SUMMARY OF BASIC RESULTS ON H^∞ CONTROL

System dynamics : $\dot{x} = A(t)\,x + B(t)\,u(t) + D(t)\,w(t)\,, \qquad x(t_0) = x_0\,, \quad \dim(x) = n$

Controlled output : $z(t) = H(t)\,x(t) + G(t)\,u(t)\,, \qquad H^T H =: Q,\ H^T G = 0,\ G^T G = I$

Measured output : $y(t) = C(t)\,x(t) + E(t)\,v(t)\,, \qquad EE^T =: N, \quad N > 0$

1. Perfect State Measurements : $u(t) = \mu(t; x(\tau), \tau \le t)$

1a. Finite horizon: $[t_0, t_f]$ and $x_0 = 0$

Cost : $L(u,w) = \left[\,\left(|x(t_f)|^2_{Q_f} + \|z\|^2\right)^{1/2} / \|w\|\,\right], \quad Q_f \ge 0\,, \quad \|z\|^2 := \int_{t_0}^{t_f} |z(t)|^2\,dt$

Optimum performance : $\gamma^* := \inf_\mu \sup_w L(\mu, w)$

Related differential game (for every $\gamma > \gamma^*$) :

$$J_\gamma(u,w) = |x(t_f)|^2_{Q_f} + \|z\|^2 - \gamma^2 \|w\|^2$$

GRDE-1 : $\dot{Z} + A'Z + ZA - Z(BB' - \gamma^{-2}DD')Z + Q = 0\,; \quad Z(t_f) = Q_f$

Let $\hat{\gamma} := \inf\{\,\gamma > 0 :$ GRDE-1 admits a solution on $[t_0, t_f]\,\}$

Theorem 1a

(i) $\gamma^* = \hat{\gamma}$

(ii) For each $\gamma > \gamma^*$, there exists a state-feedback controller that achieves the performance bound γ, which is given by

$$\mu_\gamma(t,x) = -B(t)'\,Z_\gamma(t)\,x\,,$$

where Z_γ is the unique nonnegative-definite (nnd) solution of *GRDE-1* above. Equivalently,

$$L(\mu_\gamma, w) \le \gamma\,, \quad \forall\, w \in L_2[t_0, t_f]$$

1b. Finite horizon: $[t_0, t_f]$ and $x_0 \ne 0$

Cost : $L(u,w) = \left[\,\left(|x(t_f)|^2_{Q_f} + \|z\|^2\right)^{1/2} / \left(\|w\|^2 + |x_0|^2_{Q_0}\right)^{1/2}\right], \quad Q_f \ge 0\,, \quad Q_0 \ge 0$

Optimum performance : $\gamma^* := \inf_\mu \sup_w L(\mu, w)$

Related differential game (for every $\gamma > \gamma^*$) :

$$J_\gamma(u,w) = |x(t_f)|^2_{Q_f} + \|z\|^2 - \gamma^2\left(\|w\|^2 + + |x_0|^2_{Q_0}\right)$$

GRDE-1 : $\dot{Z} + A'Z + ZA - Z(BB' - \gamma^{-2}DD')Z + Q = 0\,; \quad Z(t_f) = Q_f$

Let $\hat{\gamma} := \inf\{\,\gamma > 0 :$ GRDE-1 admits a solution $Z_\gamma(\cdot)$ on $[t_0, t_f]$ with the property $\gamma^2 Q_0 - Z_\gamma(t_0) \ge 0\}$

Theorem 1b

(i) $\gamma^* = \hat{\gamma}$

(ii) For each $\gamma > \gamma^*$, there exists a state-feedback controller that achieves the performance bound γ, which is given by

$$\mu_\gamma(t,x) = -B(t)'\,Z_\gamma(t)\,x\,,$$

where Z_γ is the unique nonnegative-definite solution of *GRDE-1* above. Equivalently,

$$L(\mu_\gamma, w) \le \gamma\,, \quad \forall\, w \in L_2[t_0, t_f] \text{ and } \forall\, x_0 \in \mathbf{R}^n$$

1c. Infinite horizon: $[t_0, \infty)$ and $x_0 = 0$; all matrices time-invariant

Cost : $L(u,w) = \left[\, \|z\| / \|w\| \,\right]$, $\|z\|^2 := \int_{t_0}^{\infty} |z(t)|^2 \, dt$

