\mu_\ell(\mathbf{x})$ is $\mathscr{H}_\infty$-optimal with respect to the cost functional (B.2) where

$$v_\ell(\mathbf{x},\tilde{x}_{\{2\}}):=b_1'N_0^{-1/2}v_{c_\ell}(\mathbf{x})-b_1'N_0^{-1}\hat{C}_1\tilde{x}_{\{2\}}+\gamma^{-2}[I-b_1'N_0^{-1}b_1]\hat{M}_{\{2\}}B_{1\{2\}}(\Pi_s^\infty)^{-1}\tilde{x}_{\{2\}}$$

is the worst-case linear disturbance. Moreover, if $w_\ell\equiv 0$, then $(\mathbf{x},\tilde{x}_{\{2\}})=(0,0)$ is exponentially stable, and if $w_\ell(t)\in\mathscr{L}_2$, then all signals are bounded and converge to zero as $t\to\infty$ (Teel, 1999). Furthermore, since $Q_1:=\tilde{Q}_{1\{2\}}$ and $Q_2:=\tilde{Q}_2$ satisfy $\tilde{x}'\tilde{Q}_1\tilde{x}+\hat{x}'\tilde{Q}_2\hat{x}\geqslant x'Qx$, we have

$$\dot{U}_\ell\leqslant -x'Qx-Ru_\ell^2+\gamma^2w_\ell'w_\ell-\gamma^2|w_\ell-v_{f_\ell}|^2-\gamma^2(\varpi_\ell-v_{c_\ell})^2+R(u_\ell-\mu_\ell)^2.$$

Integration of both sides and the fact that all system states converge to zero as $t\to\infty$ yield $U_\ell^*(x)\leqslant U_\ell(\mathbf{x},\tilde{x}_{\{2\}})$, which implies that $u_\ell=-R^{-1}B_2'P_s\mathbf{x}$ is suboptimal with respect to the cost functional (2), where $U_\ell^*(x)$ given by (11) is the optimal value function for the standard $\mathscr{H}_\infty$-optimal linear output-feedback problem. □

B.2. Global design

We now proceed to the stabilization of the nonlinear system (B.1) using the cost-to-come methodology of Appendix A. The observer for this system is

$$\dot{\Pi}_s=\Pi_s\hat{A}'+\hat{A}\Pi_s-\Pi_s[\gamma^2\hat{C}_1'N^{-1}(x_1)\hat{C}_1-\Theta_s]\Pi_s+\gamma^{-2}\hat{B}\hat{B}', \tag{B.7a}$$

$$\dot{\hat{x}}_{\{2\}}=A_{\{2\}}\hat{x}_{\{2\}}+\varphi(x_1)+\hat{\Gamma}_1(x_1,\Pi_s)\varpi+B_{2\{2\}}u, \tag{B.7b}$$

where

$$\varpi(x_1,\bar{y},\hat{x}_{\{2\}}):=N^{-1/2}(x_1)(\bar{y}-\hat{C}_1\hat{x}_{\{2\}}), \tag{B.8}$$

$\hat{C}_1\equiv A_{12\{2\}}$, and $\hat{A}(x_1)$, $\hat{B}(x_1)$ and $\hat{\Gamma}_1(x_1,\Pi_s)$ are defined by (A.3). Without loss of generality, we set $\hat{x}_{\{2\}}(0)\equiv 0$, while $x_{\{2\}}(0)$ is unknown. The matrix Π_s is the solution of (B.7a) with initial condition $\Pi_s(0)=\Pi_s^\infty>0$. We let $\Theta_s(\Pi_s):=\gamma^{-2}\Pi_s^{-1}M\Pi_s^{-1}$ where $M:=\gamma^2\Pi_s^\infty Q_1\Pi_s^\infty>0$. This choice of $\Theta_s(\Pi_s)$ guarantees that if $x_1(t)\in\mathscr{L}_\infty$, then there exist constant positive definite matrices which bound $\Pi_s(t)$ from above and from below for all $t\geqslant 0$

(Lemma A.1). Furthermore,

$$\lim_{t\to\infty} x_1(t) = 0 \Rightarrow \lim_{t\to\infty} \Pi_s(t) = \Pi_s^{\infty} \quad \text{and} \quad \lim_{t\to\infty} \Theta_s(t) = Q_1.$$

The value function for the optimal filtering problem is $W = \tilde{x}'_{\{2\}}\Pi_s^{-1}\tilde{x}_{\{2\}}$, and

$$\dot{W} = -|\tilde{x}_{\{2\}}|^2_{\Theta_s(\Pi_s)} - \gamma^2\varpi'\varpi - \gamma^2|w - v_f|^2 + \gamma^2 w'w,$$

where the worst-case filtering disturbance $v_f(x_1, \Pi_s)$ is given by (A.5). Moreover, since $\inf_u \sup_{x_{\{2\}}(0),w} \tilde{J}(u,w) \equiv \inf_u \sup_{\varpi} \mathbf{J}(u,\varpi)$, where

$$\mathbf{J}(u,\varpi) := \int_0^{\infty} [q_2(\mathbf{x},t) + r(\mathbf{x},t)u^2 - \gamma^2\varpi'\varpi]\,dt, \tag{B.9}$$

the output-feedback control problem is transformed into an equivalent full-state differential game problem in the **x**-coordinates.

