ECE 553 **SPRING 2006**

Correspondence # 9 February 10, 2006

NOTES ON OPTIMAL CONTROL

AND CALCULUS OF VARIATIONS

From: *Optimum Systems Control*

by Andrew P. Sage & Chelsea C. White, III

Prentice Hall, 2nd edition, 1977

4 The maximum principle and Hamilton-Jacobi theory

In the previous chapter, we formulated many problems in the classical calculus of variations [1]. A derivation of the Euler-Lagrange equations for both the scalar and vector cases was presented. We discussed the associated transversality conditions and some of the difficulties which we may encounter if inequality constraints are present. Several simple optimal control problems were stated and solved. In this chapter we wish to reexamine many of the problems presented in the previous chapter and obtain more general solutions for some of them. In addition, we will develop methods for handling some problems which could not be conveniently formulated by the methods in the previous chapter.

To these ends, we will present the Bolza formulation of the variational calculus using Hamiltonian methods. This will lead us into a proof of the Pontryagin maximum principle and the associated transversality conditions [2–5]. We will proceed then to a development of the Hamilton-Jacobi equations [12–14], which are equivalent to Bellman's equations of continuous dynamic programming. Finally, we will give brief mention to some limitations of dynamic programming. Examples to illustrate the methods will be presented. We will reserve the next chapter for a discussion of some of the many problems which we can formulate and solve using the maximum principle.

In order to fully develop our approach to optimization theory where the terminal time is not fixed and where the control and state vectors are not necessarily smooth functions, we must consider in more detail the first variation for such problems.

4.1
The variational approach for functions with terminal times not fixed

We now extend the variational approach introduced in Sec. 3.6 to problems having unspecified terminal times. Consider extremizing

$$J = \int_{t_0}^{t_f} \Phi[\mathbf{x}(t), \dot{\mathbf{x}}(t), t]\, dt \tag{4.1-1}$$

with respect to the set of all admissible (see Sec. 2.1) trajectories. Let t_f be the terminal time associated with optimal trajectory $\mathbf{x}$. Associated with each perturbation $\mathbf{h}$ away from the optimal trajectory is a perturbation δt_f in the terminal time. Let the first variation δJ be the part of

$$\Delta J = J(\mathbf{x} + \mathbf{h}, t_f + \delta t_f) - J(\mathbf{x}, t_f) \tag{4.1-2}$$

which is linear in $\mathbf{h}$ and δt_f. Substituting Eq. (4.1-1) into Eq. (4.1-2), taking the linear terms of ΔJ (in $\mathbf{h}$, δt_f, and $\dot{\mathbf{h}}$), and performing the usual integration by parts to reduce terms dependent on $\dot{\mathbf{h}}$ to terms dependent on $\mathbf{h}$, we obtain

$$\delta J = \Phi[\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f]\, \delta t_f + \mathbf{h}^T(t_f) \frac{\partial \Phi[\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f]}{\partial \dot{\mathbf{x}}(t_f)} + \int_{t_0}^{t_f} \mathbf{h}^T(t) \left(\frac{\partial \Phi}{\partial \mathbf{x}} - \frac{d}{dt} \frac{\partial \Phi}{\partial \dot{\mathbf{x}}} \right) dt \tag{4.1-3}$$

where for convenience, we have assumed that the initial condition is fixed and hence $\mathbf{h}(t_0) = \mathbf{0}$.

