University of Illinois

ECE 553 **SPRING 2006**

Professor Tamer Başar May 4, 2006

FINAL EXAM

DUE DATE: Tuesday, May 9, 2006, 12 noon, in my office

NOTE: There are five (5) problems for you to solve. Please provide the answers to each Problem on a **separate** sheet. The problems carry equal weight.

Problem 1

Obtain a piecewise continuously differentiable function $x_{[-1,1]}$ that minimizes the functional

$$J(x) = \int_{-1}^{1} x^2(t)[4 - \dot{x}^2(t)]^2 dt$$

subject to the end-point and interior-point constraints

$$x(-1) = 0, \quad x(0) = 1, \quad x(1) = -1$$

Is the solution you have obtained unique?

Problem 2

For each of the following two objective functions, obtain a globally minimizing control $u_{[0,1]}$ in the class of continuous functions. Discuss uniqueness of the solution in each case.
[You may use your favorite method to obtain the solutions, but be sure to justify your steps.]

a.

$$J(u) = \int_0^1 \big(1 + \int_0^t u(s)ds\big)^2 dt + \int_0^1 u^2(t)dt$$

b.

$$L(u) = \int_0^1 \big(1 + \int_t^1 u(s)ds\big)^2 dt + \int_0^1 u^2(t)dt$$

Problem 3

Given the linear plant

$$\dot{x}_1 = x_2 , \quad \dot{x}_2 = u ,$$

consider the problem of driving the state (x_1, x_2) from the *origin* at time $t = 0$ to the line $2x_1 - x_2 = 1$ at time $t = 1$.

a. We wish to obtain an **open-loop** control $u_{[0,1]}$ that accomplishes this task, while minimizing the control energy

$$J(u) = \int_0^1 u^2(t)dt .$$

Solve this optimal control problem by employing:
(i) the minimum principle of Pontryagin;
(ii) the calculus of variations.

b. Let k be the smallest scalar such that the optimal open-loop control u^o above satisfies the bound

$$|u^o(t)| \leq k, \quad \text{for all} \quad t \in [0, 1] .$$

First determine the value of k.
Now, we wish to drive the plant state from the *origin* to the same line $2x_1 - x_2 = 1$ in **minimum time**.

Prove that there exists a solution to this *time-optimal* control problem when the magnitude of the control is bounded by k, and provide a complete description for the optimal control.

Problem 4

You are given a zero-sum static game with objective function

$$J(u,w) = 4(1+u+w)^2 + u^2$$

where u is the minimizing variable taking values on the real line, and w is the maximizing variable taking values in the closed interval $[0,1]$.

a. Obtain the upper and lower values of this game.

b. Obtain a mixed-strategy saddle-point solution. Is the solution unique?

c. Let u^* be chosen as an element of the support set of the solution in part b) above for the minimizing player. Consider the two intervals $[u^* - 0.5, u^* + 0.5]$ and $[0,1]$ for the minimizing player and the maximizing player, respectively, and their uniform discretization into 5 subintervals, with the value of $J(u,w)$ on each of the 25 squares thus formed being taken equal to its value at the center of the corresponding square. Obtain the saddle-point solution of the resulting 5×5 matrix game (in pure or mixed strategies, whichever exists), and compare it with the saddle-point solution of the original game.

d. Repeat (c) above for another related 5×5 matrix game, whose *ij*'th entry is now equal to :

$$J(u^* - 0.4 + 0.2(i-1), 0.2(j-1)), \; i,j = 1,\ldots,5$$

Problem 5

We want to design H^∞ controls for the following two-dimensional system, under different information patterns:

$$\begin{aligned} \dot{x}_1 &= x_2 + w \\ \dot{x}_2 &= x_1 + x_2 + u \end{aligned} \quad ; \qquad y = x_1 + v$$

Here, x_1 and x_2 denote the two states, whose initial values are completely unknown, u is the scalar control variable, y is the measured output, w is the scalar system disturbance, and v is the measurement disturbance. With this system, we associate the following performance index, where t_f is the terminal time:

$$L(u;w,v) = \{\int_0^{t_f} [\,|x_1(t) + x_2(t)|^2 + |u(t)|^2\,]\}/\{\,|x_1(0)|^2 + |x_2(0)|^2 + \int_0^{t_f} [\,|w(t)|^2 + |v(t)|^2\,]\,\}$$

a. Let $t_f = \infty$, and the controller have access to perfect state information. Denote the optimum (minimax) performance level for this problem (that is, the optimum level of disturbance attenuation) by γ^*. Obtain the value of γ^*, and a controller that will ensure a performance level no worse than $\gamma = \gamma^* + 0.01$.

b. Repeat the above under the original imperfect state information.

c. Now take the time horizon to be finite, and particularly $t_f = 1$. Furthermore take the initial state to be known to be zero. Obtain (approximately) the γ^* for this problem under perfect state measurements by numerically solving the corresponding Riccati differential equation for different values of γ.

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