The observer dynamics (B.7) and the equivalent disturbance (B.8) are rewritten in the **x**-coordinates as

$$\dot{\Pi}_s = \pi_s(\mathbf{x}_1, \Pi_s), \tag{B.10a}$$

$$\dot{\mathbf{x}} = A\mathbf{x} + \breve{f}(\mathbf{x}_1) + \hat{G}_1(\mathbf{x}_1, \Pi_s)\varpi + B_2 u, \tag{B.10b}$$

where $\hat{G}_1(\mathbf{x}_1, \Pi_s) := [N^{1/2}(\mathbf{x}_1) \quad \hat{\Gamma}'_1(\mathbf{x}_1, \Pi_s)]'$. Our objective is to design a control law for system (B.10) which is locally optimal and globally inverse optimal. Other than the fact that (B.10) depends on Π_s, the recursive design of the nonlinear control law follows the procedure of Ezal et al. (2000). This is because Π_s, $\dot{\Pi}_s$ and $\mathbf{x}_1 = x_1$ are all available for feedback.

The linearization of this system is given by (B.5) where $\hat{B}_1 \equiv \hat{G}_1(0, \Pi_s^{\infty})$. As shown in Ezal et al. (2000), the solution P_s of the GARE (7) admits a unique Cholesky factorization of the form $P_s = L'\Delta L$, where L is a lower triangular matrix with ones along its diagonal and $\Delta := \mathrm{diag}\{\delta_1, \ldots, \delta_n\} > 0$. The transformation $z = L\mathbf{x}$ of (B.5) results in $\dot{z} = \bar{A}z + \bar{B}_1\varpi + B_2 u$, where $\bar{A} = LAL^{-1}$, $\bar{B}_1 = L\hat{B}_1$, and $\bar{Q}_2 := L'^{-1}Q_2L^{-1}$. We recall from Ezal et al. (2000) that at each step of the n-step backstepping procedure L dictates the linear portion of the nonlinear virtual control law, while Δ defines the Lyapunov function.

We initiate the design by defining $z_1 := \phi_1(\mathbf{x}_1) := \mathbf{x}_1$, $\alpha_0 := 0$, and $\bar{V}_0 := 0$. Thereafter, on the ith step of the design procedure, we define an error variable $z_{i+1} := \phi_{i+1}(\mathbf{x}_{[i+1]}, \Pi_s) := x_{i+1} - \bar{\alpha}_i(z_{[i]}, \Pi_s)$, and the control Lyapunov function $\bar{V}_i(z_{[i]}) := \bar{V}_{i-1} + \delta_i z_i^2 = z'_{[i]}\Delta_{[i]}z_{[i]} > 0$. The virtual control law $\bar{\alpha}_i(z_{[i]}, \Pi_s)$ is designed to dominate the harmful nonlinearities and disturbances at each step while simultaneously matching the optimal linear design. This procedure terminates with $z_n := \phi_n(\mathbf{x}, \Pi_s) := \mathbf{x}_n - \bar{\alpha}_{n-1}(z_{[n-1]}, \Pi_s)$, which completes the construction of the diffeomorphism $z = \Phi(\mathbf{x}, \Pi_s)$ and the control Lyapunov function (CLF)

$$\bar{V}(z) = \bar{V}_{n-1} + \delta_n z_n^2 = z'\Delta z > 0. \tag{B.11}$$

The original system (B.10) is thus transformed into

$$\dot{\Pi}_s = \pi_s(z_1, \Pi_s), \tag{B.12a}$$

$$\dot{z} = \bar{A}z + \breve{f}(z, \Pi_s) + \bar{G}_1(z, \Pi_s)\varpi + B_2 u, \tag{B.12b}$$

where $\Phi^{-1}(z, \Pi_s) - L^{-1}z$ contains only higher-order terms so that the linear part of $\Phi(\mathbf{x}, \Pi_s)$ is $L\mathbf{x}$, that is $z = L\mathbf{x} + \breve{\Phi}(\mathbf{x}, \Pi_s)$. Thus, in both z and $\mathbf{x}$ coordinates, the properties of the linearized systems are the same. Moreover, the derivative of the CLF $\bar{V}$ for system (B.12) is