In order to rephrase Eq. (4.1-3) into a convenient form, we introduce the following notation. We define

$$\delta \mathbf{x}(t_f) = \mathbf{h}(t_f) + \dot{\mathbf{x}}(t_f)\, \delta t_f. \tag{4.1-4}$$

From a Taylor series expansion of $\mathbf{x}(t_f + \delta t_f)$, we note that $\delta \mathbf{x}(t_f)$ is a close approximation to that part of $[\mathbf{x}(t_f + \delta t_f) + \mathbf{h}(t_f + \delta t_f)] - \mathbf{x}(t_f)$ which is linear in $\mathbf{h}(t_f)$ and δt_f. Submitting Eq. (4.1-4) into Eq. (4.1-3) and rearranging, the first variation becomes

$$\delta J = \left\{ \Phi[\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f] - \dot{\mathbf{x}}^T(t_f) \frac{\partial \Phi[\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f]}{\partial \dot{\mathbf{x}}(t_f)} \right\} \delta t_f + \delta \mathbf{x}^T(t_f) \frac{\partial \Phi[\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f]}{\partial \dot{\mathbf{x}}(t_f)} + \int_{t_0}^{t_f} \mathbf{h}^T(t_f) \left\{ \frac{\partial \Phi}{\partial \mathbf{x}} - \frac{d}{dt} \frac{\partial \Phi}{\partial \dot{\mathbf{x}}} \right\} dt. \tag{4.1-5}$$

In much of our work, it will be convenient to define a quantity, called the Hamiltonian, by

$$H[\mathbf{x}(t), \boldsymbol{\lambda}(t), t] = \Phi - \dot{\mathbf{x}}^T \frac{\partial \Phi}{\partial \dot{\mathbf{x}}} = \Phi + \dot{\mathbf{x}}^T \boldsymbol{\lambda} \tag{4.1-6}$$

where the Hamiltonian is not a function of $\dot{\mathbf{x}}$; $\mathbf{x}(t)$, and $\boldsymbol{\lambda}(t)$ are called the canonical variables. In terms of the Hamiltonian, the first variation of Eq. (4.1-1), which is Eq. (4.1-5) becomes

$$\delta J = -\delta \mathbf{x}^T(t_f) \boldsymbol{\lambda}(t_f) + H[\mathbf{x}(t_f), \boldsymbol{\lambda}(t_f), t_f]\, \delta t_f + \int_{t_0}^{t_f} \mathbf{h}^T(t) \left\{ \frac{\partial H}{\partial \mathbf{x}} + \frac{d\boldsymbol{\lambda}}{dt} \right\} dt. \tag{4.1-7}$$

To establish a necessary condition for a minimum, it is necessary that the integrand in Eqs. (4.1-5) and (4.1-7) vanish and also that the transversality condition, as obtained from Eq. (4.1-7),

$$-\delta \mathbf{x}^T(t_f) \boldsymbol{\lambda}(t_f) + H[\mathbf{x}(t_f), \boldsymbol{\lambda}(t_f), t_f]\, \delta t_f = 0 \tag{4.1-8}$$

be satisfied.

4.2
Weierstrass-Erdmann conditions

Thus far in our development, admissible trajectories have been constrained to be continuously differentiable with respect to x and t. This functional constraint on the class of all admissible trajectories is often unrealistically restrictive, as the following example will show. For this example, an optimal admissible solution does not exist; however, if the functional restriction on an admissible trajectory is sufficiently relaxed, existence of an optimal admissible trajectory is assured. We now examine the consequences of our new definition of an admissible trajectory which are the Weierstrass-Erdmann conditions [1].

Let us consider the problem of minimizing the cost function

$$J = \int_0^1 x^2(2 - \dot{x})^2\, dt$$

subject to

$$x(0) = 0, \qquad x(1) = 1.$$

Physically, it is clear that the absolute minimum for J is 0 and that this is obtained for

$$x(t) = 0, \qquad t \in [0, \tfrac{1}{2}]$$
$$x(t) = 2t - 1, \qquad t \in [\tfrac{1}{2}, 1]$$

which is certainly a solution to the Euler-Lagrange equation for this problem

$$x^2\ddot{x} + x\dot{x}^2 - 4x = 0.$$

There is one disturbing feature about this solution, however, in that the optimum $x(t)$ has a "corner" or discontinuous first derivative which gives rise to formal difficulty since $\ddot{x}$ is contained in the Euler-Lagrange equations. Thus, the solution of the above problem is not admissible. Certainly, this particular function $x(t)$ is continuously differentiable everywhere except at a finite number of points (in this case the single point $t = \frac{1}{2}$). Thus, in relaxing the set of admissible trajectories to allow for functions which are piecewise continuously differentiable, the function $x(t)$ is admissible and the above problem then has an optimal admissible control.

