ECE 553 **SPRING 2006**

Correspondence # 20 March 28, 2006

ASSIGNMENT 5

Reading Assignment: Correspondence # 19 (chapters 8 & 9 of *Sage & White*);
Bertsekas, pp. 101-110.;
Kwakernaak & Sivan: pp. 85-113, 253-269, 339-409;
Anderson & Moore: pp. 178-200, 207-227.
Başar & Bernhard, 1995, chapters 1, 2.

Useful Reading on Game Theory: *Başar and Olsder, 1999*, pp. 1-36; 161-179; 326-350 (corresponding pages in the 1995 edition: 1-38; 167-185; 331-353)

Advance Reading : *Başar & Bernhard, 1995*, chapters 1, 2, 4, 5, 9 (excluding sampled-data and delayed information controls, and risk sensitivity).

Problems (to be handed in): Due Date: **Wednesday, April 26**.

36. In Problem 29 of Homework set # 5, replace the deterministic state equation by the stochastic one:

$$\dot{x} = -x + u - 1 + w\ , \quad x(0) = 1$$

where w is the standard white Gaussian noise (WGN), and also replace J by its expected value, that is

$$E\{J(\mu(\cdot))\} = E\big\{[x(1)]^2 + \int_0^1 [u(t)]^2\,\mathrm{d}t\big\}$$

Derive the state-feedback controller that minimizes $E\{J\}$, and obtain an expression for the minimum value of $E\{J\}$.

37. Let a be a scalar Gaussian random variable with mean zero and variance ρ, and x be a signal described by

$$\ddot{x} = a, \quad x(0) = 1,\ \dot{x}(0) = 1$$

The signal x is not measured directly, but a noisy measurement y of x is available:

$$y = x + v$$

where v is the standard WGN. Construct a Kalman filter for estimating $x(t)$, and study the limiting behavior of the Kalman gain and the error (co)variance as $t \to \infty$.

38. You are given a scalar stochastic system:

$$\dot{x} = u + 1.5w, \quad x(0) = 1,$$

and noisy state measurements:

$$y = x + v, \quad t \geq 0,$$

where w and v are independent standard WGN's. Let the cost function be exponentially discounted:

$$J = E\{\int_0^\infty \exp^{-2t}[x^2 + u^2]dt\}.$$

Find a stationary controller, $u^*(t) = \mu^*(\hat{x}(t))$, that minimizes J. What is the corresponding value of J?

39. Determine the set of all mixed saddle-point solutions of the following two matrix games, where Player 1 (minimizer) picks a row and Player 2 (maximizer) picks a column:

(i) $\begin{pmatrix} 1 & 3 & -1 & 2 \\ -3 & -2 & 2 & 1 \\ 0 & 2 & -2 & 1 \end{pmatrix}$ *(ii)* $\begin{pmatrix} 1 & 3 & -1 & -2 \\ -3 & -2 & 2 & 1 \\ 0 & 2 & -2 & 1 \end{pmatrix}$ *(iii)* $\begin{pmatrix} 2 & 1 & 0 & -1 \\ -1 & 3 & 1 & 4 \end{pmatrix}$

40. A zero-sum game on the unit square (i.e. $0 \le u, w \le 1$) has the cost function:

$$J(u,w) = u^3 - 3uw + w^3$$

where u is the minimizing and w the maximizing variable. Determine the pure or mixed saddle-point strategies.

41. Consider the zero-sum continuous-kernel game on the unit square, defined by the kernel

$$J(u,w) = (u-w)^2 - \alpha w^2$$

where again $u \in [0,1]$ is the minimizing and $w \in [0,1]$ the maximizing variable, and α is a real-valued parameter.
Determine the pure or mixed saddle-point solutions when

i) $1 < \alpha \le 2$, *ii)* $\alpha = 0$, *iii)* $0 < \alpha \le 1$.

42. You are given the static zero-sum game with cost function

$$J(u,w) = |Ax + Bu + Dw|_Q^2 + |u|^2 - \gamma^2 |w|^2,$$

where $x \in \mathbf{R}^n$ is a given vector, $u \in \mathbf{R}^{r_1}$ and $w \in \mathbf{R}^{r_2}$ are controlled by Players 1 and 2, respectively, A, B, D are matrices of compatible dimensions, Q is a nonnegative definite matrix of dimensions $n \times n$, γ is a positive scalar parameter, and $|v|_Q^2 := v'Qv$ (where $'$ stands for *transpose*) denotes an appropriate Euclidean norm. Player 1 is the minimizer, and Player 2 is the maximizer.
(i) Write down the conditions under which the game admits a pure-strategy saddle point, and also obtain the saddle-point solution.
(ii)Consider the following computational scheme:

$$u^{(k+1)} = \arg\min_u J(u, w^{(k)}) , \quad k = 0, 1, \ldots$$
$$w^{(k+1)} = \arg\max_w J(u^{(k)}, w) , \quad k = 0, 1, \ldots$$

which we start with $u^{(0)} = 0, w^{(0)} = 0$.
Assume that the two sequences thus generated converge, say to

$$\lim_{k\to\infty} u^{(k)} = \bar{u}, \qquad \lim_{k\to\infty} w^{(k)} = \bar{w}.$$

Is the pair $(\bar{u}, \bar{w})$ necessarily a saddle-point solution?
(iii) Under what conditions on A, B, D, Q, and γ do the sequences generated by the algorithm above converge (or have converging subsequences)?

43. Consider the scalar version of the problem above, where $A = B = D = Q = 1$, $x \neq 0$, and γ is a positive parameter. Obtain **tight** bounds on γ such that:
(i) The game has a saddle point.
(ii) The algorithm introduced above converges.
(iii) The game has a finite lower value.
(iv) The game has a finite upper value.

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