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OPTIMUM SYSTEMS

CONTROL EXAMPLES

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Chapter 5

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5 Optimum systems control examples

In this chapter, we will illustrate some, but certainly not all or even most, of the optimal control problems for which closed-form analytic solutions have been obtained. The problems we will solve in this chapter are very important in their own right and illustrate the use of the maximum principle for problems in which closed-form analytic solutions may be obtained. Specifically, we will discuss the linear regulator problem, the first solution of which was due to Kalman [1, 2, 3, 4]. We then discuss the minimum time problem which has been considered by Pontryagin [5], Bellman [6], LaSalle [7], and many others [8 through 13].

A characteristic of some minimum time problems is the possibility of a singular solution. The possibility of singular solutions is well-recognized in the variational calculus literature and has been extensively discussed for control problems by Johnson [14, 15, 16] and others. Minimum fuel problems for linear differential systems are then discussed. A variety of authors, but notably Athans, have discussed various aspects of minimum fuel problems including the possibility of singular solutions [17 through 20]. Although we will not consider the minimum time–fuel–energy control of self–adjoint systems [21] due to its limited practical usefulness, we do note that such systems admit a particularly thorough analysis. For a survey of many other problems plus a lengthy bibliography, we refer to the survey papers of Paiewonsky [22] and Athans [23].

5.1
The linear regulator

We will now study a particular control problem which has as its solution a linear feedback control law. It occurs where we have a linear differential system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}, \qquad \mathbf{x}(t_o) = \mathbf{x}_o \tag{5.1-1}$$

and wish to find the control which minimizes the cost function (for t_f fixed)

$$J = \tfrac{1}{2}\mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} [\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt. \tag{5.1-2}$$

Clearly, there is no loss of generality in assuming $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ to be symmetric. We may obtain the solution to this problem via the maximum principle or the Hamilton-Jacobi equation. Here, we will use the former method. The Hamiltonian is

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \tfrac{1}{2}\mathbf{u}^T\mathbf{R}\mathbf{u} + \boldsymbol{\lambda}^T\mathbf{A}\mathbf{x} + \boldsymbol{\lambda}^T\mathbf{B}\mathbf{u}. \tag{5.1-3}$$

Application of the maximum principle requires that, for an optimum control,

$$\frac{\partial H}{\partial \mathbf{u}} = 0 = \mathbf{R}(t)\mathbf{u}(t) + \mathbf{B}^T(t)\boldsymbol{\lambda}(t) \tag{5.1-4}$$

and

$$\frac{\partial H}{\partial \mathbf{x}} = -\dot{\boldsymbol{\lambda}} = \mathbf{Q}(t)\mathbf{x}(t) + \mathbf{A}^T(t)\boldsymbol{\lambda}(t) \tag{5.1-5}$$

with the terminal condition

$$\boldsymbol{\lambda}(t_f) = \frac{\partial \theta}{\partial \mathbf{x}(t_f)} = \mathbf{S}\mathbf{x}(t_f). \tag{5.1-6}$$

Thus we require that

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\boldsymbol{\lambda}(t), \tag{5.1-7}$$

and we shall inquire whether we may convert this to a closed-loop control by assuming that the solution for the adjoint is similar to Eq. (5.1-6)

$$\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t). \tag{5.1-8}$$

If we substitute this relation into Eqs. (5.1-1) and (5.1-7), we see that we must require

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\mathbf{x}(t). \tag{5.1-9}$$

Also, from Eqs. (5.1-8) and (5.1-5) we require

$$\dot{\boldsymbol{\lambda}} = \dot{\mathbf{P}}\mathbf{x}(t) + \mathbf{P}(t)\dot{\mathbf{x}} = -\mathbf{Q}(t)\mathbf{x}(t) - \mathbf{A}^T(t)\mathbf{P}(t)\mathbf{x}(t). \tag{5.1-10}$$

By combining Eqs. (5.1-9) and (5.1-10) we have

$$[\dot{\mathbf{P}} + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t) + \mathbf{Q}(t)]\mathbf{x}(t) = 0. \tag{5.1-11}$$

Since this must hold for all nonzero $\mathbf{x}(t)$, the term premultiplying $\mathbf{x}(t)$ must be zero. Thus the $\mathbf{P}$ matrix, which we see is an $n \times n$ symmetric matrix and which has $n(n+1)/2$ different terms, must satisfy the matrix Riccati equation—which, as we shall see later, must be positive definite—

$$\dot{\mathbf{P}} = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}^T(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t) - \mathbf{Q}(t) \tag{5.1-12}$$

with a terminal condition given by Eqs. (5.1-6) and (5.1-8)

$$\mathbf{P}(t_f) = \mathbf{S}. \tag{5.1-13}$$

Thus we may solve the matrix Riccati equation backward in time from t_f to t_o, store the matrix

$$\mathbf{K}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t), \tag{5.1-14}$$

and then obtain a closed-loop control from

$$\mathbf{u}(t) = +\mathbf{K}(t)\mathbf{x}(t). \tag{5.1-15}$$

It is important to note that all components of the state vector must be accessible. We will remove this restriction in Chapter 8 when we discuss the ideal observer. A block diagram for accomplishing this solution to the regulator problem is shown in Fig. 5.1-1. If we compute the second variation, we find that

$$\delta^2 J = \tfrac{1}{2}\,\delta\mathbf{x}^T(t_f)\mathbf{S}\,\delta\mathbf{x}(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} [\delta\mathbf{x}^T(t)\mathbf{Q}(t)\,\delta\mathbf{x}(t) + \delta\mathbf{u}^T(t)\mathbf{R}(t)\,\delta\mathbf{u}(t)]\, dt. \tag{5.1-16}$$

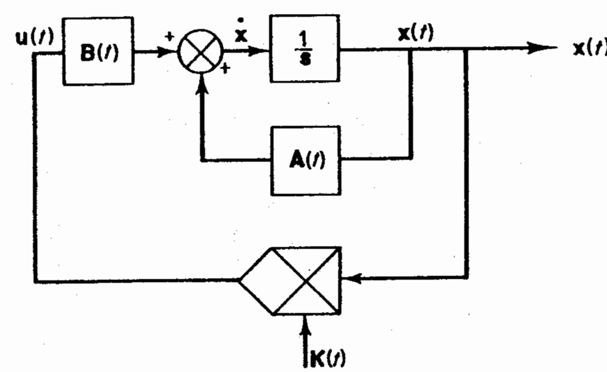

Fig. 5.1-1 Optimum linear closed-loop regulator.

Thus, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ must be at least positive semidefinite in order to establish the sufficient condition for a minimum. In addition, we know from Eq. (5.1-7) that $\mathbf{R}$ must have an inverse;† therefore, it is sufficient that $\mathbf{R}$ be positive definite and the $\mathbf{Q}$ and $\mathbf{S}$ be at least positive semidefinite.

†Approaches that allow this assumption to be relaxed can be found in [24] and [25].

In some cases it may turn out that certain elements of the S matrix are large enough to give computational difficulties. In this case, it is possible and perhaps desirable to obtain an inverse Riccati differential equation; we let

$$\mathbf{P}(t)\mathbf{P}^{-1}(t) = \mathbf{I}, \tag{5.1-17}$$

and, by differentiating, we obtain

$$\dot{\mathbf{P}}\mathbf{P}^{-1}(t) + \mathbf{P}(t)\dot{\mathbf{P}}^{-1} = 0 \tag{5.1-18}$$

such that we obtain an "inverse" matrix Riccati equation

$$\dot{\mathbf{P}}^{-1} = \mathbf{A}(t)\mathbf{P}^{-1}(t) + \mathbf{P}^{-1}(t)\mathbf{A}^T(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t) + \mathbf{P}^{-1}(t)\mathbf{Q}(t)\mathbf{P}^{-1}(t) \tag{5.1-19}$$

with

$$\mathbf{P}^{-1}(t_f) = \mathbf{S}^{-1}. \tag{5.1-20}$$

In this way, for example, it is possible to solve the Riccati equation such that $\mathbf{S}^{-1} = [0]$, the null matrix, which will require that each and every component of the state vector approach the origin as the time approaches the terminal time. The "gains" $\mathbf{K}(t)$, or at least some components of them, become infinite at the terminal time in this case. It is also necessary to assume certain controllability requirements here, as we shall see in Chapter 7.

It is possible to write the nonlinear $n \times n$ matrix Riccati equation with a terminal condition as a $2n$ vector linear differential equation with two-point boundary conditions. We will use this approach, in part, to solve a Riccati equation associated with a filtering problem in Chapter 8. Our discussion of the second variation method in Chapter 10 will also make use of a Riccati transformation.

Example 5.1-1. Consider the scalar system

$$\dot{x} = -\tfrac{1}{2}x(t) + u(t), \qquad x(t_o) = x_o$$

with the cost function

$$J = \tfrac{1}{2}sx^2(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} [2x^2(t) + u^2(t)]\, dt.$$

The Riccati equation, Eq. (5.1-12), becomes

$$\dot{p} = p + p^2 - 2, \qquad p(t_f) = s$$

which has a solution we may write as either

$$p(t) = -0.5 + 1.5 \tanh(-1.5t + \xi_1)$$

or

$$p(t) = -0.5 + 1.5 \coth(-1.5t + \xi_2)$$

where ξ_1 and ξ_2 are adjusted such that $p(t_f) = s$.

For example, if

(a) $s = 0$, $t_f = 1$, then $\xi_1 = 1.845$ radians, which gives

$$K(t) = -R^{-1}B^TP = 0.5 - 1.5 \tanh(-1.5t + 1.845).$$

Since $s = 0$, we are not particularly weighting the state at the final time, and the "gain" (and control) goes to zero at the final time.

(b) $s = 10$, $t_f = 10$, then $\xi_2 = 15.1425$ radians. In this case we are applying a great weight to the error at $t = t_f$, and the gain becomes large (-10) at the terminal time.

(c) $s = \infty$, the Riccati equation cannot be solved directly since it has an infinite initial condition. The inverse Riccati equation can be solved with zero terminal condition to give

$$K^{-1}(t) = 0.25 + 0.75 \tanh(-1.5t + 1.5t_f - 0.346).$$

As t_f becomes infinite, it is easy to show that $K(t)$ becomes unity and, as is expected, the feedback gain becomes constant. Figure 5.1-2 illustrates $K(t)$, the "Kalman gains" as they are sometimes called, for these three cases for this particular problem.

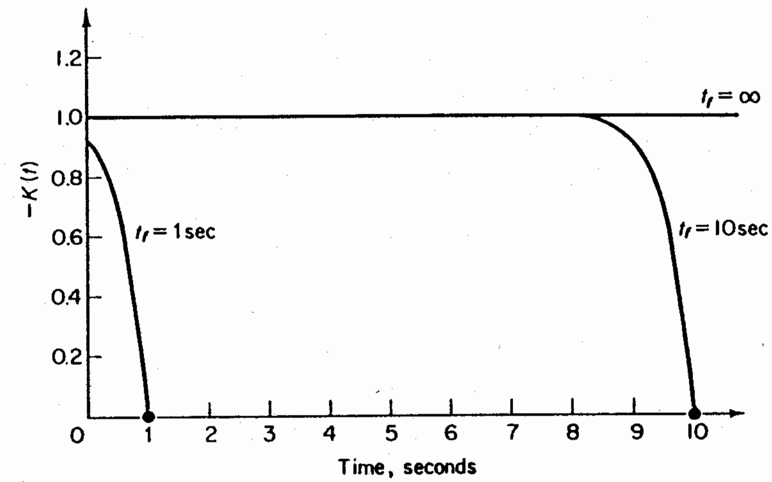

Fig. 5.1-2a (-1) times Kalman gain for controller, $s = 0$.

