ECE 553 **SPRING 2006**

Correspondence # 11 February 24, 2006

ASSIGNMENT 4

Reading Assignment: *Gelfand & Fomin*, chp. 5 (pp. 97-125) – for conjugate point condition; Correspondence # 13 (from the book by *George Leitmann*); Correspondence # 14 (chapter 5 of *Sage & White*, sections 5.3, 5.4);

Recommended Reading: *Bertsekas*, pp. 105-142.
Sethi & Thompson, chps 2-4 (for applications of the maximum principle to finance, production planning, and marketing, see chps 5-7)

Problems (to be handed in): Due Date: **Wednesday, March 8**.

Announcement : This is the last homework set before the midterm, scheduled for March 13th

22. Consider the calculus of variations problem of minimizing

$$J(x) = \int_0^{3\pi/2} [\dot{x}^2(t) - x^2(t)]\, \mathrm{d}t$$

subject to the end-point restrictions

$$x(0) = 0, \;\; x(3\pi/2) = 1.$$

This problem admits the unique extremal: $x^0(t) = -\sin t$, which also satisfies the strengthened *Legendre* condition. But, as we know, this is not a sufficient condition for x^o to be a local minimum. An additional necessary condition is *Jacobi*'s "conjugate point" condition, which was discussed in class on Wednesday (February 22); see also assigned reading from Gelfand & Fomin.

i) For the problem above, write down the *Jacobi*'s equation and show that there exists a point in the open interval $(0, 3\pi/2)$ that is conjugate to $t = 0$.

ii) Existence of a conjugate point implies that there exists an admissible variation $\eta_{[0,3\pi/2]}$ around $x^0_{[0,3\pi/2]}$ which makes $J(x^o + \epsilon\eta)$ smaller than $J(x^o)$, for all $\epsilon \in (-\epsilon_o, \epsilon_o)$, for some $\epsilon_o > 0$. Show explicitly that this is indeed the case, by choosing the variation as $\eta(t) = \sin \frac{2t}{3}$.

23. Consider the optimal control problem of minimizing

$$J(u) = \int_0^2 (u^2 + 3u - 2x)\, dt$$

subject to $\dot{x} = x + u$, $x(0) = 5$, and the control constraint $0 \le u \le 2$. **Show** that the optimal solution is of the form:

$$u^*(t) = \begin{cases} 2 & \text{if } 0 \le t < t_1 \\ \alpha(t) & \text{if } t_1 \le t \le t_2 \\ 0 & \text{if } t_2 < t \le 2 \end{cases}$$

and **find** the expression for $\alpha(t)$, and the values of t_1 and t_2.

24. Consider the optimal control problem of minimizing

$$J(u) = \int_0^2 x(t)\, dt$$

subject to $\dot{x} = u$, $x(0) = 1$, and the control constraint $-1 \leq u \leq 1$.
(i) **Show** that the optimal control is a constant. **Find** this constant, and an expression for the corresponding co-state variable $p(t)$. What is the optimum value of J?
(ii) Now consider the same problem as above, but under the additional state constraint: $x \geq 0$. **Show** that the optimal control is now of the form:

$$u^*(t) = \begin{cases} -1 & \text{if } 0 \leq t < t_1 \\ 0 & \text{if } t_1 \leq t \leq 2 \end{cases}$$

Find t_1, and an expression for the corresponding co-state variable $p(t)$. What is the optimum value of J in this case?

25. For the second-order system:

$$\dot{x} = \begin{pmatrix} -2 & 2 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \quad x(0) = x_0,$$

we wish to drive the system state $x := (x_1,\ x_2)'$ from a given initial state x_0, to the origin in *minimum time*, by using a control u that is bounded in magnitude by 1, i.e. $|u(t)| \leq 1$.
Show that the optimal control is of *bang-bang* type, and switches sign at most once.
Obtain an expression for the switching curve.

26. **Answer** the same questions as in *Problem 25* above, this time for the system:

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} y + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u(t), \quad y(0) = y_0.$$

In addition, **show** that the two problems are related by the linear transformation:

$$y(t) = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} x(t)$$

How does this linear transformation affect the switching curves of the two problems ?

27. Read Example 2.5 of Correspondence # 13 (pp. 48-53), and **obtain** *time-optimal controls* for the following two cases:
(a) We wish to transfer the state from an arbitrary initial state to the line $x_1 = 1$.
(b) We wish to transfer the state from an arbitrary initial state to the line $x_2 = 0$.

28. Consider the system whose state equations are

$$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = -x_1 + u_2,$$

with $|u_1| \leq 1$ and $|u_2| \leq 1$. **Obtain** the controller that would transfer an arbitrary initial state to the *origin* in minimum time. **Sketch** the general nature of the switching curves.

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