$$\begin{aligned}\dot{\bar{V}} = &-z'\bar{Q}_2 z - \bar{r}(z, \Pi_s)u^2 + \gamma^2\varpi'\varpi - \gamma^2|\varpi - \bar{v}_c|^2 \\ &+ \bar{r}(z, \Pi_s)(u - \bar{\mu})^2 - (\bar{r}^{-1}(z, \Pi_s) - R^{-1})\delta_n^2 z_n^2 \\ &+ 2z_n\delta_n\bar{\eta}(z, \Pi_s),\end{aligned}$$

where $\bar{r}(z, \Pi_s)$ is a design function, $\bar{\mu}(z, \Pi_s)$ is the control law

$$u = \bar{\mu}(z, \Pi_s) := -\bar{r}^{-1}(z, \Pi_s)B'_2\Delta z \tag{B.13}$$

and $\bar{v}_c(z, \Pi_s) := (1/\gamma^2)\bar{G}_1(z, \Pi_s)\Delta z$ is the worst-case (control) disturbance. The remaining dynamics, which need to be dominated by $\bar{r}(z, \Pi_s)$, are contained in the function $\bar{\eta}(z, \Pi_s)$.

We now complete the output-feedback control design by constructing a nonlinear control law with the following properties:

Benchmark Theorem. *Under Assumptions* 1–3 *and B*.1, *there exist functions* $q_1(\tilde{x}_{\{2\}}, \Pi_s) \geqslant 0$, $q_2(\mathbf{x}, \Pi_s) \geqslant 0$ *and* $r(\mathbf{x}, \Pi_s) > 0$, *such that the control law*

$$u = \mu(\mathbf{x}, \Pi_s) := -r^{-1}(\mathbf{x}, \Pi_s)B'_2\Delta\Phi(\mathbf{x}, \Pi_s), \tag{B.14}$$

applied to system (B.1) *with the robust observer* (B.7) *is locally* $\mathscr{H}_\infty$*-optimal with respect to the cost functional* (B.2) *and globally inverse* $\mathscr{H}_\infty$*-optimal with respect to the cost functional* (5). *In the absence of a disturbance*, $w \equiv 0$, *the equilibrium* $(x, \hat{x}_{\{2\}}, \Pi_s) = (0, 0, \Pi_s^{\infty})$ *is GAS and LES. In the presence of* $w(t) \in \mathscr{L}_2$, *all system signals are bounded and* $(x, \hat{x}_{\{2\}}, \Pi_s) \to (0, 0, \Pi_s^{\infty})$ *as* $t \to \infty$.

Proof. The functions $\bar{r}(z, \Pi_s)$ and $\bar{q}_2(z, \Pi_s)$ are to be selected to achieve global inverse optimality with respect to cost functional (B.9) with $q_2(\mathbf{x}, t) := \bar{q}_2(\Phi(\mathbf{x}, \Pi_s), \Pi_s(t))$ and $r(\mathbf{x}, t) := \bar{r}(\Phi(\mathbf{x}, \Pi_s), \Pi_s(t))$. For a meaningful cost $\mathbf{J}(u, \varpi)$, we restrict $\bar{q}_2(z, \Pi_s)$ to be positive definite and $\bar{r}(z, \Pi_s)$ to be positive for all $z \in \mathbb{R}^n$ and all $\Pi_s > 0$. In addition, to meet the objective of local $\mathscr{H}_\infty$-optimality, we require $\bar{r}(0, \Pi_s^{\infty}) = R > 0$ and $\bar{q}_{2_{zz}}(0, \Pi_s^{\infty}) = \bar{Q}_2 > 0$. A particular choice of $\bar{q}_2(z, \Pi_s)$ and $\bar{r}(z, \Pi_s)$ meet all of our requirements:

$$\begin{aligned}\bar{q}_2(z, \Pi_s) := &|z_{[n-1]} + \bar{Q}^{-1}_{2[n-1]}(\bar{q}_{2_{2,1}} - \bar{\eta}'_1\delta_n)z_n|^2_{\bar{Q}2[n-1]} \\ &+ (\bar{q}_{2_{2,2}} - \bar{q}'_{2,1}\bar{Q}^{-1}_{2[n-1]}\bar{q}_{2_{2,1}})z_n^2 \\ &+ (\bar{r}^{-1}(z, \Pi_s) - R^{-1} - \bar{\sigma}(z, \Pi_s))\delta_n^2 z_n^2,\end{aligned} \tag{B.15}$$

$$\bar{r}(z, \Pi_s) := \left(\sqrt{R^{-2} + \bar{\sigma}_r^2(z, \Pi_s)} + \bar{\sigma}_r(z, \Pi_s)\right)^{-1}, \tag{B.16}$$

where $\bar{\sigma}_r$, $\bar{\sigma}$, $\bar{\eta}_1$ and $\bar{\eta}_2$ satisfy $\bar{\sigma}_r(z,\Pi_s) := \bar{\sigma}(z,\Pi_s) + |z_{[n-1]}|^2$,