Example 5.1-2. Let us consider the optimum closed-loop control for a nuclear reactor system. Specifically, we wish to consider a very simple reactor model with zero temperature feedback. Only one group of delayed neutrons will be used.

The reactor kinetics are described by the equations

$$\dot{n} = \frac{(\rho - \beta)n}{\Lambda} + \lambda c, \qquad \dot{c} = \frac{\beta n}{\Lambda} - \lambda c$$

where the neutron density, n, and the precursor concentration, c, are the state variables, and the reactivity ρ is the control variable. The system has the initial conditions $n(0) = n_o$ and $c(0) = c_o$. β, Λ and λ are constants, the average fraction of precursors formed, effective neutron lifetime, and precursor decay constant.

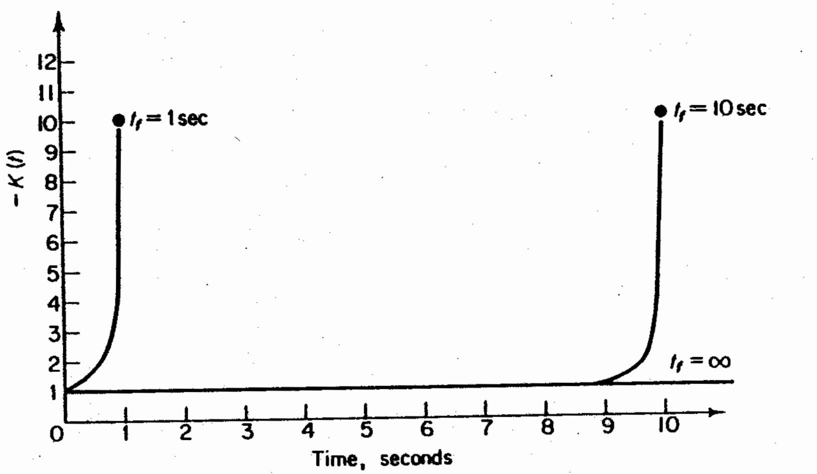

Fig. 5.1-2b (−1) times Kalman gain for controller, $s = 10$.

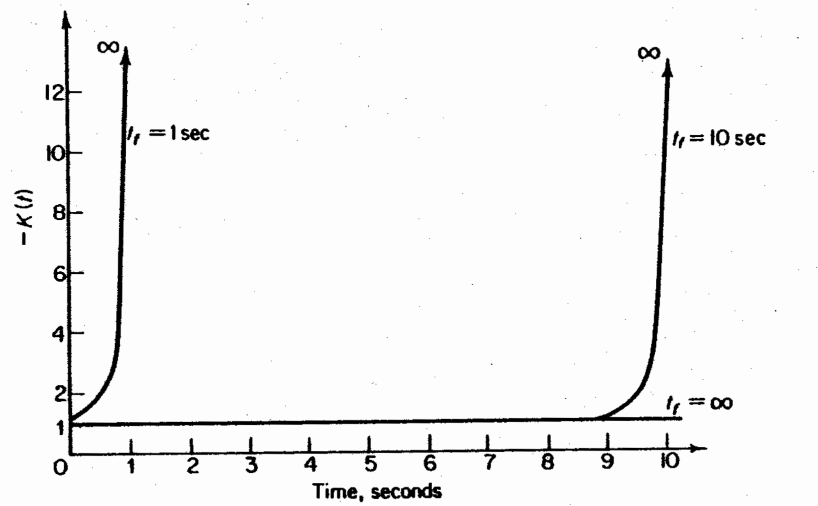

Fig. 5.1-2c (−1) times Kalman gain for controller, $s = \infty$.

The problem is to increase the power from the initial state n_o to a terminal state dn_o, where d is some constant greater than 1.0. The performance index for the system is

$$J_1 = \frac{1}{2}\int_0^{t_f} \dot{\rho}^2\, dt.$$

The control variable therefore becomes $\dot{\rho}$, and ρ, in effect, thus becomes a state variable. The kinetics equations may then be rewritten as

$$\dot{n} = \frac{(\rho - \beta)n}{\Lambda} + \lambda c$$

$$\dot{c} = \frac{\beta n}{\Lambda} - \lambda c$$

$$\dot{\rho} = u$$

where u is the control variable. Chapter 10 on quasilinearization indicates how the nonlinear two-point boundary value problem resulting from the use of optimal control theory may be used to obtain the optimum control and trajectory, which are shown in Fig. 5.1-3, for the following system parameters

$$\lambda = 0.1\ \text{sec}^{-1} \qquad n_o = 10\ \text{kW}$$
$$d = 5$$
$$\Lambda = 10^{-3}\ \text{sec} \qquad \beta = 0.0064$$
$$t_f = 0.5\ \text{sec.}$$

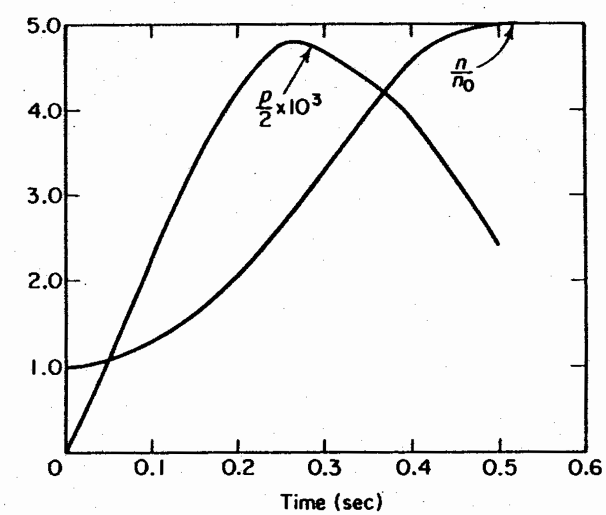

Fig. 5.1-3 Optimal control (reactivity) and trajectory (flux density) for Example (5.1-2).

We will now develop a method of feedback control about the optimal trajectory which minimizes a cost function J_2; it will be quadratic in deviation from the nominal (optimal for J_1) trajectory and control.

Having formulated a model for the nuclear reactor system and determined the optimal trajectories, we now desire to determine the linearized system coefficient matrix about the optimal trajectory. The deviations of the state and control variables about the optimal or nominal trajectories are expressed by

$$n = n_n(t) + \Delta n(t), \qquad c = c_n(t) + \Delta c(t)$$
$$\rho = \rho_n(t) + \Delta \rho(t), \qquad u = u_n(t) + \Delta u(t).$$

The state vector is

$$\Delta \mathbf{x}^T(t) = [\Delta n(t), \Delta c(t), \Delta \rho(t)].$$

The linearized model becomes

$$\Delta \dot{\mathbf{x}} = \begin{bmatrix} a_{11}(t) & \lambda & a_{13}(t) \\ \dfrac{\beta}{\Lambda} & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \Delta \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Delta u$$

$$= \mathbf{A}(t)\, \Delta \mathbf{x}(t) + \mathbf{b}(t)\, \Delta u(t)$$

where

$$a_{11}(t) = \frac{\rho_n(t) - \beta}{\Lambda}, \qquad a_{13}(t) = \frac{n_n(t)}{\Lambda}.$$

To complete our design of the closed-loop controller, we must evaluate $\mathbf{A}(t)$ and $\mathbf{b}(t)$ about the optimum or nominal trajectories, select the $\mathbf{R}$, $\mathbf{Q}$, and $\mathbf{S}$ matrices, and solve the associated Riccati equation. The nominal trajectory, control, and time-varying gains are then stored and used to complete the closed-loop controller design.

The choice of the $\mathbf{R}, \mathbf{Q}$, and $\mathbf{S}$ matrices to minimize

$$J_2 = \tfrac{1}{2}\, \Delta \mathbf{x}^T(t_f)\mathbf{S}\Delta \mathbf{x}(t_f) + \tfrac{1}{2} \int_{t_0}^{t_f} [\Delta \mathbf{x}^T(t)\mathbf{Q}(t)\Delta \mathbf{x}(t) + r(t)\Delta u^2(t)]\, dt$$

is somewhat arbitrary and can perhaps best be done here by experimentation. We can accomplish this only after we have obtained a knowledge of possible disturbances which may drive the system off the nominal trajectory. Let us assume that we will use

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10^4 \end{bmatrix}, \qquad \mathbf{S} = 0, \qquad r = 1.$$

In Chapter 10 the second variation and neighboring optimal methods of control-law computation will lead us to a method for choosing the proper weighting matrices for a variety of cases, in particular, for relating J_1 and J_2.

The control, $\Delta u(t)$, is computed from

$$\Delta u(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\Delta \mathbf{x}(t)$$
$$= -[p_{31}(t)\, \Delta n(t) + p_{32}(t)\, \Delta c(t) + p_{33}(t)\, \Delta \rho(t)]$$

where it is necessary to solve the 3×3 matrix Riccati equation, having six different first-order differential equations, to obtain $\mathbf{P}(t)$. Figure 5.1-4 illustrates the Kalman gains, $-\mathbf{K}^T(t) = [p_{31}(t), p_{32}(t), p_{33}(t)]$, for this example. Figure 5.1-5 indicates how the complete closed-loop controller is obtained. It is interesting to note that, in an actual physical problem, the precursor concentration is not measurable, and therefore we need to add an "observer" of this particular state variable. We also need to discuss many more aspects of this problem such as disturbances and parameter variations. We will postpone further consideration of these important questions until we establish some foundation in state and parameter estimation and optimal adaptive

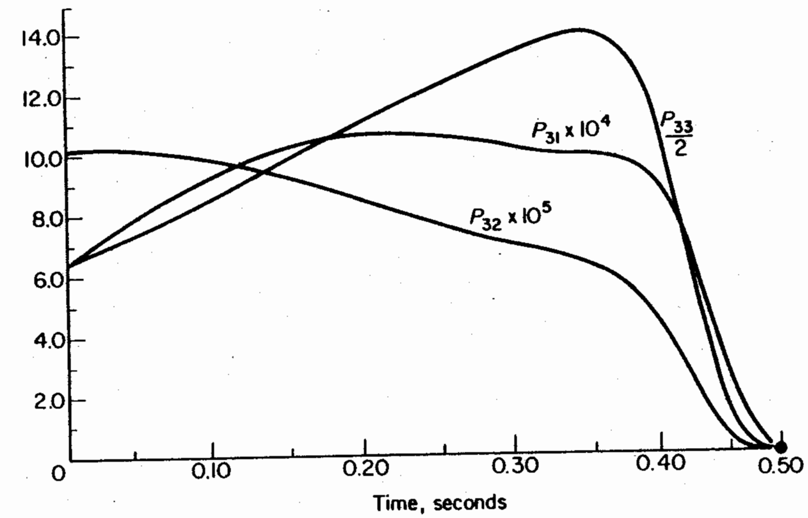

Fig. 5.1-4 Kalman gains for Example (5.1-2).

