# ECE 553

## – Optimum Control Systems –

## Spring 2006

## Recap of Lectures 5-9

Isoperimetric Problem

Differential Constraints

Optimal Control: Pontryagin's MP

LQR (linear-quadratic regulator)

Corner Conditions / Broken Extremals

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# Isoperimetric Problem

$$\min J(x) = \int_{t_o}^{t_f} \phi[x, \dot{x}, t]dt \quad \text{s.t.} \quad \int_{t_o}^{t_f} m[x, \dot{x}, t]dt = k$$

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Let $x^o$ be a solution to this isoperimetric CV problem, with fixed initial and terminal conditions $(x_o, x_f)$

Further let, when evaluated at $x^o$,

$$\alpha(t) := m_x - \tfrac{d}{dt} m_{\dot{x}} \neq 0 \text{ for some } t \in (t_o, t_f)$$

$\Rightarrow$ **the constraint is regular at** $x^o$

Then, there exists a scalar $\lambda$ such that $x^o$ satisfies the E-L equation associated with the Lagrangian $L(x, \dot{x}, t; \lambda) := \phi(x, \dot{x}, t) + \lambda m(x, \dot{x}, t)$, that is

$$L_x(x^o, \dot{x}^o, t; \lambda) - \frac{d}{dt} L_{\dot{x}}(x^o, \dot{x}^o, t; \lambda) = 0$$

If $x(t_f)$ is free, then provided that $m_{\dot{x}} \neq 0$ at $t = t_f$, we have the NBC :

$$L_{\dot{x}}(x^o(t_f), \dot{x}^o(t_f), t_f; \lambda) = 0$$

# **Isoperimetric Problem** (continued)

$$\min J(x) = \int_{t_o}^{t_f} \phi[x, \dot{x}, t]dt \quad \text{s.t.} \quad \int_{t_o}^{t_f} m[x, \dot{x}, t]dt = k$$

$$L(x, \dot{x}, t; \lambda) := \phi(x, \dot{x}, t) + \lambda\, m(x, \dot{x}, t)$$

## A Second-Order Necessary Condition

$$\int_{t_o}^{t_f} \left[ L_{\dot{x}\dot{x}} \dot{n}'^2 + \left( L_{xx} - \frac{d}{dt} L_{x\dot{x}} \right) n'^2 \right] dt \geq 0 \qquad (\star)$$

for all continuously differentiable $n'$ satisfying

$$\int_{t_o}^{t_f} \left( m_x - \frac{d}{dt} m_{\dot{x}} \right) n'^2 \, dt = 0 \quad \left[ \int_{t_o}^{t_f} \alpha(t) n'(t)^2 dt = 0 \right]$$

A sufficient condition for this necessary condition is

$$L_{\dot{x}\dot{x}} \geq 0 \qquad \text{and} \qquad L_{xx} - \frac{d}{dt} L_{x\dot{x}} \geq 0$$

If one of these is positive, then $(\star)$ is positive, and we have a **weak local minimum**.

# Differential (Pointwise) Constraints

$$\min J(x_1, x_2) = \int_{t_o}^{t_f} \phi[x_1, x_2, \dot{x}_1, \dot{x}_2, t]dt$$

$$\text{s.t. } \Lambda(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = 0 \;\; \forall t \in [t_o, t_f]$$

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Let $\underline{x}^o = (x_1^o, x_2^o)$ be a solution to this CV problem with differential constraints, with fixed initial and terminal conditions ($x_1, x_2$ specified at $t_o$ and $t_f$. Then there exists a continuously differentiable function $\lambda(\cdot)$ such that $\underline{x}^o$ satisfies the E-L equation associated with the Lagrangian

$$L(\underline{x}^o, \underline{\dot{x}}^o, t; \lambda(t)) := \phi[\underline{x}^o, \underline{\dot{x}}^o, t] + \lambda(t)\Lambda(\underline{x}^o, \underline{\dot{x}}^o, t)$$

that is

$$L_{x_i} - \frac{d}{dt} L_{\dot{x}_i} = 0\,, \quad i = 1, 2\,; \quad \text{at } \; \underline{x} = \underline{x}^o$$

**provided that**

$\Lambda_{x_1}^o, \Lambda_{x_2}^o, \Lambda_{\dot{x}_1}^o, \Lambda_{\dot{x}_2}^o$ do not vanish simultaneously.