$$\bar{\sigma}(z,\Pi_s) := \bar{\eta}_1 \bar{Q}^{-1}_{2[n-1]}\bar{\eta}_1' - 2\bar{q}'_{2_{2,1}}\bar{Q}^{-1}_{2[n-1]}\delta_n^{-1}\bar{\eta}_1' + 2\delta_n^{-1}\bar{\eta}_2,$$

$$\bar{\eta}(z,\Pi_s) \equiv \bar{\eta}_1(z_{[n-1]},\Pi_s)z_{[n-1]} + \bar{\eta}_2(z,\Pi_s)z_n,$$

$$z'\bar{Q}_2 z \equiv z'_{[n-1]}\bar{Q}_{2[n-1]}z_{[n-1]} + 2z'_{[n-1]}\bar{q}_{2_{2,1}}z_n + \bar{q}_{2_{2,2}}z_n^2,$$

$\bar{\sigma}(0,\Pi_s^\infty) = 0$, $\bar{\eta}_1(0,\Pi_s^\infty) = 0$, and $\bar{\eta}_2(0,\Pi_s^\infty) = 0$. Moreover, this choice guarantees that $\bar{q}_2(z,\Pi_s)$ is positive definite and radially unbounded for all $z \neq 0$ and $\Pi_s > 0$. Evaluating the derivative of (B.11),

$$\dot{\bar{V}} = -\bar{q}_2(z,\Pi_s) - \bar{r}(z,\Pi_s)u^2 + \gamma^2\varpi'\varpi - \gamma^2|\varpi - \bar{v}_c|^2 + \bar{r}(z,\Pi_s)(u-\bar{\mu})^2,$$

we conclude that $\bar{V}(z)$ is the value function for the cost functional (B.9). Likewise, $V(\mathbf{x},\Pi_s) := \bar{V}(\Phi(\mathbf{x},\Pi_s))$ is the value function in the $\mathbf{x}$-coordinates, and (B.14) is the locally optimal and globally inverse optimal full-state feedback control law for system (B.10).

With this control law, the system composed of (B.1) and the observer dynamics (B.7) is

$$\dot{\xi}_s = \mathscr{A}_1(\xi_s) + \mathscr{B}_1(\xi_s)w + \mathscr{B}_2 u, \tag{B.17}$$

where $\varpi = N^{-1/2}(x_1)(\tilde{x}_2 + g_1(x_1)w)$ and $\xi_s := [\mathbf{x}'\tilde{x}'_{\{2\}} \quad \mathbf{\Pi}_s']'$. The value function for the extended system is defined as

$$U(\xi_s) := V(\mathbf{x},\Pi_s) + W(\tilde{x}_{\{2\}},\Pi_s) \geqslant 0 \tag{B.18}$$

and its time derivative is

$$\dot{U} = -q_1(\tilde{x}_{\{2\}}\,\Pi_s) - q_2(\mathbf{x},\Pi_s) - r(\mathbf{x},\Pi_s)u^2 + \gamma^2 w'w - \gamma^2|w - v_f|^2 - \gamma^2|\varpi - v_c|^2 + r(\mathbf{x},\Pi_s)(u-\mu)^2,$$

where $\mu(\mathbf{x},\Pi_s)$ is defined by (B.14), $v_c(\mathbf{x},\Pi_s) := \bar{v}_c(\Phi(\mathbf{x},\Pi_s),\Pi_s)$,

$$q_1(\tilde{x}_{\{2\}},\Pi_s) := |\tilde{x}_{\{2\}}|^2_{\Theta_s(\Pi_s)}, \qquad q_2(\mathbf{x},\Pi_s) := \bar{q}_2(\Phi(\mathbf{x},\Pi_s),\Pi_s),$$

and $r(\mathbf{x},\Pi_s) := \bar{r}(\Phi(\mathbf{x},\Pi_s),\Pi_s)$.

Note that $U(\xi_s)$ is positive definite and radially unbounded in $(\mathbf{x}(t),\tilde{x}_{\{2\}}) \in \mathbb{R}^{2n-1}$ for each $\Pi_s > 0$, and that there exists a function $\rho(\cdot) \in \mathscr{K}_\infty$, such that $\rho(|x_1|) \leqslant V(\mathbf{x},\Pi_s) \leqslant U(\xi_s)$ for all $\mathbf{x} \in \mathbb{R}^n$, $\tilde{x}_{\{2\}} \in \mathbb{R}^{n-1}$, and all $\Pi_s \in \mathbb{R}^{n-1\times n-1}$. However, since $U(\xi_s)$ is not radially unbounded in Π_s, we need show that the solution $\xi_s(t)$ is defined for all $t \in [0,\infty)$.