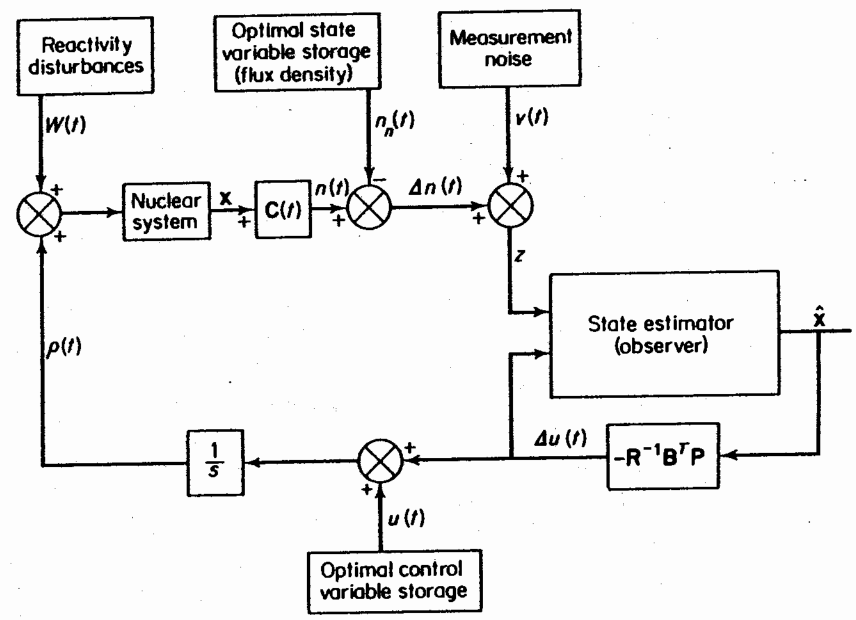

Fig. 5.1-5 Structure of controller for Example (5.1-2).

control. We have, in this example, illustrated how a basically nonlinear problem may be linearized, and a linear time-varying closed-loop controller obtained, if a nominal trajectory is known. Since this can be accomplished for a variety of problems, we see that the linear regulator problem is indeed an important one.

Example 5.1-3. We now consider the optimal control of a distributed parameter system. By a spatial discretization technique, we will reduce the distributed parameter optimal control problem to a form of the linear regulator problem.

Consider the one-dimensional diffusion equation

$$\frac{\partial x(y,t)}{\partial t} = \frac{\partial^2 x(y,t)}{\partial y^2} + u(y,t) \tag{5.1-21}$$

with initial condition $x(y, t_o = 0) = x_o(y)$,

and $\dfrac{\partial x(y,t)}{\partial y} = 0$ at $y = 0$, $\dfrac{\partial x(y,t)}{\partial y} = 0$ at $y = y_f$.

We desire to find the control $u(y, t)$ which minimizes the cost function

$$J = \frac{1}{2}\int_o^{t_f}\int_o^{y_f} [Q'x^2(y,t) + R'u^2(y,t)]\,dy\,dt.$$

We wish to obtain an approximate solution of Eq. (5.1-21) where $u(y, t)$ is assumed to be available. We shall establish a spatially discretized model in which the size of the space increment is $\Delta y = y_f/n$, where n is an integer. Physically, this corresponds to cutting a slab of length y_f into n slices. We shall use central difference formulas and obtain a spatially discrete model that can be described by vector differential equations. Let us use the notation

$$\frac{\partial x(y,t)}{\partial t} = \dot{x}_i(t) \quad \text{where} \quad i = 1, 2, \ldots, n \tag{5.1-22}$$

and then use central difference formulas to obtain

$$\frac{\partial^2 x(y,t)}{\partial y^2} \cong \frac{x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)}{(\Delta y)^2} \tag{5.1-23}$$

where

$$x_{i+1}(t) = x(y + \Delta y, t), \qquad x_i(t) = x(y, t), \qquad x_{i-1}(t) = x(y - \Delta y, t).$$

Therefore, using Eqs. (5.1-22) and (5.1-23) in Eq. (5.1-21), we obtain

$$\dot{x}_i(t) = \frac{x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)}{(\Delta y)^2} + u_i(t) \tag{5.1-24}$$

where $i = 1, 2, \ldots, n$.

By considering different values of $i, i = 1, 2, \ldots, n$, and using Eq. (5.1-24), we obtain n first-order linear differential equations which approximate Eq. (5.1-21). These are

$$\begin{aligned}
\dot{x}_1(t) &= \frac{1}{(\Delta y)^2}[x_2(t) - 2x_1(t) + x_0(t)] + u_1(t) \\
\dot{x}_2(t) &= \frac{1}{(\Delta y)^2}[x_3(t) - 2x_2(t) + x_1(t)] + u_2(t) \\
&\;\;\vdots \\
\dot{x}_{n-1}(t) &= \frac{1}{(\Delta y)^2}[x_n(t) - 2x_{n-1}(t) + x_{n-2}(t)] + u_{n-1}(t) \\
\dot{x}_n(t) &= \frac{1}{(\Delta y)^2}[x_{n+1}(t) - 2x_n(t) + x_{n-1}(t)] + u_n(t).
\end{aligned} \tag{5.1-25}$$

We may use the boundary conditions to obtain $x_0(t)$ and $x_{n+1}(t)$. Then, by using a first difference approximation to the initial boundary condition, we obtain

$$\frac{x_{i+1}(t) - x_i(t)}{(\Delta y)} = 0 \quad \text{for} \quad y = 0 \quad \text{or, equivalently,} \quad i = 0. \tag{5.1-26}$$

We have therefore established the boundary condition

$$x_0(t) = x_1(t). \tag{5.1-27}$$

In a similar fashion, we may easily show that

$$x_n(t) = x_{n+1}(t). \tag{5.1-28}$$

If use is made of Eqs. (5.1-27) and (5.1-28) in the set of ordinary differential equations given in Eq. (4.3-28), we obtain

$$\begin{aligned}
\dot{x}_1(t) &= \frac{1}{(\Delta y)^2}[x_2(t) - x_1(t)] + u_1(t) \\
\dot{x}_2(t) &= \frac{1}{(\Delta y)^2}[x_3(t) - 2x_2(t) + x_1(t)] + u_2(t) \\
\dot{x}_3(t) &= \frac{1}{(\Delta y)^2}[x_4(t) - 2x_3(t) + x_2(t)] + u_3(t) \\
&\;\;\vdots \\
\dot{x}_{n-1}(t) &= \frac{1}{(\Delta y)^2}[x_n(t) - 2x_{n-1}(t) + x_{n-2}(t)] + u_{n-1}(t) \\
\dot{x}_n(t) &= \frac{1}{(\Delta y)^2}[-x_n(t) + x_{n-1}(t)] + u_n(t).
\end{aligned} \tag{5.1-29}$$

We will now represent this set of ordinary linear differential equations by the vector differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_o \tag{5.1-30}$$

where: $\mathbf{x}$ is an n-dimensional state vector; $\mathbf{u}$ is an n-dimensional control vector; $\mathbf{A}$ is the $n \times n$ tridiagonal matrix,

$$\mathbf{A} = \frac{1}{(\Delta y)^2}\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & & 0 & 0 & 1 & -1
\end{bmatrix}; \tag{5.1-31}$$

and where $\mathbf{B}$ is the identity matrix of order n, $\mathbf{B} = \mathbf{I}$. It is an easy task to verify that this linear system is always stable.

A discrete approximate form of the performance function is

$$J = \tfrac{1}{2}\Delta y \int_0^{t_f} \{\sum_{i=1}^{n-1} [Q'x_i^2(t) + R'u_i^2(t)] + \tfrac{1}{2}[Q'x_o(t) + Q'x_n(t)$$
$$+ R'u_o(t) + R'u_n(t)]\}\, dt$$

where n is the last discretized spatial stage. We may rewrite this as

$$J = \tfrac{1}{2}\Delta y \int_0^{t_f} [\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)]\, dt.$$

For this problem, the Hamiltonian is

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) = \tfrac{1}{2}\Delta y \mathbf{x}^T\mathbf{Q}\mathbf{x} + \tfrac{1}{2}\Delta y \mathbf{u}^T\mathbf{R}\mathbf{u} + \boldsymbol{\lambda}^T\mathbf{A}\mathbf{x} + \boldsymbol{\lambda}^T\mathbf{B}\mathbf{u}.$$

Application of the maximum principle to this problem immediately yields the two-point boundary value problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad x_i(0) = 1 + \alpha i y_f/n, \qquad i = 1, 2, \ldots, n$$
$$-\dot{\boldsymbol{\lambda}} = \Delta y \mathbf{Q}\mathbf{x}(t) + \mathbf{A}^T\boldsymbol{\lambda}, \qquad \boldsymbol{\lambda}(t_f) = 0$$
$$\mathbf{u} = -\frac{1}{\Delta y}\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}.$$

We shall solve this problem by generating the Riccati equation where, as before in Sec. 5.1, we assume $\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t)$. Thus, the optimal control is a linear feedback control determined by solving

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A} - \frac{1}{\Delta y}\mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) + \Delta y\mathbf{Q} + \mathbf{A}^T\mathbf{P}(t) = 0, \qquad \mathbf{P}(t_f) = 0$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \frac{1}{\Delta y}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}(t), \qquad x_i(0) = 1 + \alpha i y_f/n, \qquad i = 1, 2, \ldots, n$$

$$\mathbf{u}(t) = -\frac{1}{\Delta y}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t)\mathbf{x}(t).$$

Let us consider the following two cases:

Case A $\quad t_f = 1.0, \quad y_f = 4.0, \quad \mathbf{B} = \mathbf{I}, \quad Q' = R' = 1$

$\qquad \Delta t = 0.01, \quad \Delta y = 1.0, \quad \alpha = 1$

$$\mathbf{A} = \frac{1}{1}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \qquad \mathbf{Q} = \mathbf{R} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Case B $\quad t_f = 1.0, \quad \Delta t = 0.01, \quad \mathbf{B} = \mathbf{I}, \quad Q' = R' = 1$

$\qquad y_f = 4.0, \quad \Delta y = 0.5, \quad \alpha = 1$

$$\mathbf{A} = 4\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$\mathbf{Q} = \mathbf{R} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

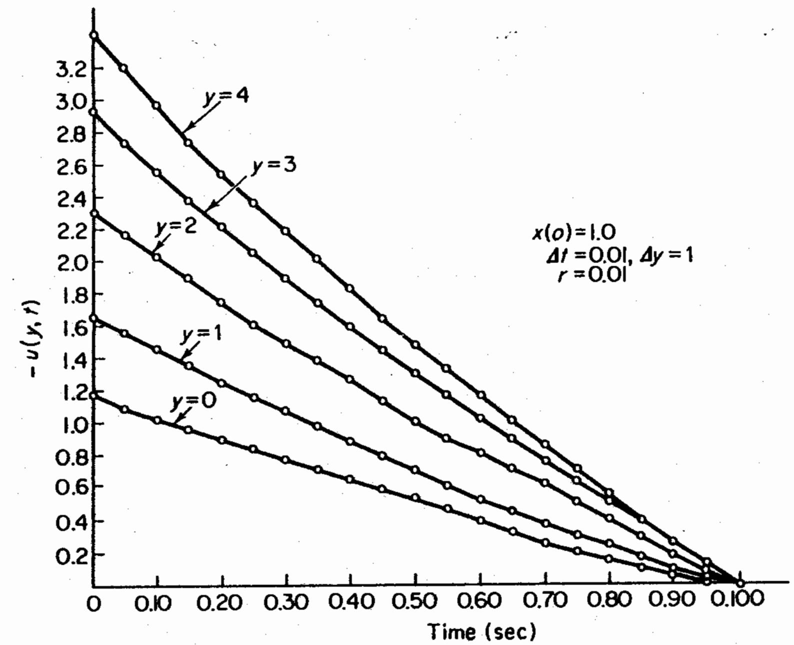

Fig. 5.1-6 Optimal control versus spatial coordinate and time, Example 5.1-3.