## **Differential (Pointwise) Constraints** (continued)

– natural extension to multiple differential constraints –

$$\min J(\underline{x}) = \int_{t_o}^{t_f} \phi[\underline{x}, \dot{\underline{x}}, t]dt\,, \quad \dim(\underline{x}) = n$$

$$\text{s.t.}\ \ \Lambda_j(\underline{x}, \dot{\underline{x}}, t) = 0 \ \ j = 1, 2, \ldots, r < n, \quad \forall t \in [t_o, t_f]$$

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Lagrangian: $L(\underline{x}, \dot{\underline{x}}, t; \underline{\lambda}(t)) := \phi + \sum_{j=1}^{r} \lambda_j(t)\Lambda_j$

Solution $\underline{x}^o$ satisfies the E-L eqn associated with $L$ (the multivariable version):

$$L_{x_i} - \frac{d}{dt}L_{\dot{x}_i} = 0\,, \ \ i = 1, 2, \ldots, n\,; \ \ \text{at} \ \ \underline{x} = \underline{x}^o$$

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NBCs : If $x_i(t_f)$ is free, $L_{\dot{x}_i}|_{t=t_f} = 0$ for $\underline{x} = \underline{x}^o$.

Transversality conditions : If $t_f$ is free and $x_i(t_f)$ is to lie on the curve $c_i(\cdot)$, for each $i = 1, \ldots, n$, we have

$$\left[\sum_{i=1}^{n} L_{\dot{x}_i}\left[\dot{c}_i - \dot{x}_i\right] + L\right]\Bigg|_{t=t_f} = 0 \quad \text{for } \underline{x} = \underline{x}^o$$

along with $x_i^o(t_f) = c_i(t_f)\,, \ \ i = 1, \ldots, n$.

# Optimal Control: Pontryagin's MP

$$\min J(\underline{u}) = q(t_f, \underline{x}(t_f)) + \int_{t_o}^{t_f} g(\underline{x}(t), \underline{u}(t), t)\, dt$$

$$\text{s.t.}\;\; \dot{\underline{x}} = f(\underline{x}, \underline{u}, t)\,, \;\; \underline{x}(t_o) = \underline{x}_o\,, \;\; t \in [t_o, t_f]\,,$$

$$\underline{x}(t) \in \mathcal{R}^n\,, \;\; \underline{u}(t) \in \mathcal{R}^r \;\; \forall t \in [t_o, t_f]$$

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Convert to CV problem with $n$ differential constraints, and $n + r$ variables. Lagrangian ⇒

$$L(\underline{x}, \underline{u}, t; \underline{\lambda}) = q_t + \sum_{i=1}^{n} q_{x_i} \dot{x}_i + g - \sum_{i=1}^{n} \lambda_i [f_i - \dot{x}_i]$$

$$= q_t + \sum_{i=1}^{n} [q_{x_i} + \lambda_i] \dot{x}_i + H(\underline{x}, \underline{u}, t; -\underline{\lambda})$$

⇒ A form of the Minimum Principle $\quad (p_i := -\lambda_i)$

$$\frac{d}{dt} p_i(t) = -H_{x_i}(\underline{x}, \underline{u}, t; \underline{p})\,, \;\; i = 1, \ldots, n$$

$$\frac{d}{dt} x_i(t) = H_{p_i}(\underline{x}, \underline{u}, t; \underline{p})\,, \;\; i = 1, \ldots, n$$

$$H_{u_j}(\underline{x}, \underline{u}; \underline{p}) = 0\,, \; j = 1, \ldots, r\,; \; p_i(t_f) = q_{x_i}(t_f, \underline{x}(t_f))$$