The Weierstrass-Erdmann corner conditions furnish us with necessary conditions for an optimal trajectory to have a discontinuous derivative at a point in the control interval of interest. Specifically, consider the problem of finding a trajectory among the class of all continuously differentiable functions on $[a, b]$ having a corner at $c \in (a, b)$ which satisfies fixed initial and final boundary values such that the functional

$$J(\mathbf{x}) = \int_a^b \Phi[\mathbf{x}(t), \dot{\mathbf{x}}(t), t]\, dt$$

has an extremum. It is of course clear that, for $t \in [a, c]$ and $t \in [c, b]$, the function $\mathbf{x}(t)$ must satisfy the Euler-Lagrange equations for a minimum

$$\frac{d}{dt}\frac{\partial \Phi}{\partial \dot{\mathbf{x}}} - \frac{\partial \Phi}{\partial \mathbf{x}} = 0.$$

We may rewrite the cost function as a sum of two cost functions:

$$J(\mathbf{x}) = \int_a^c \Phi[\mathbf{x}(t), \dot{\mathbf{x}}(t), t]\, dt + \int_c^b \Phi[\mathbf{x}(t), \dot{\mathbf{x}}(t), t]\, dt$$
$$= J_1(\mathbf{x}) + J_2(\mathbf{x}).$$

We may now take the first variation $\delta J_1(\mathbf{x})$ and $\delta J_2(\mathbf{x})$ separately. We assume, for the moment only, that a and b are fixed, and we require that the $\dot{\mathbf{x}}(t)$ calculated from $J_1(\mathbf{x})$ and $J_2(\mathbf{x})$ is the same at $t = c$ which is unknown. Since c is arbitrary, the first variation of $J_1(\mathbf{x})$ is

$$\delta J_1(\mathbf{x}) = -\delta \mathbf{x}^T(a)\frac{\partial \Phi[\mathbf{x}(a), \dot{\mathbf{x}}(a), a]}{\partial \dot{\mathbf{x}}(a)} +$$
$$\left\{\Phi[\mathbf{x}(c), \dot{\mathbf{x}}(c), c] - \dot{\mathbf{x}}^T(c)\frac{\partial \Phi[\mathbf{x}(c), \dot{\mathbf{x}}(c), c]}{\partial \dot{\mathbf{x}}(c)}\right\} \delta c + \delta \mathbf{x}^T(c)\frac{\partial \Phi[\mathbf{x}(c), \dot{\mathbf{x}}(c), c]}{\partial \dot{\mathbf{x}}(c)} +$$
$$\int_a^c \mathbf{h}^T(t)\left\{\frac{\partial \Phi}{\partial \mathbf{x}} - \frac{d}{dt}\frac{\partial \Phi}{\partial \dot{\mathbf{x}}}\right\} dt.$$

Since $\mathbf{x}(t)$ satisfies the Euler-Lagrange equations for an extremal and since $\delta\mathbf{x}(a) = 0$, we have

$$\delta J_1(\mathbf{x}) = \delta \mathbf{x}^T(\tau)\frac{\partial \Phi[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau), \tau]}{\partial \dot{\mathbf{x}}(\tau)} +$$
$$\left\{\Phi[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau), \tau] - \dot{\mathbf{x}}^T(\tau)\frac{\partial \Phi[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau), \tau]}{\partial \dot{\mathbf{x}}(\tau)}\right\} \delta\tau \qquad (\text{for } \tau = c - 0).$$