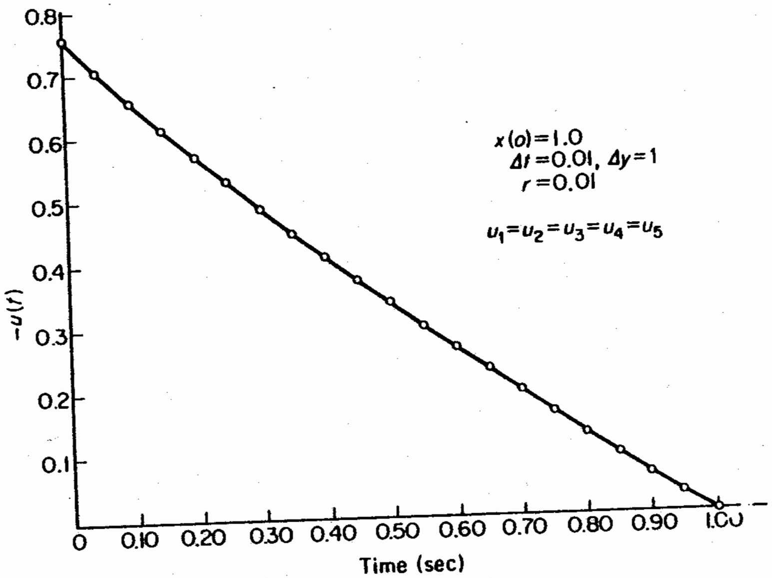

Fig. 5.1-7 Optimal control versus time for $\alpha = 0$, Example 5.1-3.

Solution of the two cases considered yields essentially the same result. This indicates that, for the particular initial condition $x(y, 0) = 1 + y$, a model with five coordinates yields as good a solution as a model with ten coordinates. Thus we may safely assume that, for this particular initial condition, lumping the distributed system into five states is a satisfactory thing to do. As we change the initial distribution, $x(y, 0) = 1 + \alpha y$, by changing α, the number of necessary states to provide a good lumped model changes. For $\alpha = 0$, where the initial condition is uniform throughout y, a single state suffices for an exact model since $\partial^2 x(y, t)/\partial y^2$ is then always zero, and the distributed system degenerates to a lumped system for this particular case.

Figure 5.1-6 illustrates a plot of the optimal control versus time and distance y for $\alpha = 1$. A plot of the optimal control versus time is shown for the spatially independent case when $\alpha = 0$ in Fig. 5.1-7.

5.2 The linear servomechanism

The linear regulator problem considered in the preceding section can be generalized in several ways. We can assume that we desire to find the control in such a way as to cause the output to track or follow a desired output state, $\eta(t)$. We may also assume that there is a forcing function (not the control) for the system differential equations. Therefore, we will consider the mini-

mization of

$$J = \tfrac{1}{2}\|\boldsymbol{\eta}(t_f) - \mathbf{z}(t_f)\|^2_{\mathbf{S}} + \tfrac{1}{2}\int_{t_0}^{t_f} [\|\boldsymbol{\eta}(t) - \mathbf{z}(t)\|^2_{\mathbf{Q}(t)} + \|\mathbf{u}(t)\|^2_{\mathbf{R}(t)}]\, dt \qquad (5.2\text{-}1)$$

for the system which contains a deterministic input or plant "noise" vector $\mathbf{w}(t)$

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{w}(t), \quad \mathbf{x}(t_o) = \mathbf{x}_o \qquad (5.2\text{-}2)$$

$$\mathbf{z}(t) = \mathbf{C}(t)\mathbf{x}(t). \qquad (5.2\text{-}3)$$

The requirements on the various matrices are the same as in the preceding section. We proceed in exactly the same fashion as for the regulator problem. The Hamiltonian is, from Eq. (4.3-34),

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) = \tfrac{1}{2}\|\boldsymbol{\eta}(t) - \mathbf{C}(t)\mathbf{x}(t)\|^2_{\mathbf{Q}(t)} + \tfrac{1}{2}\|\mathbf{u}(t)\|^2_{\mathbf{R}(t)} + \boldsymbol{\lambda}^T(t)[\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{w}(t)]. \qquad (5.2\text{-}4)$$

We employ the maximum principle and set $\partial H/\partial \mathbf{u} = \mathbf{0}$ to obtain

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\boldsymbol{\lambda}(t) \qquad (5.2\text{-}5)$$

and

$$\frac{\partial H}{\partial \mathbf{x}} = -\dot{\boldsymbol{\lambda}} = \mathbf{C}^T(t)\mathbf{Q}(t)[\mathbf{C}(t)\mathbf{x}(t) - \boldsymbol{\eta}(t)] + \mathbf{A}^T(t)\boldsymbol{\lambda}(t) \qquad (5.2\text{-}6)$$

with the terminal condition

$$\boldsymbol{\lambda}(t_f) = \mathbf{C}^T(t_f)\mathbf{S}[\mathbf{C}(t_f)\mathbf{x}(t_f) - \boldsymbol{\eta}(t_f)]. \qquad (5.2\text{-}7)$$

In order to attempt to determine a closed-loop control, we assume

$$\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t) - \boldsymbol{\xi}(t). \qquad (5.2\text{-}8)$$

We substitute this relation into the canonic equations and determine the requirements for a solution. By a procedure analogous to that of the preceding section, we easily obtain the following requirements:

$$\dot{\mathbf{P}} = -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}^T(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t) - \mathbf{C}^T(t)\mathbf{Q}(t)\mathbf{C}(t) \qquad (5.2\text{-}9)$$

$$\mathbf{P}(t_f) = \mathbf{C}^T(t_f)\mathbf{S}\mathbf{C}(t_f), \qquad (5.2\text{-}10)$$

and

$$\dot{\boldsymbol{\xi}} = -[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)]^T\boldsymbol{\xi} + \mathbf{P}(t)\mathbf{w}(t) - \mathbf{C}^T(t)\mathbf{Q}(t)\boldsymbol{\eta}(t) \qquad (5.2\text{-}11)$$

$$\boldsymbol{\xi}(t_f) = \mathbf{C}^T(t_f)\mathbf{S}\boldsymbol{\eta}(t_f). \qquad (5.2\text{-}12)$$

Thus we see that the linear servomechanism problem is composed of two parts: a linear regulator part, plus a prefilter to determine the optimal driving function from the desired value, $\boldsymbol{\eta}(t)$, of the system output. The optimum control law is linear and is obtained from Eq. (5.2-5) as

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)[\mathbf{P}(t)\mathbf{x}(t) - \boldsymbol{\xi}(t)]. \qquad (5.2\text{-}13)$$

Unfortunately, the optimal control is, in practice, often computationally unrealizable because it involves $\xi(t)$ which must be solved backward from t_f to t_o and, therefore, requires a knowledge of $\eta(t)$ and $\mathbf{w}(t)$ for all time $t \in [t_o, t_f]$. This is quite often not known at the initial time t_o.

Example 5.2-1. Let us consider the minimization of the cost function

$$J = \tfrac{1}{2}\int_0^{t_f} [(x_1 - \eta_1)^2 + u^2]\, dt$$

for the system described by

$$\dot{x}_1 = x_2, \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = u, \qquad x_2(0) = x_{20}.$$

We first use Eqs. (5.2-9) and (5.2-10) to obtain the Riccati equation for this example

$$\dot{p}_{11} = p_{12}^2 - 1, \qquad p_{11}(t_f) = 0$$
$$\dot{p}_{12} = -p_{11} + p_{12}p_{22}, \qquad p_{12}(t_f) = 0$$
$$\dot{p}_{22} = -2p_{12} + p_{22}^2, \qquad p_{22}(t_f) = 0.$$

If we allow t_f to become infinite, we obtain the solution $p_{11} = p_{22} = \sqrt{2}$, $p_{12} = 1$. Thus we have for the closed-loop control

$$u = -\mathbf{R}^{-1}\mathbf{B}^T[\mathbf{P}\mathbf{x} - \xi] = -x_1 - \sqrt{2}x_2 + \xi_2$$

where we must determine ξ by solving Eqs. (5.2-11) and (5.2-12) which become for this example

$$\dot{\xi}_1 = \xi_1 - \eta_1, \qquad \xi_1(t_f) = 0$$
$$\dot{\xi}_2 = -\xi_1 + \sqrt{2}\xi_2, \qquad \xi_2(t_f) = 0.$$

If $\eta_1 = \alpha$, a constant, for t greater than zero, we are justified in obtaining the equilibrium solution for the ξ equation if $t_f = \infty$ by setting $\dot{\xi} = \mathbf{0}$ to obtain $\xi_2 = 0.707\xi_1 = \eta_1 = \alpha$. If $\eta_1 = 1 - e^{-t}$, we will then find by a simple limiting process that for $t_f = \infty$,

$$\xi_2(t) = 1 + \frac{1}{2 + \sqrt{2}}e^{-t}, \qquad t \geq 0.$$

We may realize this solution as shown in Fig. 5.2-1.

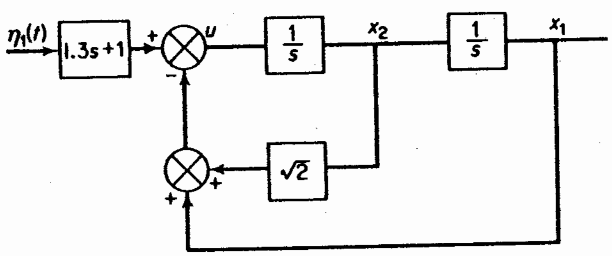

Fig. 5.2-1 Block diagram of optimum servomechanism for Example 5.2-1.

We note that if $\mathbf{w}(t) = \eta(t) = \mathbf{0}$, or for that matter, any vector constant in time, the servomechanism problem reduces to a regulator problem except that it is an "output" regulator problem rather than a "state" regulator problem because of the presence of the output matrix $\mathbf{C}(t)$. It is not necessary for the system to be controllable in order to find a solution to the regulator problem. The only exception to this is in the limiting cases where $\mathbf{S}$ becomes infinite or where t_f becomes infinite. It is, however, necessary that the system be observable in order for a solution to the output regulator problem to exist. We will expand considerably on these ideas when we consider controllability, observability, and the reachable zone problem in Chapter 7.

5.3 Bang bang control and minimum time problems

Maximum effort control problems have become increasingly important in a variety of applications. It is natural that we ask under what circumstances optimal controls will always be maximum effort, or *bang bang*. To do this, we will restrict each component of the control vector, $\mathbf{u}(t)$, to some bounded interval. Let us consider the nonlinear differential system where the control enters in a linear fashion

$$\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}(t), t] + \mathbf{G}[\mathbf{x}(t), t]\mathbf{u}(t), \qquad \mathbf{x}(t_o) = \mathbf{x}_o \tag{5.3-1}$$
$$a_i \leq u_i \leq b_i, \qquad \forall\, i \tag{5.3-2}$$

and assume a performance index which likewise contains only linear terms in the control variable, such that the Hamiltonian will also be linear in $\mathbf{u}(t)$.

$$J = \theta[\mathbf{x}(t_f), t_f] + \int_{t_o}^{t_f} \{\phi[\mathbf{x}(t), t] + \mathbf{h}^T[\mathbf{x}(t), t]\mathbf{u}(t)\}\, dt \tag{5.3-3}$$

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = \phi[\mathbf{x}(t), t] + \mathbf{h}^T[\mathbf{x}(t), t]\mathbf{u}(t) + \boldsymbol{\lambda}^T(t)\{\mathbf{f}[\mathbf{x}(t), t] + \mathbf{G}[\mathbf{x}(t), t]\mathbf{u}(t)\}. \tag{5.3-4}$$

Since the Hamiltonian is linear in the control vector, $\mathbf{u}(t)$, minimization of the Hamiltonian with respect to $\mathbf{u}(t)$ requires that

$$u_i = \begin{cases} a_i & \text{if } \{\mathbf{h}^T[\mathbf{x}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{G}[\mathbf{x}(t), t]\}_i > 0 \\ b_i & \text{if } \{\mathbf{h}^T[\mathbf{x}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{G}[\mathbf{x}(t), t]\}_i < 0. \end{cases} \tag{5.3-5}$$

Thus we see that when the control vector appears linearly in both the equation of motion of the differential system and the performance index, and if, in addition, each component of the control vector is bounded, the optimal control is bang bang. The only exception to this occurs in cases where

$$\mathbf{h}^T[\mathbf{x}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{G}[\mathbf{x}(t), t] = \mathbf{0}, \tag{5.3-6}$$

for then the Hamiltonian is not a function of $\mathbf{u}(t)$ and cannot be minimized with respect to $\mathbf{u}(t)$. When Eq. (5.3-6) holds for more than isolated points in time, the optimization problem is said to possess a singular solution, a problem which we will discuss in detail in the next section. A singular solution is possible with respect to a particular control component, u_i, if the ith component of Eq. (5.3-6) is zero.