Let us assume that there exists a finite $t^* > 0$, such that $|\xi_s(t)| \to \infty$ as $t \to t^*$. Integrating $\dot{U}$ with $u = \mu(\mathbf{x},\Pi_s)$ and $w(t) \in \mathscr{L}_2$, we have $U(\xi_s(t)) \leqslant U(\xi_s(0)) + \gamma^2\|w(t)\|_2^2$ for all $t \in [0,t^*)$. Hence, for each $x(0) \in \mathbb{R}^n$, with $\hat{x}_{\{2\}}(0) = 0_{n-1}$ and $\Pi_s(0) = \Pi_s^\infty$, there exists a positive constant β_2, such that $U(\xi_s(t)) \leqslant \beta_2$ and $\rho(|x_1|) \leqslant V(\mathbf{x},\Pi_s) \leqslant \beta_2$ for all $t \in [0,t^*)$. Therefore, $|x_1(t)| \leqslant \rho^{-1}(\beta_2)$ for all $t \in [0,t^*)$, which by Lemma A.1 and causality implies that $\Pi_s(t)$ is bounded from above and below by positive definite matrices for all $t \in [0,t^*)$. Furthermore, since $V(\mathbf{x},\Pi_s) \leqslant \beta_2$ and $W(\tilde{x}_{\{2\}},\Pi_s) \leqslant \beta_2$, both $\mathbf{x}(t)$ and $\tilde{x}_{\{2\}}(t)$ are uniformly bounded in $[0,t]$ for all $t \in [0,t^*)$. Hence, by the continuity of solutions, there does not exist a finite time after which the solutions are not defined, and all signals are bounded. Therefore, we have $\mathbf{x}(t) \to 0$, and $\tilde{x}_{\{2\}}(t) \to 0$ as $t \to \infty$ (Teel, 1999). Hence, $x(t) \to 0$ as $t \to \infty$. Furthermore, by Lemma A.1, we have $\Pi_s(t) \to \Pi_s^\infty$ and $\Theta_s(\Pi_s(t)) \to Q_1$ as $t \to \infty$.

The proof of inverse optimality is immediate from the fact that the value function U satisfies the HJI equality

$$U_{\xi_s}\mathscr{A}_1 + \frac{1}{4}U_{\xi_s}\left(\frac{1}{\gamma^2}\mathscr{B}_1\mathscr{B}_1' - \mathscr{B}_2 r^{-1}(\xi_s)\mathscr{B}_2'\right)U'_{\xi_s} + \hat{q}(\xi_s) = 0 \tag{B.19}$$

with the optimal control and disturbance pair

$$u = \mu(\mathbf{x},\Pi_s) \equiv -\frac{1}{2}r^{-1}(\xi_s)\mathscr{B}_2' U'_\xi,$$

and

$$w = v(\mathbf{x},\tilde{x}_{\{2\}},\Pi_s) \equiv \frac{1}{2\gamma^2}\mathscr{B}_1' U'_\xi,$$

where

$$v(\mathbf{x},\tilde{x}_{\{2\}},\Pi_s) := g_1'(x_1)N^{-1/2}(x_1)v_c(\mathbf{x},\Pi_s) - g_1'(x_1)N^{-1}(x_1)\hat{C}_1\tilde{x}_{\{2\}} + \gamma^{-2}[I - g_1'(x_1)N^{-1}(x_1)g_1(x_1)] \times G_{1\{2\}}(x_1)\Pi_s^{-1}\tilde{x}_{\{2\}},$$

$r(\xi_s) \equiv r(\mathbf{x},\Pi_s)$, and $\hat{q}(\xi_s) := q_1(\tilde{x}_{\{2\}},\Pi_s) + q_2(\mathbf{x},\Pi_s)$.

At the equilibrium, $\xi_{s_0} := (0, 0'_{n-1}, 0'_{n-1}, \mathbf{\Pi}_s^{\infty\prime})$, and (B.18) satisfies

$$\xi_s'[U_{\xi_s\xi_s}(\xi_{s_0})]\xi_s = \tilde{x}'_{\{2\}}(\Pi_s^\infty)^{-1}\tilde{x}_{\{2\}} + \mathbf{x}'P_s\mathbf{x} \equiv U_\ell(\mathbf{x},\tilde{x}_{\{2\}}),$$

where $U_\ell(\mathbf{x},\tilde{x}_{\{2\}})$ is the value function (B.6) for the linear problem. Hence, the control law (B.14) achieves local $\mathscr{H}_\infty$-optimality with respect to the cost functional (B.2). Moreover, by Lemma B.1, the control law (B.14) also achieves local suboptimality with respect to the cost functional (2). □