In a similar fashion, we can show that the first variation for the extremal solution of $J_2(\mathbf{x})$ is

$$\delta J_2(\mathbf{x}) = -\delta \mathbf{x}^T(\tau)\frac{\partial \Phi[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau), \tau]}{\partial \dot{\mathbf{x}}(\tau)} -$$
$$\left\{\Phi[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau), \tau] - \dot{\mathbf{x}}^T(\tau)\frac{\partial \Phi[\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau), \tau]}{\partial \dot{\mathbf{x}}(\tau)}\right\} \delta\tau \qquad (\text{for } \tau = c + 0).$$

In order to obtain the extremum, the extremal solution must satisfy

$$\delta J(\mathbf{x}) = \delta J_1(\mathbf{x}) + \delta J_2(\mathbf{x}) = 0$$

Thus

$$\left.\frac{\partial \Phi}{\partial \dot{\mathbf{x}}}\right|_{t=c-0} = \left.\frac{\partial \Phi}{\partial \dot{\mathbf{x}}}\right|_{t=c+0} \qquad (4.2\text{-}1)$$

$$\left.\Phi - \dot{\mathbf{x}}^T\frac{\partial \Phi}{\partial \dot{\mathbf{x}}}\right|_{t=c-0} = \left.\Phi - \dot{\mathbf{x}}^T\frac{\partial \Phi}{\partial \dot{\mathbf{x}}}\right|_{t=c+0} \qquad (4.2\text{-}2)$$

since $\delta\mathbf{x}$ and δt_f are arbitrary. These requirements, Eqs. (4.2-1) and (4.2-2), are called the *Weierstrass-Erdmann corner conditions* and must hold at any point c where the extremal has a corner. If we use the Hamiltonian canonical variables

$$H = \Phi - \dot{\mathbf{x}}^T\frac{\partial \Phi}{\partial \dot{\mathbf{x}}} = \Phi + \boldsymbol{\lambda}^T\dot{\mathbf{x}}$$

$$\boldsymbol{\lambda} = -\frac{\partial \Phi}{\partial \dot{\mathbf{x}}},$$

we immediately see that the Weierstrass-Erdmann conditions simply require H and $\boldsymbol{\lambda}$ to be continuous on the optimum trajectory at all points where there are corners.

4.3 The Bolza problem—no inequality constraints

In Sec. 3.7 we considered the solution of Lagrange problems with equality constraints of the form $\mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, t) = \mathbf{0}$ for all t in the control interval of interest. A special case of this equality constraint which is well-recognized as a model of a large and important class of physical systems is

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t], \qquad (4.3\text{-}1)$$

where the m-vector $\mathbf{u}$ represents the control function to be selected and the n-vector $\mathbf{x}$ represents the resulting trajectory. We will assume that $\mathbf{f}$ has continuous partial derivatives with respect to $\mathbf{x}$ and $\mathbf{u}$. Often it is the case that such smoothness assumptions guarantee that for any piecewise continuous function $\mathbf{u}$, there exists a unique, admissible trajectory $\mathbf{x}$ to Eq. (4.3-1). We therefore define the set of admissible control functions to be the class of piecewise continuous functions and assume that, for an admissible $\mathbf{u}$ and a given initial condition $\mathbf{x}(t_o)$, Eq. (4.3-1) defines a unique, admissible solution over the control interval of interest.

Throughout the remainder of this section and the following section, we will consider the development of necessary conditions for Bolza problems subject to the equality constraint in Eq. (4.3-1). Subsections (4.3-1) and (4.3-2) will consider the fixed final time and the unspecified final time cases when no restrictions are imposed on the value that $\mathbf{u}$ can take at each time t during the control interval of interest. Section 4.4 considers two cases of inequality constraints on the control function and its associated trajectory over the control interval.

4.3-1
Continuous optimal control problems—fixed beginning and terminal times—no inequality constraints

We now consider the problem of determining an admissible control function $\mathbf{u}$ in order to minimize the criterion

$$J = \theta[\mathbf{x}(t), t]\Big|_{t=t_o}^{t=t_f} + \int_{t_o}^{t_f} \phi[\mathbf{x}(t), \mathbf{u}(t), t]\, dt, \tag{4.3-2}$$

where θ and ϕ possess continuous partial derivatives in $\mathbf{x}$ and $\mathbf{u}$.