For this problem, the canonic equations are obtained as

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} = \mathbf{f}[\mathbf{x}(t), t] + \mathbf{G}[\mathbf{x}(t), t]\mathbf{u}(t) \tag{5.3-7}$$

$$-\dot{\boldsymbol{\lambda}} = \frac{\partial H}{\partial \mathbf{x}} = \frac{\partial \phi[\mathbf{x}(t), t]}{\partial \mathbf{x}} + \frac{\partial \mathbf{h}^T[\mathbf{x}(t), t]}{\partial \mathbf{x}}\mathbf{u}(t) + \frac{\partial \mathbf{f}^T[\mathbf{x}(t), t]}{\partial \mathbf{x}}\boldsymbol{\lambda}(t) + \frac{\partial \{\mathbf{G}[\mathbf{x}(t), t]\mathbf{u}(t)\}^T}{\partial \mathbf{x}(t)}\boldsymbol{\lambda}(t) \tag{5.3-8}$$

where $\mathbf{u}(t)$ is determined via Eq. (5.3-5). Since we have not specifically stated the end conditions, we have carried the general problem about as far as possible. When we specify information concerning the desired states at the terminal time and the initial condition vector, we have, as before, a two-point boundary value problem with half of the conditions specified at the initial time and half at the terminal time. A possible method of solution of the canonic equations for this formulation consists of reversing time in the canonic equations. Starting at the determined or specified terminal vector, which often is the origin of the state vector, we integrate back from this point with a constant control until a switching point is obtained from Eq. (5.3-5). Since no terminal conditions are present for half of the state variables, the method is, of necessity, cut and try. Chapter 10 provides more systematic methods for solving this type of two-point boundary value problem.

We shall now illustrate various solutions to a particular case which results in bang bang control—the minimum time problem for constant linear systems with a scalar input. In this problem, we desire to transfer an n vector constant differential system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \qquad \mathbf{x}(t_o) = \mathbf{x}_o \tag{5.3-9}$$

to the origin, $\mathbf{x}(t_f) = \mathbf{0}$, in minimum time, such that we have for the cost function

$$J = \int_{t_o}^{t_f} (1)\, dt = t_f - t_o \tag{5.3-10}$$

with the restriction that

$$-1 \leq u(t) \leq +1. \tag{5.3-11}$$

The Hamiltonian for our problem is

$$H[\mathbf{x}(t), u(t), \boldsymbol{\lambda}(t)] = 1 + \boldsymbol{\lambda}^T(t)\mathbf{A}\mathbf{x}(t) + \boldsymbol{\lambda}^T(t)\mathbf{b}u(t). \tag{5.3-12}$$

We must minimize the Hamiltonian with respect to a choice of $u(t)$, so we require

$$u(t) = -\text{sign}\,[\boldsymbol{\lambda}^T(t)\mathbf{b}]. \tag{5.3-13}$$

Thus the Hamiltonian with the control optimum is

$$H[\mathbf{x}(t), \boldsymbol{\lambda}(t)] = 1 + \boldsymbol{\lambda}^T(t)\mathbf{A}\mathbf{x}(t) - |\boldsymbol{\lambda}^T(t)\mathbf{b}|. \tag{5.3-14}$$

Since the terminal time is free, and since H does not depend explicitly on t, we know from Eqs. (4.3-21) and (4.3-39) that

$$H[\mathbf{x}(t), \boldsymbol{\lambda}(t)] = 0, \qquad \forall t \in [t_o, t_f] \tag{5.3-15}$$

on the optimal trajectory. The canonic equations are

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) = \mathbf{A}(t)\mathbf{x}(t) - \mathbf{b}\,\text{sign}\,[\boldsymbol{\lambda}^T(t)\mathbf{b}] \tag{5.3-16}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}} = -\mathbf{A}^T\boldsymbol{\lambda}(t). \tag{5.3-17}$$

To avoid a singular solution, we must ensure that $\boldsymbol{\lambda}^T(t)\mathbf{b}$ cannot be zero over a time interval of nonzero length. From Eq. (5.3-17) we see that this is almost certainly the case unless $\boldsymbol{\lambda}(t_o)$ were identically $\mathbf{0}$, which is not possible. The only other requirement is that the system be controllable as shall be defined in Chapter 7. If the system were not controllable, we could not, in general, transfer its motion to the origin. For this problem we shall discover stronger requirements than controllability which we must place on the system if we are to transfer it to the origin. These requirements come about because of the restriction $|u| \leq 1$ which always results in initial states in an unstable plant which cannot be transferred to the origin. The solution to Eq. (5.3-17) is

$$\boldsymbol{\lambda}(t) = e^{-\mathbf{A}^T(t-t_f)}\boldsymbol{\lambda}(t_f). \tag{5.3-18}$$

It is convenient for us to rewrite Eq. (5.3-16) in terms of the time to go by letting $t_o = 0$ and

$$\tau = t_f - t \tag{5.3-19}$$

$$\boldsymbol{\xi}(\tau) = \mathbf{x}(t) = \mathbf{x}(t_f - \tau). \tag{5.3-20}$$

This gives us

$$\frac{d\boldsymbol{\xi}}{d\tau} = -\mathbf{A}\boldsymbol{\xi}(\tau) + \mathbf{b}\,\text{sign}\,[\boldsymbol{\lambda}^T(t_f)e^{\mathbf{A}\tau}\mathbf{b}] \tag{5.3-21}$$

which has its solution, since $\boldsymbol{\xi}(0) = \mathbf{x}(t_f) = \mathbf{0}$,

$$\boldsymbol{\xi}(\tau) = \int_0^\tau e^{-\mathbf{A}(\tau-p)}\mathbf{b}\,\text{sign}\,[\boldsymbol{\lambda}^T(t_f)e^{\mathbf{A}p}\mathbf{b}]\, dp. \tag{5.3-22}$$

A state $\mathbf{x}(t_o) = \mathbf{x}_o$ from which the origin can be reached in a specified minimum time t_f may now be obtained if we substitute a value of $\boldsymbol{\lambda}(t_f)$ in Eq. (5.3-22) and then calculate $\mathbf{x}(t_o) = \mathbf{x}(0) = \boldsymbol{\xi}(t_f)$. This alludes to a relation between $\mathbf{x}(t_o)$ and $\boldsymbol{\lambda}(t_f)$ and suggests that an alternate name for $\boldsymbol{\lambda}$ might well be "influence function." Chapter 10 will develop this very point in considerably more detail.

Since it is clear that the direction and not the magnitude of the $\boldsymbol{\lambda}(t_f)$ vector determines sign $[\boldsymbol{\lambda}^T(t_f)e^{\mathbf{A}p}\mathbf{b}]$, all states which can be reached in a given minimum time may be determined if we allow $\boldsymbol{\lambda}(t_f)$ to assume all values over a unit sphere. At points where $\boldsymbol{\lambda}^T(t_f)e^{\mathbf{A}\tau}\mathbf{b}$ is $\mathbf{0}$, we have a switching point. It is possible to show [7] that if the eigenvalues of $\mathbf{A}$ are real, there are, at most, $n - 1$ switchings or changes of sign of the control. We will now give several simple examples of calculations of the minimum time control.

Example 5.3-1. Let us now consider the simplest nontrivial time optimal control problem, the time optimal control of a pure inertia or double integration system. The system dynamics are described by

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t), \qquad \begin{matrix}x_1(0) = x_{10}\\ x_2(0) = x_{20}.\end{matrix}$$

We need to compute the transition matrix which is

$$e^{\mathbf{A}t} = e^{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots = \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}.$$

Thus, from Eq. (5.3-22) we have

$$\xi(\tau) = \begin{bmatrix}1 & -\tau\\ 0 & 1\end{bmatrix}\int_0^{\tau}\begin{bmatrix}p\\ 1\end{bmatrix}\text{sign}\,[\lambda_1(t_f)p + \lambda_2(t_f)]\,dp.$$

For a fixed $\lambda(t_f)$, a switch may occur only at $\tau_s = -\lambda_2(t_f)/\lambda_1(t_f)$, verifying, in part, the statement made earlier that, at most, $n - 1$ switchings occur in a plant with n real poles. If we attempt to compute the switching line, we have at the switching point

$$\mathbf{x}(t_s) = \xi(\tau_s) = \begin{bmatrix}1 & -\tau_s\\ 0 & 1\end{bmatrix}\int_0^{\tau_s}\begin{bmatrix}p\\ 1\end{bmatrix}\text{sign}\,\{\lambda_1(t_f)[p - \tau_s]\}\,dp$$

$$= -\begin{bmatrix}1 & -\tau_s\\ 0 & 1\end{bmatrix}\int_0^{\tau_s}\begin{bmatrix}q\\ 1\end{bmatrix}\text{sign}\,[\lambda_1(t_f)]\,dq = \{\text{sign}\,\lambda_1(t_f)\}\begin{bmatrix}\frac{\tau_s^2}{2}\\ -\tau_s\end{bmatrix}.$$

Thus we see that, if $\lambda_1(t_f)$ is greater than zero, the switch points are $x_1 = \tau_s^2/2$, $x_2 = -\tau_s$; whereas if $\lambda_1(t_f)$ is less than zero, the switch points are $x_1 = -\tau_s^2/2$, $x_2 = \tau_s$. Therefore, the equation for the switching boundary is

$$x_1 + \tfrac{1}{2}x_2|x_2| = 0$$

which has the parabolic shape previously obtained in Fig. 4.5-1. It is now evident to us that the control law for this minimum time problem is

$$u = -\text{sign}\,[x_1(t) + \tfrac{1}{2}x_2(t)|x_2(t)|].$$

Example 5.3-2. Let us consider the slightly more complicated minimum time problem for the plant with real eigenvalues

$$\begin{aligned}\dot{x}_1 &= x_2, & x_1(0) &= x_{10}\\ \dot{x}_2 &= -\alpha x_2 + u, & x_2(0) &= x_{20}\end{aligned}$$

or

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -\alpha\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t), \qquad \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = x_o.$$

For this problem, it is a simple matter to compute the transition matrix as

$$e^{\mathbf{A}t} = \begin{bmatrix}1 & \frac{1}{\alpha}(1 - e^{-\alpha t})\\ 0 & e^{-\alpha t}\end{bmatrix}$$

and the equation for the switching boundary as

$$\xi(\tau_s) = \begin{bmatrix}1 & \frac{1}{\alpha}(1 - e^{\alpha\tau_s})\\ 0 & e^{\alpha\tau_s}\end{bmatrix}\int_0^{\tau_s}\begin{bmatrix}\frac{1}{\alpha}(1 - e^{-\alpha p})\\ e^{-\alpha p}\end{bmatrix} \times$$

$$\text{sign}\left\{\lambda_1(t_f)\Big[\frac{1}{\alpha}(1 - e^{\alpha p})\Big] + \lambda_2(t_f)[e^{-\alpha p}]\right\}dp.$$

Upon evaluation of this expression, we see that the switching boundary and control are given by the equations

$$x_1(t) + \frac{1}{\alpha}x_2(t) - \left\{\frac{\text{sign}\,[x_2(t)]}{\alpha^2}\right\}\{\ln\,[1 + \alpha|x_2(t)|]\} = 0$$

$$u(t) = -\text{sign}\left[x_1(t) + \frac{1}{\alpha}x_2(t) - \left\{\frac{\text{sign}\,[x_2(t)]}{\alpha^2}\right\}\{\ln\,[1 + \alpha|x_2(t)|]\}\right].$$

These results are, of course, quite similar to those in the previous examples. There is, at most, one switching point, and the equation for the switching boundary is relatively simple. The switching boundary is much more complicated if the $\mathbf{A}$ matrix has complex or complex-conjugate eigenvalues as we shall now see.