Optimum performance : $\gamma_\infty^* := \inf_\mu \sup_w L(\mu, w)$

Related differential game (for every $\gamma > \gamma_\infty^*$) :

$$J_\gamma(u,w) = \|z\|^2 - \gamma^2 \|w\|^2$$

GARE-1 : $A'Z + ZA - Z(BB' - \gamma^{-2} DD')Z + Q = 0$

Let $\hat{\gamma}_\infty := \inf \{ \gamma > 0 : \text{ GARE-1 admits a nonnegative-definite solution } \}$

Theorem 1c. Let (A,B) be stabilizable, and (A,H) be detectable. Then,

(i) γ_∞^* is finite, and $\gamma_\infty^* = \hat{\gamma}_\infty$

(ii) For each $\gamma > \gamma_\infty^*$, GARE-1 admits a unique minimal nonnegative-definite solution, $\bar{Z}_\gamma^+$.

(iii) For each $\gamma > \gamma_\infty^*$, there exists a time-invariant state-feedback controller that achieves the performance bound γ, which is given by

$$\mu_\gamma(x) = -B' \, \bar{Z}_\gamma^+ \, x \, ,$$

Equivalently,

$$L(\mu_\gamma, w) \leq \gamma \, , \quad \forall \, w \in L_2[t_0, \infty)$$

(iv) The state-feedback controller above leads to a BIBS stable system, that is the matrix $A - BB'\bar{Z}_\gamma^+$ is Hurwitz.

1d. Infinite horizon: $[t_0, \infty)$ and $x_0 \neq 0$; all matrices time-invariant

Cost : $L(u,w) = \left[\, \|z\| / \left(\|w\|^2 + |x_0|_{Q_0}^2 \right)^{1/2} \right]$, $Q_0 \geq 0$, $\|z\|^2 := \int_{t_0}^{\infty} |z(t)|^2 \, dt$

Optimum performance : $\gamma_\infty^* := \inf_\mu \sup_w L(\mu, w)$

Related differential game (for every $\gamma > \gamma_\infty^*$) :

$$J_\gamma(u,w) = \|z\|^2 - \gamma^2 \left(\|w\|^2 + |x_0|_{Q_0}^2 \right)$$

GARE-1 : $A'Z + ZA - Z(BB' - \gamma^{-2} DD')Z + Q = 0$

Let $\hat{\gamma}_\infty := \inf \{ \gamma > 0 : \text{ GARE-1 admits a nnd solution } \bar{Z}_\gamma \text{ with the property } Q_0 - \gamma^{-2} \bar{Z}_\gamma \geq 0 \}$

Theorem 1d. Let (A,B) be stabilizable, and (A,H) be detectable. Then,

(i) γ_∞^* is finite, and $\gamma_\infty^* = \hat{\gamma}_\infty$

(ii) For each $\gamma > \gamma_\infty^*$, GARE-1 admits a unique minimal nonnegative-definite solution, $\bar{Z}_\gamma^+$.

(iii) For each $\gamma > \gamma_\infty^*$, there exists a time-invariant state-feedback controller that achieves the performance bound γ, which is given by

$$\mu_\gamma(x) = -B' \, \bar{Z}_\gamma^+ \, x \, ,$$

Equivalently,

$$L(\mu_\gamma, w) \leq \gamma \, , \quad \forall \, w \in L_2[t_0, \infty) \text{ and } \forall \, x_0 \in \mathbf{R}^n$$

(iv) The state-feedback controller above leads to a BIBS stable system, that is the matrix $A - BB'\bar{Z}_\gamma^+$ is Hurwitz.

2. Noisy Output Measurements $\quad u(t) = \mu(t; y(\tau), \tau \le t)$

2a. Finite horizon: $[t_0, t_f]$ and x_0 unknown, but has nominal value $\bar{x}_0$

Cost : $L(u,w) = \left[\left(|x(t_f)|^2_{Q_f} + \|z\|^2 \right)^{1/2} / \left(\|w\|^2 + \|v\|^2 + |x_0 - \bar{x}_0|^2_{Q_0} \right)^{1/2} \right] \quad Q_f \ge 0\,,\ Q_0 > 0$

Optimum performance : $\gamma_I^* := \inf_\mu \sup_w L(\mu, w)$

Related differential game (for every $\gamma > \gamma_I^*$) :

$$J_\gamma(u,w) = |x(t_f)|^2_{Q_f} + \|z\|^2 - \gamma^2 \left(\|w\|^2 + \|v\|^2 + |x_0 - \bar{x}_0|^2_{Q_0} \right)$$