Appendix C. Proof of the Main Theorem

With the output-feedback worst-case disturbance

$$\hat{v}(\xi,\zeta) := \frac{1}{2\gamma^2}(\mathscr{B}_{11}' U'_\xi + \mathscr{B}_{21}'\mathscr{U}'_\zeta + \varepsilon\mathscr{B}_{11}'\mathscr{U}'_\xi) \equiv \frac{1}{2\gamma^2}\mathscr{G}_1'(\xi_c)\mathscr{V}'_{\xi_c} \tag{C.1}$$

and

$$\Sigma(\xi_c,u) := \mathscr{B}_{11}\mathscr{B}_{11}'U'_\xi + \mathscr{B}_{11}\mathscr{B}_{21}'\mathscr{U}'_\zeta + 2\gamma^2(\mathscr{A}_{11} + \mathscr{A}_{12}\zeta + \mathscr{B}_{12}u),$$

we compute

$$\dot{\mathscr{V}} = U_\xi(\mathscr{A}_{11} + \mathscr{A}_{12}\zeta + \mathscr{B}_{12}u) + \mathscr{U}_\zeta(\mathscr{A}_{21} - \mathscr{A}_{22}\zeta) + \gamma^2 w'w - \gamma^2|w - \hat{v}|^2 + \frac{1}{4\gamma^2}(\mathscr{B}'_{11}U'_\xi + \mathscr{B}'_{21}\mathscr{U}'_\zeta)' \times(\mathscr{B}'_{11}U'_\xi + \mathscr{B}'_{21}\mathscr{U}'_\zeta) + \frac{\varepsilon^2}{4\gamma^2}\,\mathscr{U}_\xi\mathscr{B}_{11}\mathscr{B}'_{11}\mathscr{U}'_\xi + \frac{\varepsilon}{2\gamma^2}\,\mathscr{U}_\xi\Sigma(\xi_c, u).$$

Rewriting $\mathscr{U}_\zeta$ as $\mathscr{U}_\zeta \equiv 2\gamma^2(\zeta - \mathscr{A}_{22}^{-1}\mathscr{A}_{21} - (1/2\gamma^2)\mathscr{A}_{22}^{-1}\mathscr{B}_{21}\mathscr{B}'_{11}U'_\xi)'\mathscr{A}_{22}^{-1}$, completing the squares with respect to ζ and u, and using the HJI identity (B.19), we compute

$$\dot{\mathscr{V}} = -\hat{q}(\xi) - r(\xi)u^2 - \gamma^2|\zeta - h(\xi)|^2 + \gamma^2 w'w - \gamma^2|w - \hat{v}(\xi,\zeta)|^2 + r(\xi)(u - \mu(\xi))^2 + \frac{\varepsilon^2}{4\gamma^2}\mathscr{U}_\xi\mathscr{B}_{11}\mathscr{B}'_{11}\mathscr{U}'_\xi + \frac{\varepsilon}{2\gamma^2}\mathscr{U}_\xi\Sigma(\xi_c, u), \quad \text{(C.2)}$$

where $h(\xi) := \mathscr{A}_{22}^{-1}\mathscr{A}_{21} + (1/2\gamma^2)\mathscr{A}_{22}^{-1}\mathscr{B}_{21}\mathscr{B}'_1 U'_\xi$. For any fixed positive constant $\varepsilon_1 < 1$, we can rewrite (C.2) as

$$\dot{\mathscr{V}} = -(1 - \varepsilon_1)\hat{q}(\xi) - r(\xi)u^2 - (1 - \varepsilon_1)\gamma^2|\zeta - h(\xi)|^2 + \gamma^2 w'w - \gamma^2|w - \hat{v}(\xi,\zeta)|^2 + r(\xi)(u - \mu(\xi))^2 - E(\varepsilon_1; \xi_c),$$

where

$$E(\varepsilon_1; \xi_c) := \varepsilon_1\hat{q}(\xi) + \varepsilon_1\gamma^2|\zeta - h(\xi)|^2 - (\varepsilon^2/4\gamma^2)\mathscr{U}_\xi\mathscr{B}_{11}\mathscr{B}'_{11}\mathscr{U}'_\xi - \frac{\varepsilon}{2\gamma^2}\mathscr{U}_\xi\Sigma(\xi_c, \mu(\xi)).$$

However, since $\dot{\mathscr{V}}$ is not negative definite for all ξ_c, it is not possible to conclude the global stability of system (20). Instead, it is possible to prove semiglobal stability (with respect to ε).