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Note: $H := g + \underline{p}^T f$ *Hamiltonian* / $\underline{p}$ is *co-state*

## Optimal Control: Pontryagin's MP (continued)

$$\min J(\underline{u}) = q(t_f, \underline{x}(t_f)) + \int_{t_o}^{t_f} g(\underline{x}(t), \underline{u}(t), t)\, dt$$

$$\text{s.t. } \dot{\underline{x}} = f(\underline{x}, \underline{u}, t)\,, \quad \underline{x}(t_o) = \underline{x}_o\,, \quad t \in [t_o, t_f]\,,$$

$$\underline{x}(t) \in \mathcal{R}^n\,, \quad \underline{u}(t) \in \mathcal{R}^r \quad \forall t \in [t_o, t_f]$$

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- If $x_i(t_f)$ is not free, but given to equal $x_{if}$, replace the bdary condition on $p_i$ with $x_i^o(t_f) = x_{if}$.
- If $f$ and $g$ do not explicitly depend on t,

$\frac{d}{dt} H = \sum_{i=1}^n H_{x_i} \dot{x}_i + \sum_{j=1}^r H_{u_j} \dot{u}_j + \sum_{i=1}^n H_{p_i} \dot{p}_i \equiv 0$

$\Rightarrow$ H is a constant on optimum trajectories.
(holds even if $\underline{u}^o$is not an inner solution)

- If $t_f$ is free, $\underline{x}(t_f) = \underline{c}(t_f)$, and $q \equiv 0$, using transversality conditions:

$$H(\underline{x}, \underline{u}, t; \underline{p})|_{t=t_f} = \sum_{i=1}^n p_i(t_f) \dot{c}(t_f)$$

- $t_f$ is free, no other terminal constraint, $q \not\equiv 0 \Rightarrow$

$$\Big[L - \sum_{i=1}^n L_{\dot{x}_i} \dot{x}_i\Big]\Big|_{t=t_f} = 0 \Rightarrow \big[H + q_t\big]\big|_{t=t_f} = 0$$

- Minimum time problem ($q \equiv t$)

$$\Rightarrow \quad H|_{t=t_f} = -1$$

# LQR (linear-quadratic regulator)

– generalized version –

$$\dot{x} = Ax + Bu + c\,, \quad x(t_o) = x_o \text{ given}$$

$$J(u) = |x(t_f)|^2_{Q_f} + 2x^T(t_f)q_f +$$

$$+ \int_{t_o}^{t_f} \big(|x(t)|^2_Q + 2x^T Mu + 2x^T q\big)dt$$

$$R > 0, \;\; Q \geq 0, \; Q_f \geq 0, \; \begin{pmatrix} Q & M \\ M^T & R \end{pmatrix} \geq 0$$

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Two-point boundary value problem:

$$\dot{p} = -Qx - Mu - q - p^T A\,, \quad p(t_f) = Q_f x(t_f) + q_f$$

$$\dot{x} = Ax + Bu + c\,, \quad x(t_o) = x_o$$

and $\quad \frac{1}{2}H_u = Ru + M^T x + B^T p = 0 \quad \Rightarrow$

$$u = -R^{-1}\big(M^T x + B^T p\big)$$

Solution: $\quad p = Px + k\,, \quad P(t_f) = Q_f, k(t_f) = q_f$

$$\dot{P} + PA + A^T P - (PB + M)R^{-1}(PB + M)^T + Q = 0$$

$$\dot{k} + A^T k + Pc - (PB + M)R^{-1}B^T k + q = 0\,, \quad k(t_f) = q_f$$

$$\min J(u) = x_o^T P(t_o)x_o + 2k^T(t_o)x_o + 2m(t_o)$$

$$\dot{m} + k^T c - \tfrac{1}{2}k^T BR^{-1}B^T k = 0\,, \quad m(t_f) = 0$$

## LQR (linear-quadratic regulator)

### – generalized version –

$$\dot{x} = Ax + Bu + c\,, \quad x(t_o) = x_o \text{ given}$$

$$J(u) = |x(t_f)|^2_{Q_f} + 2x^T(t_f)q_f + \\ + \int_{t_o}^{t_f} \left(|x(t)|^2_Q + 2x^T M u + 2x^T q\right) dt$$

