# ASSIGNMENT 1

**Reading Assignment:** Correspondences # 1 & # 2; *Gelfand & Fomin*, chp. 1.

**Problems** (to be handed in): Due Date: **Wednesday, February 1**.

**1.** Given the parametrized (in $\lambda$) family of functions

$$x = x(t) \equiv \lambda t(1-t)\,, \qquad 0 \le t \le 1\,,$$

determine tight upper and lower bounds on $\lambda$ so that
i) $x \in N^s_\epsilon(0)$ for $\epsilon = \frac{1}{2}$, ii) $x \in N^w_\epsilon(0)$ for $\epsilon = \frac{1}{2}$,
where $N^s_\epsilon(0)$ and $N^w_\epsilon(0)$ are, respectively, the strong and weak $\epsilon$ neighborhoods of the function that is identically *zero*.
*Note:* Given a continuously differentiable function $x(t)$ on an interval $[t_o, t_f]$, it is in a strong $\epsilon$ neighborhood of *zero* if, and only if, $\|x\|_s < \epsilon$, and it is in a weak $\epsilon$ neighborhood of *zero* if, and only if, $\|x\|_w < \epsilon$, where $\|x\|_s$ and $\|x\|_w$ are respectively the strong and weak norms of $x$, defined by

$$\|x\|_s := \max_{t\in[t_o,t_f]} \{|x(t)| + |\dot{x}(t)|\} \quad \text{and} \quad \|x\|_w := \max_{t\in[t_o,t_f]} |x(t)|\,.$$

(iii) For a *strong norm*, the above is not the only possible definition. Another possibility is:

$$\|x\|_s := \max_{t\in[t_o,t_f]} \{|x(t)|\} + \max_{t\in[t_o,t_f]} \{|\dot{x}(t)|\}$$

Now obtain the solution to (i) with this definition of a strong norm.

**2.** Show that even though $x = x(t) \equiv 0$ satisfies the Euler-Lagrange equation associated with the following calculus of variations problem, it cannot yield a strong local minimum for it:

$$\min_{x[0,\pi]} \int_0^\pi x^2(t)\big(1 - \dot{x}^2(t)\big)\,\mathrm{d}t, \quad x(0) = x(\pi) = 0.$$

(*Hint:* Consider, as a competing candidate, the function $x = x(t) \equiv (1/\sqrt{n})\sin nt$, where $n$ is an integer.)

**3.** Write down the Euler-Lagrange equation associated with the following calculus of variations problem, and obtain its solution:

$$\min_{x[0,\ln 2]} \int_0^{\ln 2} x(t)[1 + \dot{x}^2(t)]^{1/2}\,\mathrm{d}t, \quad x(0) = 1,\ x(\ln 2) = \frac{5}{4}.$$

**4.** A linear differential system is described by $\dot{x} = Ax + Bu$ , where $x$ and $u$ are 2-dimensional vectors (state and control, respectively), and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We wish to find a control vector $u(t) = \big(u_1(t),\ u_2(t)\big)^T$, over the interval $[0, 2]$, such that the performance index

$$J = \int_0^2 |u(t)|^2 \,\mathrm{d}t$$

is minimized, subject to the partial end-point constraints $x(0) = (1,\ 1)^T,\ x_1(2) = 0$, where $x_1$ denotes the first component of $x$.
Solve this problem (based on first-order necessary conditions) by converting it to a problem in the calculus of variations.

**5.** Make suitable assumptions (on $\phi$) and derive an Euler-Lagrange equation type first order condition, along with the associated natural boundary conditions, for the free end-point calculus of variations problem:

$$\min_{x_{[0,1]}} \int_0^1 \phi[t, x(t), \dot{x}(t), \ddot{x}(t)] \,\mathrm{d}t$$

*Note:* Here we seek a minimum in the class of twice continuously differentiable functions, defined on the interval $[0, 1]$.

**6.** You are given a control system described by

$$\ddot{x}(t) = u(t), \quad x(0) = -2\sqrt{2}, \ \ \dot{x}(0) = 5\sqrt{2}\,,$$

which will be steered to the target set $\Gamma = \{(x, \dot{x}) \in \mathbf{R}^2 : x^2 + \dot{x}^2 = 1\}$ (a circle) in unit time, while minimizing the control energy

$$J = \int_0^1 u^2(t) \,\mathrm{d}t\ .$$

Convert this into a calculus of variations problem, and obtain its solution (based on the first-order conditions you obtained above for *Problem 5*, and their slight generalizations).

**7.** Let

$$J(x) = \psi(x(t_f)) + \int_{t_0}^{t_f} \phi[t, x(t), \dot{x}(t)] \,\mathrm{d}t$$

where $t_0$, $t_f$ are fixed, $x(t_0)$ and $x(t_f)$ are free, and $\psi$, $\phi$ are twice continuously differentiable in their arguments. Consider the problem of minimizing $J$ subject to the *mixed* end-point constraint

$$x(t_0) + 2x(t_f) = 1.$$

If $x^o$ denotes an optimal solution to this problem, obtain the first-order necessary condition(s), along with the requisite boundary conditions.

**8.** Find the extremals of the functional

$$J(x) = \int_0^{\pi/2} [\ddot{x}^2 - x^2 + t^2] \,\mathrm{d}t$$

subject to the boundary conditions:

$$x(0) = 1\,, \quad \dot{x}(0) = 0\,, \quad x(\pi/2) = 0\,, \quad \dot{x}(\pi/2) = 1\,.$$

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