By Assumption 1, whenever $\xi(t)$ (and therefore $x_1(t)$) exists in a compact set for all $t \geqslant 0$, there exist positive constants c_1, c_2 and c_3, independent of ε, such that

$$|U_\xi| \leqslant c_1(|x_1| + |\tilde{x}_{\{2\}}| + |\tilde{x}_{\{2\}}), \quad \text{(C.3a)}$$

$$|\mathscr{U}_\xi| \leqslant c_2(|x_1| + |\tilde{x}_{\{2\}}| + |\tilde{x}_{\{2\}} + |\zeta - h(\xi)|), \quad \text{(C.3b)}$$

$$|\Sigma(\xi_c, \mu(\xi))| \leqslant c_3(|x_1| + |\tilde{x}_{\{2\}}| + |\tilde{x}_{\{2\}}| + |\zeta - h(\xi)|). \quad \text{(C.3c)}$$

Furthermore, since $\Theta_s(\Pi_s) = \gamma^{-2}\Pi_s^{-1}M\Pi_s^{-1} > 0$, and (A.7) holds, the state-penalty function $\hat{q}(\xi) = |\tilde{x}_{\{2\}}|^2_{\Theta(\Pi_s)} + q_2(x_1, \hat{x}_{\{2\}}, \Pi_s) \geqslant 0$ satisfies

$$\hat{q}(\xi) \geqslant c_{q_1} U(\xi) \geqslant c_{q_2}(|x_1|^2 + |\tilde{x}_{\{2\}}|^2 + |\tilde{x}_{\{2\}}|^2) \geqslant 0 \quad \text{(C.4)}$$

for all ξ in the compact set and some constants $c_{q_1} > 0$ and $c_{q_2} > 0$.

Let $\beta_2 := 2\beta + \gamma^2 c^2_{w_2}$ for some known $\varepsilon_2 > 0$, and suppose that $\mathscr{V}(\xi_c(t)) \leqslant \beta_2 + \varepsilon_2$ for all $t \geqslant 0$. Recalling that $\rho(|x_1|) \leqslant V(\mathbf{x}, \Pi_s) \leqslant U(\xi) \leqslant \mathscr{V}(\xi_c)$ where $\rho(\cdot) \in \mathscr{K}_\infty$, we note that $|x_1(t)| \leqslant \rho^{-1}(\beta_2 + \varepsilon_2)$ for all $t \geqslant 0$. Hence, by Lemma A.1, we have $\Pi_s(t) \in \Omega_\pi(\rho^{-1}(\beta_2 + \varepsilon_2))$ for all $t \geqslant 0$, where

$$\Omega_\pi(c_y) := \{\Pi_s > 0 \mid 0 < \beta_\pi^-(c_y)I \leqslant \Pi_s^{-1} \leqslant \beta_\pi^+(c_y)I < \infty\}$$

and $\beta_\pi^-(c_y)$ and $\beta_\pi^+(c_y)$ are strictly positive, continuous functions of $c_y > 0$. Furthermore, the set $\Omega_\pi(c_y)$ has the property that $\Omega_\pi(c_y^0) \subseteq \Omega_\pi(c_y)$ for all $c_y^0 \leqslant c_y$.

It then follows from $\mathscr{V}(\xi_c(t)) \leqslant \beta_2 + \varepsilon_2$ that both $\xi(t)$ and $\zeta(t)$ belong to compact sets for all $t \geqslant 0$, i.e., $\xi(t) \in \Omega_\xi(\beta_2 + \varepsilon_2) \cap \Omega_\pi(\rho^{-1}(\beta_2 + \varepsilon_2))$ and $\zeta(t) \in \Omega_\zeta(\beta_2 + \varepsilon_2, \varepsilon) \cap \Omega_\pi(\rho^{-1}(\beta_2 + \varepsilon_2))$. Therefore, $\xi_c(t) \in \Omega(\beta_2 + \varepsilon_2, \varepsilon) \cap \Omega_\pi(\rho^{-1}(\beta_2 + \varepsilon_2))$ for all $t \geqslant 0$. Moreover, by properties (C.3) and (C.4), there exists a set of sufficiently small values of ε such that $E(\varepsilon_1, \xi_c) \geqslant 0$ for all $\xi \in \Omega_\xi(\beta_2 + \varepsilon_2) \cap \Omega_\pi(\rho^{-1}(\beta_2 + \varepsilon_2))$. The complement of this set is

$$\mathscr{E}_2^*(\beta, c_{w_2}, \varepsilon_1, \varepsilon_2) := \{\varepsilon > 0 \,|\, E(\varepsilon_1; \xi_c) < 0,\ \forall \xi \in \Omega_\xi(\beta_2 + \varepsilon_2) \cap \Omega_\pi(\rho^{-1}(\beta_2 + \varepsilon_2))\}.$$

The upper bound ε_2^* on the admissible values of ε is $\varepsilon_2^* := \inf_\varepsilon \mathscr{E}_2^*(\beta, c_{w_2}, \varepsilon_1, \varepsilon_2)$. If $\xi \in \Omega_\xi(\beta_2 + \varepsilon_2) \cap \Omega_\pi(\rho^{-1}(\beta_2 + \varepsilon_2))$ and $\varepsilon \in (0, \varepsilon_2^*)$, then

$$\dot{\mathscr{V}} \leqslant -(1 - \varepsilon_1)\hat{q}(\xi) - r(\xi)u^2 - (1 - \varepsilon_1)\gamma^2|\zeta - h(\xi)|^2 + \gamma^2 w'w - \gamma^2|w - \hat{v}(\xi, \zeta)|^2 + r(\xi)(u - \mu(\xi))^2. \quad \text{(C.5)}$$