Example 5.3-3. We will now consider the minimum time control of a sinusoidal oscillator. The equations of motion are

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t), \qquad \begin{matrix}x_1(0) = x_{10}\\ x_2(0) = x_{20}\end{matrix}$$

and the transition matrix is

$$e^{\mathbf{A}t} = \begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$$

which yields for the switching boundary

$$\xi(\tau_s) = \begin{bmatrix}\cos\tau_s & -\sin\tau_s\\ \sin\tau_s & \cos\tau_s\end{bmatrix}\int_0^{\tau}\begin{bmatrix}\sin p\\ \cos p\end{bmatrix}\text{sign}\,[\lambda_1(t_f)\sin p + \lambda_2(t_f)\cos p]\,dp.$$

Switching occurs when

$$\lambda_1(t_f)\sin p + \lambda_2(t_f)\cos p = 0$$

which indicates that there may be an infinite number of switchings for this problem. It is considerably more difficult to evaluate $\xi(\tau_s)$ in this case than in the previous two examples. It is not difficult, however, to verify that the proper switching line and control are as shown in Fig. 5.3-1. A typical trajectory with four switchings is also shown in the figure.

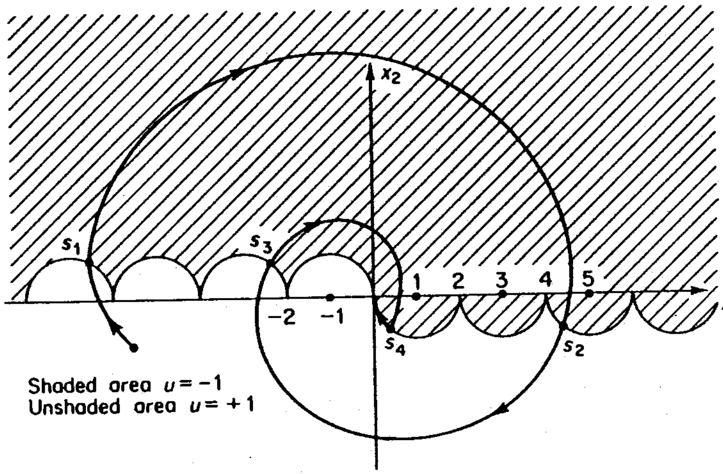

Fig. 5.3-1 Switching line for time optimum control of oscillator and typical trajectory, Example 5.3-3.

Example 5.3-4. As a final example of minimum time control of linear systems with a scalar input control, consider the unstable system with two positive real eigenvalues of +1:

$$\dot{x}_1 = x_2, \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = -x_1 + 2x_2 + u, \qquad x_2(0) = x_{20}.$$

The transition matrix for this unstable system is

$$e^{At} = \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} e^t$$

which gives for the switching boundary

$$\xi(\tau) = \begin{bmatrix} 1+\tau & -\tau \\ \tau & 1-\tau \end{bmatrix} e^{-\tau} - \int_0^\tau \begin{bmatrix} pe^p \\ (1+p)e^p \end{bmatrix} \operatorname{sign}\{p\lambda_1(t_f) + [p+1]\lambda_2(t_f)\}\, dp.$$

We see that there is, at most, one switching which occurs at

$$p = \tau_s = -\frac{\lambda_2(t_f)}{[\lambda_1(t_f) + \lambda_2(t_f)]}.$$

Rather than use the $\xi(t)$ equation to determine the switching boundary and reachable zone, an ancillary approach will be used which gives insight into an additional method of determining solutions to time optimal control problems.

If we run in reverse time from the origin with a (+1) control (the sign { } term in the expression for $\xi(\tau)$ is +1), we obtain the switching curve

$$x_1(\tau) = +(\tau + 1)e^{-\tau} - 1$$
$$x_2(\tau) = +\tau e^{-\tau}.$$

A similar expression results for applying a (−1) control. The switching line is shown in Fig. 5.3-2. By solving the original equations of motion in the forward direction, we see that many initial values for x_{10} and x_{20} lead to trajectories which cannot hit the switching line. Therefore we have a region of state space inside which we can get to the origin. If the initial conditions are outside of this region, we cannot hit the origin no matter what control strategy we apply as long as $|u(t)| \leq 1$.

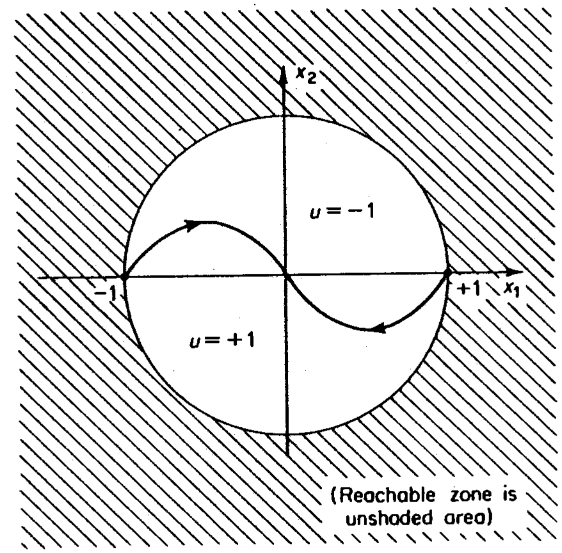

Fig. 5.3-2 Switching line for unstable plant.

Considerable insight can be gained into the analysis of optimal control problems, particularly the minimum time problem, by investigating such problems geometrically. We now examine the geometry of the linear minimum time problem, following the approach of [26], [27], and [28]. Let $K(t)$, called the *set of attainability*, denote the set of all trajectories at time t associated with admissible control functions having values at each time t constrained to the set $\mho \subset R^m$. Thus,

$$K(t) = \{\mathbf{x}(t, \mathbf{u}):\ \mathbf{u} \text{ is admissible, } \mathbf{u}(s) \in \mho, s \leq t\},$$

where $\mathbf{x}(\cdot, \mathbf{u})$ designates the unique trajectory associated with control function $\mathbf{u}$. It can be shown that for each t, if $\mho$ is compact (closed and bounded), then $K(t)$ is convex and compact [28]. Additionally, $K(t)$ inherits a type of continuity (in the sense of the Hausdorff metric (see Appendix B)) from the smoothness characteristics of the admissible trajectories. It is easy to visualize $K(t)$ expanding (which it does under reasonable assumptions [28]) and touching a given target set for the first time at time t^*, where: t^* is then the minimum time; a point of $K(t^*)$ which touches the target set is associated with an optimal trajectory; and a control function taking the initial condition to a point of intersection is an optimal control. It is easily argued that an optimal control $\mathbf{u}^*$ must take the initial condition to the boundary of $K(t^*)$. Otherwise, due to the continuity of $K(t)$, some point in $K(t)$ would have intersected the target set at some time $t < t^*$, contradicting the optimality of $\mathbf{u}^*$. Thus, an optimal control is necessarily an extremal control at time t^*, where we define an extremal control at time t_1 as a control taking the initial point to the boundary of the set of attainability at time t_1. The analysis of extremal controls forms the basis for the geometric development of necessary conditions. Two key results are as follows: Control $\mathbf{u}$ is extremal at time t_1 if and only if there exists a nontrivial solution of

$$\dot{\boldsymbol{\lambda}}(t) = -\mathbf{A}^T(t)\boldsymbol{\lambda}(t) \tag{5.3-23}$$

such that

$$\boldsymbol{\lambda}^T(t)\mathbf{B}(t)\mathbf{u}(t) = \min_{\mathbf{v}\in\mathfrak{U}} \boldsymbol{\lambda}^T(t)\mathbf{B}(t)\mathbf{v}, \tag{5.3-24}$$

for all $t \in [t_0, t_1]$; corollary to this fact is the second result: If $\mathbf{u}$ is extremal at time t_1, then $\mathbf{u}$ is extremal for all time $t \in [t_0, t_1]$. These facts imply that the trajectory $\mathbf{x}^*(\cdot, \mathbf{u}^*)$ associated with an optimal control $\mathbf{u}^*$ lies on the boundary of $K(t)$ for all $t \in [t_0, t^*]$. Necessary conditions for a control to be optimal then follow directly from the first key result: If $\mathbf{u}^*$ is an optimal control, then there exists a nontrivial solution of

$$\dot{\boldsymbol{\lambda}}(t) = -\mathbf{A}^T(t)\boldsymbol{\lambda}(t) \tag{5.3-25}$$

such that

$$\boldsymbol{\lambda}^T(t)\mathbf{B}(t)\mathbf{u}^*(t) = \min_{\mathbf{v}\in\mathfrak{U}} \boldsymbol{\lambda}^T(t)\mathbf{B}(t)\mathbf{v}, \tag{5.3-26}$$

for all $t \in [t_0, t^*]$. We now sketch a proof of the first result, using an argument following [28] which gives a geometric interpretation to the adjoint solution $\boldsymbol{\lambda}(t)$. Proof of the corollary is straightforward and is presented in [28].

Let $\mathbf{u}$ be an extremal control at time t_1 with associated trajectory $\mathbf{x}(\cdot, \mathbf{u})$; thus, $\mathbf{x}(t_1, \mathbf{u})$ is on the boundary of $K(t_1)$. The convexity of $K(t_1)$ then implies that $\mathbf{x}(t_1, \mathbf{u})$ necessarily and sufficiently satisfies the inequality

$$\boldsymbol{\lambda}^T(t_1)[\mathbf{x}(t_1, \mathbf{v}) - \mathbf{x}(t_1, \mathbf{u})] \geq 0, \tag{5.3-27}$$

where $\boldsymbol{\lambda}(t_1)$ is the unit inward normal to the hyperplane tangent to $K(t_1)$ at $\mathbf{x}(t_1, \mathbf{u})$, and $\mathbf{v}$ is any admissible control such that $\mathbf{v}(\tau) \in \mathfrak{U}$ for all $\tau \leq t_1$. Define the adjoint response

$$\dot{\boldsymbol{\lambda}}(t) = -\mathbf{A}^T(t)\boldsymbol{\lambda}(t)$$

having solution

$$\boldsymbol{\lambda}^T(t) = \boldsymbol{\lambda}^T(t_0)\boldsymbol{\Phi}^{-1}(t, t_0),$$

where $\boldsymbol{\Phi}$ is the fundamental matrix associated with the solution of the state trajectory; i.e.,

$$\mathbf{x}(t, \mathbf{u}) = \boldsymbol{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\, ds.$$

It then follows that

$$\boldsymbol{\lambda}^T(t)\mathbf{x}(t, \mathbf{u}) = \boldsymbol{\lambda}^T(t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\lambda}^T(s)\mathbf{B}(s)\mathbf{u}(s)\, ds. \tag{5.3-28}$$

Suppose that Eq. (5.3-24) is not true over some interval $(t_a, t_b) \subset [t_0, t_1]$, $t_b > t_a$, and let $\tilde{\mathbf{u}}$ be such that it satisfies Eq. (5.3-26) for all $t \in [t_0, t_1]$. It then follows from Eq. (5.3-28) that

$$\boldsymbol{\lambda}^T(t_1)\mathbf{x}(t_1, \tilde{\mathbf{u}}) < \boldsymbol{\lambda}^T(t_1)\mathbf{x}(t_1, \mathbf{u}),$$

which violates Eq. (5.3-27) and contradicts the assumption that $\mathbf{u}$ is an extremal at t_1. Thus, if $\mathbf{u}$ is an extremal at t_1, then Eq. (5.3-24) holds for all $t \in [t_0, t_1]$.