We use the method of Lagrange multipliers discussed in the last chapter to adjoin the system differential equality constraint to the cost function, which gives us

$$J = \theta[\mathbf{x}(t), t]\Big|_{t=t_o}^{t=t_f} + \int_{t_o}^{t_f} \{\phi[\mathbf{x}(t), \mathbf{u}(t), t] + \boldsymbol{\lambda}^T(t)[\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] - \dot{\mathbf{x}}]\}\, dt. \tag{4.3-3}$$

We define a scalar function, the Hamiltonian, as

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = \phi[\mathbf{x}(t), \mathbf{u}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]. \tag{4.3-4}$$

Thus the cost function becomes

$$J = \theta[\mathbf{x}(t), t]\Big|_{t=t_o}^{t=t_f} + \int_{t_o}^{t_f} \{H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}\}\, dt. \tag{4.3-5}$$

If we integrate the last term in the integrand of Eq. (4.3-5) by parts, we obtain

$$J = \{\theta[\mathbf{x}(t), t] - \boldsymbol{\lambda}^T(t)\mathbf{x}(t)\}\Big|_{t=t_o}^{t=t_f} + \int_{t_o}^{t_f} \{H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] + \dot{\boldsymbol{\lambda}}^T\mathbf{x}(t)\}\, dt. \tag{4.3-6}$$

We now take the first variation of J for variations in the control vector and, consequently, in the state vector about the optimal control and optimal

state vector. This gives us

$$\delta J = \left\{\delta\mathbf{x}^T\left[\frac{\partial\theta}{\partial\mathbf{x}} - \boldsymbol{\lambda}\right]\right\}\Big|_{t=t_o}^{t=t_f} + \int_{t_o}^{t_f} \left\{\delta\mathbf{x}^T\left[\frac{\partial H}{\partial\mathbf{x}} + \dot{\boldsymbol{\lambda}}\right] + \delta\mathbf{u}^T\left[\frac{\partial H}{\partial\mathbf{u}}\right]\right\} dt. \tag{4.3-7}$$

A necessary condition for a minimum is that the first variation in J vanish for arbitrary variations $\delta\mathbf{x}$ and $\delta\mathbf{u}$. Thus we have as the necessary condition for a minimum the very important relations

$$\delta\mathbf{x}^T\left[\frac{\partial\theta}{\partial\mathbf{x}} - \boldsymbol{\lambda}\right] = 0, \quad \text{for} \quad t = t_o, t_f \tag{4.3-8}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial\mathbf{x}}, \qquad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \frac{\partial H}{\partial\boldsymbol{\lambda}} \tag{4.3-9}$$

$$\frac{\partial H}{\partial\mathbf{u}} = \mathbf{0}. \tag{4.3-10}$$

We now consider in more detail the transversality conditions expressed in Eq. (4.3-8).

For a large class of optimal control problems, the initial state of the system is specified but the terminal state is unspecified. In that case, Eq. (4.3-8) yields the transversality conditions as

$$\mathbf{x}(t_o) = \mathbf{x}_o, \qquad \boldsymbol{\lambda}(t_f) = \frac{\partial\theta[\mathbf{x}(t_f), t_f]}{\partial\mathbf{x}(t_f)} \tag{4.3-11}$$

