FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS

The simplest problem in the calculus of variations is the optimization problem:

$$\min_{x \in X} J(x)$$

where

$$J(x) = \int_{t_0}^{t_f} \phi[x(t), \dot{x}(t), t] \, \mathrm{d}t$$

$$X = \{x \in D[t_0, t_f] \, : \, x(t_0) = x_0, \, x(t_f) = x_f\}$$

Here, t_0, t_f, x_0, x_f are fixed constants, $D[t_0, t_f]$ denotes the class of continuously differentiable functions on $[t_0, t_f]$, and ϕ is a function that is continuously differentiable in x and $\dot{x}$, and continuous in t.

We will see in the lecture tomorrow that if $x^o_{[t_0,t_f]}$ solves this *fixed end-point* calculus of variations problem, then there exists a constant C, such that the following integral equation is satisfied:

$$\phi_{\dot{x}}(x^o(t), \dot{x}^o(t), t) = \int_{t_0}^{t} \phi_x(x^o(s), \dot{x}^o(s), s) \, \mathrm{d}s + C$$

This integral equation is called the *Euler-Lagrange equation in integral form.* If ϕ is twice continuously differentiable in x and $\dot{x}$, and continuously differentiable in t, then we have its differential equation version:

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi_{\dot{x}}(x^o(t), \dot{x}^o(t), t) = \phi_x(x^o(t), \dot{x}^o(t), t)$$

which has to be solved subject to the boundary conditions: $x(t_0) = x_0, \, x(t_f) = x_f$. This is the standard (familiar) Euler-Lagrange equation of the calculus of variations. It is also sometimes called the *Euler-Lagrange equation in differential form,* which I derived in the lecture yesterday.

In the derivation of the Euler-Lagrange equation in integral form, we will make use of a fundamental lemma, which is stated and proved below.

Lemma 1

Let $[t_0, t_f]$ be a given interval, and $\mathbf{N}$ be the class of all piecewise continuously differentiable functions η on $[t_0, t_f]$, satisfying the end-point restrictions $\eta(t_0) = \eta(t_f) = 0$. Let M be a piecewise continuous function defined on the same interval. Then,

$$\int_{t_0}^{t_f} M(t)\dot{\eta}(t) \, \mathrm{d}t = 0 \, , \quad \text{for} \quad \text{all} \quad \eta \in \mathbf{N},$$

if, and only if,

$$M(t) = c \quad \text{(a constant)} \qquad t \in [t_0, t_f],$$

except possibly at a finite number of points in $[t_0, t_f]$.

Proof. Choose

$$\eta(t) = \int_{t_0}^{t} [c - M(s)]\,\mathrm{d}s$$

where

$$c := \frac{1}{t_f - t_0} \int_{t_0}^{t_f} M(s)\,\mathrm{d}s$$

(this makes $\eta \in \mathbf{N}$). Then,

$$\int_{t_0}^{t_f} M(t)\dot{\eta}(t)\,\mathrm{d}t = \int_{t_0}^{t_f} M(t)[c - M(t)]\,dt.$$

Adding to this the identically zero quantity

$$-c \int_{t_0}^{t_f} [c - M(t)]\,\mathrm{d}t\ ,$$

we obtain

$$-\int_{t_0}^{t_f} M(t)\dot{\eta}(t)\,\mathrm{d}t = \int_{t_0}^{t_f} [c - M(t)]^2\,\mathrm{d}t = 0,$$

where the last equality (to zero) follows from the hypothesis of the Lemma. Hence,

$$M(t) = c \qquad \text{a.e.} \quad t \in [t_0, t_f]$$

which follows as a necessary condition (on M) for the hypothesis of the Lemma to be valid. Sufficiency follows trivially, by direct substitution. ◇

This result is due to Dubois and Reymond, who also proved the weaker (but still useful) version given below.

Lemma 2

Let $[t_0, t_f]$ be a given interval, and $\mathbf{N}$ be the class of all continuously differentiable functions η on $[t_0, t_f]$, satisfying the end-point restrictions $\eta(t_0) = \eta(t_f) = 0$. Let m be a piecewise continuous function defined on the same interval. Then,

$$\int_{t_0}^{t_f} m(t)\eta(t)\,\mathrm{d}t = 0\ , \quad \text{for} \quad \text{all} \quad \eta \in \mathbf{N},$$

if, and only if,

$$m(t) = 0 \qquad t \in [t_0, t_f],$$

except possibly at a finite number of points in $[t_0, t_f]$.

Proof. Follows from Lemma 1, by "integration by parts". It can also be proven directly (try it !).◇

Note. One can use Lemma 2 directly, to arrive at the differential equation form of the Euler-Lagrange equation, which I did on Monday.

Historical Note. The differential equation version of the Euler-Lagrange equation is due to the Swiss mathematician Leonard Euler (1707-1783), who derived it in 1744 using polygonal approximation. The first derivation using variational calculus was given by the French mathematician Joseph Louis de Lagrange (1736-1813) who, however, did not prove Lemma 2; the proof was later furnished by Dubois and Reymond.

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