Conversely, assume an admissible control $\mathbf{u}$, $\mathbf{u}(\tau) \in \mathfrak{U}$ for $\tau \leq t_1$, satisfies Eq. (5.3-24) for all $t \in [t_0, t_1]$ but that $\mathbf{x}(t_1, \mathbf{u})$ is an interior point of $K(t_1)$. There then exists an admissible control $\tilde{\mathbf{u}}$, $\tilde{\mathbf{u}}(\tau) \in \mathfrak{U}$ for $\tau \leq t_1$, such that $\boldsymbol{\lambda}^T(t_1)\mathbf{x}(t_1, \mathbf{u}) > \boldsymbol{\lambda}^T(t_1)\mathbf{x}(t_1, \tilde{\mathbf{u}})$. By hypothesis,

$$\boldsymbol{\lambda}^T(t)\mathbf{B}(t)\tilde{\mathbf{u}}(t) \leq \boldsymbol{\lambda}^T(t)\mathbf{B}(t)\mathbf{u}(t)$$

for all $t \in [t_0, t_1]$, which implies from Eq. (5.3-28) that $\boldsymbol{\lambda}^T(t_1)\mathbf{x}(t_1, \mathbf{u}) \leq \boldsymbol{\lambda}^T(t_1)\mathbf{x}(t_1, \tilde{\mathbf{u}})$; thus, $\mathbf{u}$ is an extremal control at t_1, and the result is proved.

Of course, the necessary conditions of Eqs. (5.3-25) and (5.3-26) are only restatements of a slight generalization of Eqs. (5.3-12) and (5.3-17), where we note that all that is subject to minimization with respect to $\mathbf{u}$ in Eq. (5.3-12) is the term examined in Eq. (5.3-26).

We noted from Eqs. (5.3-5) and (5.3-13) that when $\mathfrak{U}$ is a rectangle or cube, as defined in Eqs. (5.3-2) and (5.3-11), the set of all allowable control values could be substantially restricted without affecting minimum cost. This bang bang result can be generalized to include compact sets $\mathfrak{U} \subset R^m$ as follows. Define the convex hull of $\mathfrak{U}$, denoted by $\mathcal{H}(\mathfrak{U})$, as the smallest convex set in R^m containing $\mathfrak{U}$. Let $\mathfrak{U}'$ be the smallest subset of $\mathfrak{U}$ such that $\mathcal{H}(\mathfrak{U}') = \mathcal{H}(\mathfrak{U})$. The bang bang principle then states that it is sufficient to consider only control values in $\mathfrak{U}'$ without affecting minimum cost [27, 28]. Since the convex hull of the vertices of a rectangle equals (the convex hull of) the rectangle, the results stated in Eqs. (5.3-5) and (5.3-12) are special cases of the bang bang principle.

5.4
Singular solutions

Let us consider the problem of obtaining the optimal control which minimizes the performance index

$$J = \tfrac{1}{2}\int_0^2 x^2\,dt \tag{5.4-1}$$

for the linear differential system

$$\dot{x} = u(t), \qquad x(0) = 1 \tag{5.4-2}$$

where the magnitude of the control effort is restricted by $|u(t)| \leq 1$.

The Hamiltonian for this simple problem is

$$H[x(t), u(t), \lambda(t)] = \tfrac{1}{2}x^2(t) + \lambda(t)u(t), \tag{5.4-3}$$

and it is clear that, for nonzero $\lambda(t)$, we should make

$$u(t) = -\text{sign}\,\lambda(t) = \begin{cases} +1 & \lambda(t) < 0 \\ -1 & \lambda(t) > 0. \end{cases} \tag{5.4-4}$$

This indicates, as expected, that the optimal control operates on the boundary. It is possible, however, for the control to be within the ± 1 boundary, in which case we should set

$$\frac{\partial H}{\partial u} = 0 = \lambda(t). \tag{5.4-5}$$

Thus we see that $\lambda(t) = 0$ is a possible solution, in which case the Hamiltonian does not depend on $u(t)$ at all, and therefore cannot be minimized with respect to a choice of $u(t)$. Problems such as this are called *singular problems*. More generally, an extremal arc of the optimal control problem is said to be singular if the determinant of the matrix H_{uu}, $|H_{uu}|$, vanishes at any point along the arc [29].

We may solve the above singular solution problem as follows. The canonic equations are

$$\dot{x} = \frac{\partial H}{\partial \lambda} = u(t), \qquad \dot{\lambda} = -\frac{\partial H}{\partial x} = -x(t). \tag{5.4-6}$$

We assume that initially $\lambda(t)$ is positive, such that $u(t) = -\text{sign}\,\lambda(t) = -1$. The equations of motion are then

$$x(t) = 1 - t, \qquad \lambda(t) = \lambda(0) - t + \frac{t^2}{2}. \tag{5.4-7}$$

On the singular arc, we have found that $\lambda(t) = 0$, and this requires that $\dot{\lambda}$ be zero, as well as $x(t)$. But from the foregoing, we see that $x(t)$ is zero at $t = 1$, and all the requirements for a singular solution for $t \in [1, 2]$ are established if the initial condition on the adjoint $\lambda(t_o)$ is equal to $\tfrac{1}{2}$. The

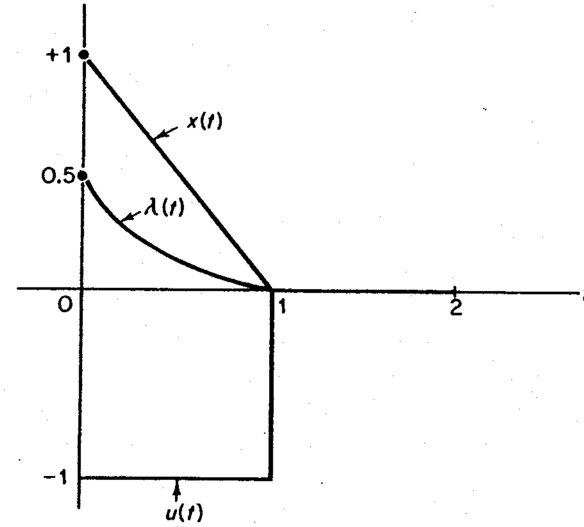

Fig. 5.4-1 State and control variables for singular solution.

optimal control and trajectory are shown in Fig. 5.4-1. Clearly, this satisfies the transversality condition that $\lambda(2) = 0$.

We may obtain an alternate approach to this problem by converting the control inequality constraint to an equality constraint as outlined in Chapters 3 and 4. We therefore use $1 - u^2(t) = \alpha^2(t)$ and write the Hamiltonian as

$$H[x(t), u(t), \lambda(t), \alpha(t)] = \tfrac{1}{2}x^2(t) + \lambda_1(t)u(t) + \lambda_2(t)[1 - u^2(t) - \alpha^2(t)]. \tag{5.4-8}$$

We then find

$$\frac{\partial H}{\partial u} = 0 = \lambda_1(t) - 2\lambda_2(t)u(t) \tag{5.4-9}$$

$$\frac{\partial H}{\partial x} = -\dot{\lambda}_1 = x(t) \tag{5.4-10}$$

$$\frac{\partial H}{\partial \alpha} = 0 = -2\lambda_2(t)\alpha(t). \tag{5.4-11}$$

Thus we need to solve the following equations

$$\begin{aligned} &\dot{x} = u, \qquad x(0) = 1 \\ &\dot{\lambda}_1 = -x(t), \qquad \lambda_1(2) = 0 \\ &1 - u^2(t) = \alpha^2(t) \\ &\lambda_2(t)\alpha(t) = 0 \\ &\lambda_1(t) = 2\lambda_2(t)u(t). \end{aligned} \tag{5.4-12}$$

From these equations, we get precisely the same results as when we incorporate the inequality constraint into the maximum principle. The only difference is that, in a computational solution of the problem, the singular control does not arise except as the solution becomes exact (after a large number of iterations in practice) when $\lambda_2(t) = \lambda_1(t) = \alpha$, $\forall t \in [1, 2]$. An alternate method of eliminating the singular solution is to use the penalty-function technique and write the cost function as

$$J = \tfrac{1}{2}x_2^2(t_f)s + \tfrac{1}{2}\int_0^2 x_1^2(t)\,dt \tag{5.4-13}$$

for the system (where there is no inequality control constraint and where H denotes the Heaviside step) described by

$$\dot{x}_1 = u, \qquad x_1(0) = 1 \tag{5.4-14}$$

$$\dot{x}_2 = K\mathrm{H}(1 - u^2), \qquad x_2(0) = 0 \tag{5.4-15}$$

$$K > 0, \qquad \mathrm{H}(a) = \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{if } a < 0. \end{cases}$$

We shall now extend these ideas to more complex systems and performance indices. In the previous section, we considered the general nonlinear problem wherein the control entered the Hamiltonian in a linear fashion, Eq. (5.3-4). The possibility of a singular solution exists if

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{0} = \mathbf{h}[\mathbf{x}(t), t] + \mathbf{G}^T[\mathbf{x}(t), t]\boldsymbol{\lambda}(t) = \mathbf{0} \tag{5.4-16}$$

such that the Hamiltonian does not depend on $\mathbf{u}(t)$

$$H[\mathbf{x}(t), \boldsymbol{\lambda}(t), t] = \phi[\mathbf{x}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{f}[\mathbf{x}(t), t]. \tag{5.4-17}$$

If the Hamiltonian is insensitive to the control for any finite range of time, then a singular solution does, in fact, exist. It is generally easy to verify whether a singular solution possibility does or does not exist. It is normally considerably more difficult to actually determine an optimum singular solution.

Let us now generalize the work of this section to include the time-varying linear system with a quadratic performance index in the state variables only. We will consider the determination of the control vector $\mathbf{u}(t)$ which minimizes

$$J = \tfrac{1}{2}\mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} \mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t)\,dt \tag{5.4-18}$$

for the system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \qquad \mathbf{x}(t_o) = \mathbf{x}_o, \tag{5.4-19}$$

where t_f is fixed and where the control vector is bounded. The Hamiltonian

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = \tfrac{1}{2}\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \boldsymbol{\lambda}^T(t)\mathbf{A}(t)\mathbf{x}(t) + \boldsymbol{\lambda}^T(t)\mathbf{B}(t)\mathbf{u}(t) \tag{5.4-20}$$

is again linear in the control variable, and for nonzero $\boldsymbol{\lambda}^T(t)\mathbf{B}(t)$, we know that $\mathbf{u}(t)$ operates on the boundary of the admissible input set. It is still possible for

$$\frac{\partial H}{\partial \mathbf{u}} = 0 = \mathbf{B}^T(t)\boldsymbol{\lambda}(t) \tag{5.4-21}$$

to be a solution which minimizes the Hamiltonian. In this case, we have

$$H[\mathbf{x}(t), \boldsymbol{\lambda}(t), t] = \tfrac{1}{2}\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \boldsymbol{\lambda}^T(t)\mathbf{A}(t)\mathbf{x}(t), \tag{5.4-22}$$

and if the Hamiltonian is not a function of $\mathbf{u}(t)$ for some finite range of t, we have a singular solution.

Recent effort has been concerned with the development of necessary, sufficient, and necessary and sufficient conditions for singular solutions from the second variation of the cost function. Of particular interest are results in [30, 31, 32]. A recent survey of singular problems can be found in [33].