$$R > 0, \;\; Q \geq 0, \; Q_f \geq 0, \; \begin{pmatrix} Q & M \\ M^T & R \end{pmatrix} \geq 0$$

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Solution is not only local, but also global minimum

Completion of squares:

$$J(u) \equiv \int_{t_o}^{t_f} \left| u + R^{-1}(M^T x + B^T P x + B^T k) \right|^2_R dt \\ + x_o^T P(t_o) x_o + 2k^T(t_o) x_o + 2m(t_o)$$

This is valid whenever the RDE for P admits a unique nnd solution on $[t_o, t_f]$. Hence a sufficient condition for existence of a unique minimizing solution is $R > 0$ along with existence of a unique solution to the RDE. If $M = 0$ (the standard LQR, when also $q = 0$), RDE has a unique solution if, in addition to $R > 0$, $Q \geq 0$ and $Q_f \geq 0$.

## Corner Conditions / Broken Extremals

Solutions to CV problems are not always continuously differentiable everywhere. They could have corners, in which case an extremal is known as a broken extremal. The additional conditions that need to be satisfied for a piecewise continuously differentiable curve to be optimal are known as Weierstrass-Erdmann corner conditions.

The first follows from E-L eqn in integral form:

$$\phi_{\dot{x}}(x^o(t), \dot{x}^o(t), t) = \int_{t_o}^{t} \phi_x(x^o(s), \dot{x}^o(s), s)\, ds + \text{const}$$

If $x^o$ has a corner at $t = c$, by continuity of the RHS:

$$\phi_{\dot{x}}(x^o(c^-), \dot{x}^o(c^-), c^-) = \phi_{\dot{x}}(x^o(c^+), \dot{x}^o(c^+), c^+)$$

which is the 1st WE corner condition. The 2nd one is

$$\left[\phi - \dot{x}\phi_{\dot{x}}\right]\Big|_{t=c^-} = \left[\phi - \dot{x}\phi_{\dot{x}}\right]\Big|_{t=c^+}$$

which follows, again by continuity, from

$$\phi - \dot{x}\phi_{\dot{x}} = \int_{t_o}^{t} \phi_s ds + \text{const}$$

another form of the E-L equation.

# Corner Conditions / Broken Extremals

– extensions –

- In case of multiple variables, the second one is replaced by

  continuity of $\phi - \sum_{i=1}^{n} \dot{x}_i \phi_{\dot{x}_i}$ at a corner

- For CV problems with isoperimetric or differential constraints, simply replace $\phi$ with the appropriate Lagrangian. Hence, we need continuity of

  $$L_{\underline{\dot{x}}} \quad \text{and} \quad L - \underline{\dot{x}}^T L_{\underline{\dot{x}}} \quad \text{at a corner}$$

- For an optimal control problem, discontinuity in $u$ (such as bang-bang control) leads to a broken extremal in $x$. Note that

  $$L_{\underline{\dot{x}}} = -\underline{p}, \quad \text{and} \quad L - \underline{\dot{x}}^T L_{\underline{\dot{x}}} = H$$

  Therefore, we have as necessary conditions that both the Hamiltonian $H$ and the co-state vector $\underline{p}$ have to be continuous across a corner ($t = c$):

  $$p_i(c^-) = p_i(c^+), \quad i = 1, \ldots, n$$

  $$H|_{t=c^-} = H|_{t=c^+}$$