Let us now suppose that (21) holds for some $\varepsilon \in (0, \varepsilon_2^*)$, then $\mathscr{V}(\xi(0), \zeta(0)) \leqslant 2\beta < \beta_2$. Suppose further that there exists a set Ω_T, such that

$$\xi_c(t) \notin \Omega(\beta_2, \varepsilon) \cap \Omega_\pi(\rho^{-1}(\beta_2)), \quad \forall t \in \Omega_T. \quad \text{(C.6)}$$

Let $t^* := \min_t \Omega_T$. This implies, by the continuity of solutions, that $\beta_2 < \mathscr{V}(\xi_c(t^*)) < \beta_2 + \varepsilon_2$. Therefore, by causality, we have $\xi(t) \in \Omega_\xi(\beta_2 + \varepsilon_2) \cap \Omega_\pi(\rho^{-1}(\beta_2 + \varepsilon_2))$ and $\xi_c(t) \in \Omega(\beta_2 + \varepsilon_2, \varepsilon) \cap \Omega_\pi(\rho^{-1}(\beta_2 + \varepsilon_2))$ for all $t \in [0, t^*]$. Moreover, inequality (C.5) holds for all $t \in [0, t^*]$. Integrating both sides of (C.5), we find that $\mathscr{V}(\xi(t^*), \zeta(t^*)) \leqslant 2\beta + \gamma^2 c_w^2 = \beta_2 < \beta_2 + \varepsilon_2$, or $t^* \notin \Omega_T$, which contradicts our supposition (C.6). Therefore, $\Omega_T \equiv \emptyset$. Hence, if (21) holds, and $w(t) \in \mathscr{W}_2(c_{w_2})$, then $\xi_c(t) \in \Omega(\beta_2, \varepsilon) \cap \Omega_\pi(\rho^{-1}(\beta_2))$ and the dissipation inequality (C.5) is valid for all $t \geqslant 0$. The fact that $\xi_c(t) \to \xi_{c_0}$ as $t \to \infty$ follows from Teel (1999) and Lemma A.1.

It follows from (C.5) that whenever (21) holds for all $\varepsilon \in (0, \varepsilon_2^*)$, then $\mathscr{V}(\xi_c)$ satisfies the HJI inequality

$$\mathscr{V}_{\xi_c}\mathscr{F}(\xi_c) + \frac{1}{4}\mathscr{V}_{\xi_c}\left[\frac{1}{\gamma^2}\mathscr{G}_1\mathscr{G}'_1 - \mathscr{G}_2 r^{-1}\mathscr{G}'_2\right]\mathscr{V}'_{\xi_c} + \mathscr{Q} \leqslant 0,$$

where $\mathscr{Q} := (1 - \varepsilon_1)\hat{q}(\xi) + (1 - \varepsilon_1)\gamma^2|\zeta - h(\xi)|^2$, with the optimal control (19), that is, $u = \mu(\xi_c) \equiv -\frac{1}{2}r^{-1}(\xi_c)\mathscr{G}'_2\mathscr{V}'_{\xi_c}$, and the worst-case disturbance (C.1). Hence, the control law (19) is semiglobally (in ε) inverse $\mathscr{H}_\infty$-optimal with respect to the cost functional (22).

Finally, it follows from $\xi'_c[\mathscr{V}_{\xi_c\xi_c}(\xi_{c_0})]\xi_c \equiv U_\ell(\xi) + O(\varepsilon)$ that the leading quadratic terms of the series expansion of $\mathscr{V}(\xi_c)$ are $O(\varepsilon)$-close to the $\mathscr{H}_\infty$-optimal value function for the linear problem, which implies that the control law (19) is near-optimal with respect to cost functional (4). Near-suboptimality of (19) with respect to (2) is derived from Lemma 1, and the fact that $U_\ell^* \leqslant U_\ell$. □

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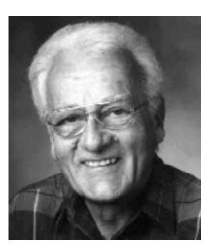

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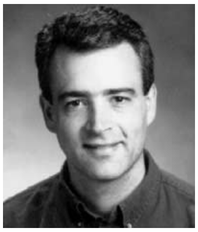

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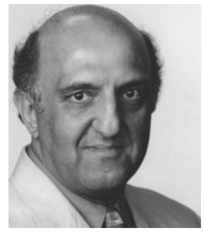

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