Example 5.4-1. Let us consider the following particular example [14]. The cost function is

$$J = \tfrac{1}{2}\int_0^{t_f} x_1^2\,dt$$

for the system

$$\dot{x}_1 = x_2(t) + u(t), \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = -u(t), \qquad x_2(0) = x_{20},$$

where it is desired to drive the system to the origin at a fixed time, t_f, $x_1(t_f) = x_2(t_f) = 0$ with a bounded control $|u(t)| \le K$.

The Hamiltonian is

$$H[\mathbf{x}(t), u(t), \boldsymbol{\lambda}(t)] = \lambda_1(t)[x_2(t) + u(t)] - \lambda_2(t)u(t) + \tfrac{1}{2}x_1^2(t).$$

Singular arcs are such that

$$\frac{\partial H}{\partial u} = 0 = \lambda_1(t) - \lambda_2(t)$$

for a finite time interval with the canonic equations

$$\dot{x}_1 = x_2(t) + u(t), \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = -u(t), \qquad x_2(0) = x_{20}$$
$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -x_1(t), \qquad x_1(t_f) = 0$$
$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1(t), \qquad x_2(t_f) = 0.$$

For a singular solution, $\lambda_1(t) = \lambda_2(t)$; therefore, $\dot{\lambda}_1 = \dot{\lambda}_2$, which yields from the canonic equations for the adjoint variables $x_1(t) = \lambda_1(t)$. Also, on the singular arc

$$\frac{d^2}{dt^2}\frac{\partial H}{\partial u} = 0 = -\dot{x}_1 + \dot{\lambda}_1 = -x_2(t) - u(t) - x_1(t).$$

Since the Hamiltonian depends only implicitly on time, it must be constant about an optimal control and trajectory and, therefore, on a singular arc

$$H[\mathbf{x}(t), \boldsymbol{\lambda}(t)] = x_1(t)x_2(t) + \tfrac{1}{2}x_1^2(t) = \text{constant}.$$

Thus we have discovered that, on the singular arc, the closed-loop control is

$$u(t) = -x_1(t) - x_2(t),$$

and the singular arcs are described by

$$x_1(t)x_2(t) + \tfrac{1}{2}x_1^2(t) = \text{constant}.$$

On a singular arc, the optimum closed-loop control yields the system equations:

$$\dot{x}_1 = x_2 - (x_1 + x_2) = x_1(t)$$

$$\dot{x}_2 = x_1(t) + x_2(t)$$

so

$$x_1(t) = e^{-(t-t_1)}x_1(t_1)$$

$$x_2(t) = \left.\frac{e^{(t-t_1)} - e^{-(t-t_1)}}{2}\right| x_1(t_1) + e^{(t-t_1)}x_2(t_1).$$

A typical trajectory involves three parts. An extreme $\pm K$ value of the control is used initially to transfer the system to the singular arc. The singular control $u(t) = -x_1(t) - x_2(t)$ is then applied until another application of the extreme value of the control transfers the state of the system to the origin. We may now consider two cases:

(a) $K \longrightarrow \infty$ such that the control is unbounded. In this case, impulse controls are used initially to transfer to the singular arc which is followed until the time $x_1(t) + x_2(t) = 0$, where application of another impulse transfers the system to the origin. Since the system must arrive at $x_1(t) + x_2(t) = 0$ at $t = t_f$, and since the impulse transfers the state to the origin in zero time, the particular value of the constant Hamiltonian which determines the particular singular arc is determined. Figure 5.4-2 illustrates a possible trajectory in this case and also shows the straight line directions of motion due to application of impulse control inputs.

(b) $t_f \longrightarrow \infty$ such that $H[\mathbf{x}(t), \boldsymbol{\lambda}(t)] = 0$. In this case, the singular arcs are the two lines $x_1(t) = 0$ and $x_1(t) + 2x_2(t) = 0$, as we easily obtain by setting the Hamiltonian equal to zero for the singular solution. On the line $x_1(t) = 0$, the closed-loop control becomes $u(t) = -x_2(t)$ and, consequently,

$$x_1(t) = 0, \qquad x_2(t) = e^{(t-t_1)}x_2(t_1).$$

On the line $x_1(t) + 2x_2(t) = 0$, the closed-loop control is $u(t) = x_2(t)$ and

$$x_1(t) = x_1(t) + [1 - e^{-(t-t_1)}]x_2(t_1), \qquad x_2(t) = e^{-(t-t_1)}x_2(t_1).$$

If there is an inequality constraint on u, $|u(t)| \leq K$, then the largest values S_1 and S_2 that the singular arc can have are also limited. The

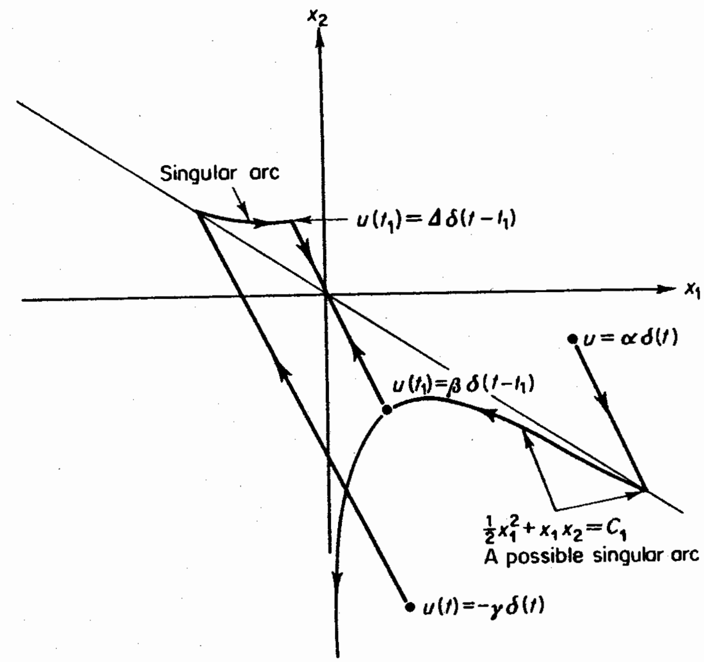

Fig. 5.4-2 Possible system trajectories for Example 5.4-1(a).

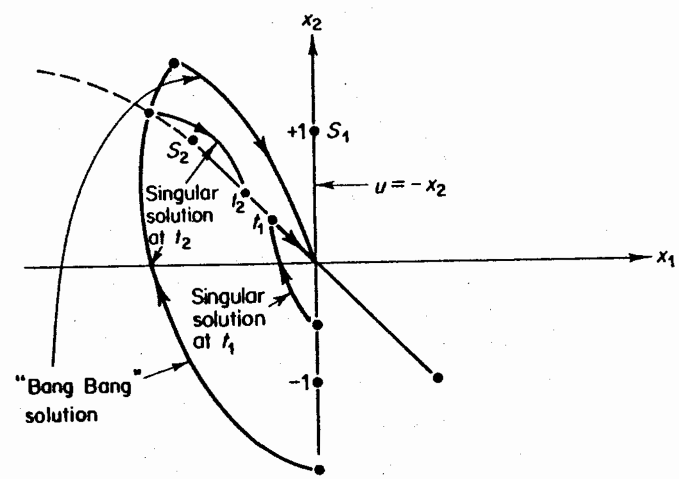

Fig. 5.4-3 Possible system trajectories for Example 5.4-1(b).

motion on the singular arc where $u(t) = -x_2(t)$ is unstable in that it is directed away from the origin. In this case, the optimal control is at the extreme value until the singular arc is reached, at which time the singular arc is followed until the origin is reached. As shown in Fig. 5.4-3, which illustrates the singular arc and several optimal trajectories, there is no guarantee that the singular control is minimizing or that it will always enter into the optimal solution even if the possibility of a singular solution does exist. In this example too large an initial condition produces a bang bang control to drive the system to the singular arc. The resulting cost function turns out to be less than the cost function with pure bang bang control.

Example 5.4-2. [33, 34] Historically, singular solution problems initially arose in aerospace applications. Consider the following formulation of the fundamental problem of space navigation. Let (x_1, x_2, x_2) and (v_1, v_2, v_3) designate the Cartesian spatial and velocity coordinates for a space vehicle at time t, and assume its rocket thrust acts in a direction having direction cosines (l_1, l_2, l_3). Let the control $m(t)$ be the mass rate of propellant consumption. The equations of motion are

$$\dot{v}_i = (cml_i)/M + g_i(x_1, x_2, x_3, t)$$
$$\dot{x}_i = v_i, \qquad i = 1, 2, 3$$
$$\dot{M} = -m, \quad 0 \leq m \leq \bar{m},$$

where g_i, $i = 1, 2, 3$, are the components of the gravitational field, c is the rocket exhaust velocity, and M is the rocket mass. The direction cosines can be expressed as

$$l_1 = \sin\theta \cos\phi$$
$$l_2 = \sin\theta \sin\phi$$
$$l_3 = \cos\theta,$$

where the controls θ and ϕ are the spherical polar coordinates. The control problem is to determine θ, ϕ, and m so as to minimize fuel consumption,

$$J = -M(t_f),$$

subject to the following boundary conditions: The initial and final states $x_i(t_o)$ and $x_i(t_f)$, $i = 1, 2, 3$, are specified; the initial and final velocities $v_i(t_o)$ and $v_i(t_f)$, $i = 1, 2, 3$, are specified; the initial mass of the rocket $M(t_o)$ is specified; and the initial and final times can be free or fixed. It is easily shown that the Hamiltonian for this problem is linear in the fuel consumption rate $m(t)$; therefore, the control function will be singular. Recent applications of singular solution theory to aerospace problems can be found in [35, 36].

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Problems

1. Obtain the closed-loop control for the regulator problem where the dynamics are

$$\dot{x}_1 = x_2, \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = u, \qquad x_2(0) = x_{20}$$
$$J = \tfrac{1}{2}\mathbf{x}^T(t_f)\begin{bmatrix} s_{11} & 0 \\ 0 & s_{22} \end{bmatrix}\mathbf{x}(t_f) + \tfrac{1}{2}\int_0^{t_f}\left\{u^2(t) + \mathbf{x}^T(t)\begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}\mathbf{x}(t)\right\} dt.$$

For $t_f = \infty$ we may solve the Riccati equation $\dot{\mathbf{P}} = 0$ and obtain two values for $\mathbf{P}$. How do we determine which one to use?

2. In Problem 1, let $q_{11} = q_{22} = 0$ and $s_{11} = s_{22} = \infty$. Determine the closed-loop control for this case as a function of t_f and $\mathbf{x}(t)$.

3. In Problem 1, let $z(t) = x_1(t) + x_2(t)$. Determine the closed-loop control which minimizes

$$J = \tfrac{1}{2}sz^2(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}[u^2(t) + qz^2(t)]\, dt.$$

4. In Problem 3, let $q = 0$, $s = \infty$. Determine the closed-loop control for this case as a function of t_f and $\mathbf{x}(t)$.

5. Determine the switching surfaces for the minimum time control which transfers the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 2 \\ 0 & -1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \qquad |u| \le 1$$

to the origin.

6. Determine the switching surfaces for the minimum time control which transfers

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}u(t), \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \qquad |u| \le 1$$

to the origin.

7. Show that Problem 5 is related to Problem 6 by the linear transformation

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}\mathbf{x}(t).$$

How does this linear transformation affect the switching surfaces of the two problems?

8. Determine the switching surfaces for the minimum time control which transfers to the origin the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = u, \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \qquad |u| \le 1.$$

9. Discuss in detail the nature of the optimum control which minimizes

$$J = \tfrac{1}{2}\int_0^{t_f} x^2(t)\, dt, \qquad |u| \le 1,\ t_f \text{ free},$$