GRDE-2 : $\dot{\Sigma} = A\Sigma + \Sigma A' - \Sigma(C'N^{-1}C - \gamma^{-2}H'H)\Sigma + DD'\,; \quad \Sigma(t_0) = Q_0^{-1}$

FILTER : $\dot{\hat{x}} = \left[A - (BB' - \gamma^{-2}DD')Z\right] \hat{x} + [I - \gamma^{-2}\Sigma Z]^{-1}\Sigma C' N^{-1}(y - C\hat{x})\,; \quad \hat{x}(t_0) = \bar{x}_0$

Let $\hat{\gamma}_I := \inf\{\, \gamma > 0 :$ GRDE-1 and GRDE-2 admit solutions on $[t_0, t_f]$, such that $\rho(\Sigma Z) < \gamma^2 \,\}$

Theorem 2a

(i) $\gamma_I^* = \hat{\gamma}_I$

(ii) For each $\gamma > \gamma_I^*$, there exists an n-dimensional controller that achieves the performance bound γ, which is given by

$$\mu_\gamma(t,y) = -B(t)'\, Z_\gamma(t)\, \hat{x}_\gamma(t)\,,$$

where $\hat{x}_\gamma$ is generated by the FILTER above. Equivalently,

$$L(\mu_\gamma, w) \le \gamma\,, \quad \forall\, w \in L_2[t_0, t_f],\ \forall\, v \in L_2[t_0, t_f],\ \text{and}\ \forall\, x_0 \in \mathbf{R}^n$$

2b. Infinite horizon: $(-\infty, \infty)$ and $x(t_0) \to 0$ as $t_0 \to -\infty$; all matrices time-invariant

Cost : $L(u,w) = \left[\|z\| / \left(\|w\|^2 + \|v\|^2 \right)^{1/2} \right] \qquad \|z\|^2 := \int_{-\infty}^{\infty} |z(t)|^2\, dt$

Optimum performance : $\gamma_{I\infty}^* := \inf_\mu \sup_w L(\mu, w)$

Related differential game (for every $\gamma > \gamma_{I\infty}^*$) :

$$J_\gamma(u,w) = \|z\|^2 - \gamma^2 \left(\|w\|^2 + \|v\|^2 \right)$$

GARE-2 : $A\Sigma + \Sigma A' - \Sigma(C'N^{-1}C - \gamma^{-2}H'H)\Sigma + DD' = 0$

Let $\hat{\gamma}_{I\infty} := \inf\{\, \gamma > 0 :$ GARE-1 and GARE-2 admit nnd solutions $\bar{Z}_\gamma$ and $\bar{\Sigma}_\gamma$ with the property $\rho(\bar{\Sigma}_\gamma \bar{Z}_\gamma) < \gamma^2 \,\}$

Theorem 2b. Let (A, B) and (A, D) be stabilizable, and (A, H) and (A, C) be detectable. Then,

(i) $\gamma_{I\infty}^*$ is finite, and $\gamma_{I\infty}^* = \hat{\gamma}_{I\infty}$

(ii) For each $\gamma > \gamma_{I\infty}^*$, GARE-1 admits a unique minimal nnd solution, $\bar{Z}_\gamma^+$, and GARE-2 admits a unique minimal nnd solution, $\bar{\Sigma}_\gamma^+$, which further satisfy the spectral radius condition $\rho(\bar{\Sigma}_\gamma^+ \bar{Z}_\gamma^+) < \gamma^2$.

(iii) For each $\gamma > \gamma_{I\infty}^*$, there exists an n-dimensional controller that achieves the performance bound γ, which is given by

$$\mu_\gamma(y) = -B'\, \bar{Z}_\gamma^+\, \hat{x}(t)\,, \qquad \text{where}$$

$$\dot{\hat{x}} = \left[A - (BB' - \gamma^{-2}DD')\bar{Z}_\gamma^+\right]\hat{x} + [I - \gamma^{-2}\bar{\Sigma}_\gamma^+ \bar{Z}_\gamma^+]^{-1}\bar{\Sigma}_\gamma^+ C' N^{-1}(y - C\hat{x})$$

Equivalently,

$$L(\mu_\gamma, w) \le \gamma\,, \quad \forall\, w \in L_2(-\infty, \infty)\ \text{and}\ \forall\, v \in L_2(-\infty, \infty)$$

(iv) The controller above leads to a BIBS stable $2n$-dimensional system, that is the matrices $A - BB'\bar{Z}_\gamma^+$ and $A - \bar{\Sigma}_\gamma^+ C' N^{-1} C$ are Hurwitz.

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