since $\delta\mathbf{x}(t_o) = \mathbf{0}$, $\mathbf{x}(t_o)$ is fixed, and $\delta\mathbf{x}(t_f)$ is completely arbitrary. In another broad class of problems $\mathbf{x}(t_o)$ and $\mathbf{x}(t_f)$ are fixed. In this case $\delta\mathbf{x}(t_o)$ and $\delta\mathbf{x}(t_f)$ must be zero, and $\mathbf{x}(t_o)$ and $\mathbf{x}(t_f)$ are the boundary conditions for the two-point boundary value problem. For many estimation problems, neither $\mathbf{x}(t_o)$ nor $\mathbf{x}(t_f)$ are fixed and $\theta = 0$. In that case, Eq. (4.3-8) yields $\boldsymbol{\lambda}(t_o) = \boldsymbol{\lambda}(t_f) = 0$ as the boundary conditions for the problem since $\delta\mathbf{x}(t_o)$ and $\delta\mathbf{x}(t_f)$ are arbitrary. In still another case, we might have $\mathbf{x}(t_o) = \mathbf{x}_o$, $\theta = 0$, and $\|\mathbf{x}(t_f)\|^2 = 1$. In this event, it is easy for us to show that the final transversality conditions are obtained if we solve the two scalar equations, each in n variables

$$\delta\mathbf{x}^T(t_f)\mathbf{x}(t_f) = 0, \qquad \delta\mathbf{x}^T(t_f)\boldsymbol{\lambda}(t_f) = 0. \tag{4.3-12}$$

We now give a more general and precise interpretation to the transversality conditions. For the general case where the initial manifold is

$$\mathbf{M}[\mathbf{x}(t_o), t_o] = 0 \tag{4.3-13}$$

and the terminal manifold is

$$\mathbf{N}[\mathbf{x}(t_f), t_f] = 0, \tag{4.3-14}$$

we adjoin these conditions to the θ function by means of Lagrange multipliers, $\boldsymbol{\xi}$ and $\boldsymbol{\nu}$ and obtain for the cost function

$$J = \theta[\mathbf{x}(t), t]\Big|_{t=t_o}^{t=t_f} - \boldsymbol{\xi}^T\mathbf{M}[\mathbf{x}(t_o), t_o] + \mathbf{v}^T\mathbf{N}[\mathbf{x}(t_f), t_f] +$$
$$\int_{t_o}^{t_f} \{H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}\}\, dt. \tag{4.3-15}$$

We now apply the usual variational techniques to obtain for the transversality conditions at the initial time:

$$\boldsymbol{\lambda}(t_o) = \frac{\partial\theta}{\partial\mathbf{x}} + \left(\frac{\partial\mathbf{M}^T}{\partial\mathbf{x}}\right)\boldsymbol{\xi}, \qquad \mathbf{M}[\mathbf{x}(t), t] = 0, \qquad t = t_o. \tag{4.3-16}$$

The n initial conditions are obtained from this, with r parameters to be found in Eq. (4.3-16) such that we satisfy the r conditions of Eq. (4.3-13). In a similar fashion, the terminal condition is

$$\boldsymbol{\lambda}(t_f) = \frac{\partial\theta}{\partial\mathbf{x}} + \left(\frac{\partial\mathbf{N}^T}{\partial\mathbf{x}}\right)\mathbf{v}, \qquad \mathbf{N}[\mathbf{x}(t), t] = 0, \qquad t = t_f; \tag{4.3-17}$$

n terminal conditions are obtained from this with q parameters $\mathbf{v}$ found in Eq. (4.3-17) such that the q conditions of Eq. (4.3-14) are satisfied.

The n vector differential equation obtained from Eq. (4.3-9) will be called the adjoint equation. Equation (4.3-10) provides the coupling relation between the original plant dynamics, Eq. (4.3-1), and the adjoint equation, the $\dot{\boldsymbol{\lambda}}$ equation of Eq. (4.3-9). This coupling equation was obtained from

$$\delta J = \ldots + \int_{t_o}^{t_f} \left\{\delta\mathbf{u}^T\frac{\partial H}{\partial\mathbf{u}} + \ldots\right\} dt,$$

and it is important to note that $\delta\mathbf{u}$ must be completely arbitrary in order for us to draw the conclusion that $\partial H/\partial\mathbf{u} = 0$ to obtain the optimal control. For the problem posed here where the admissible control set is infinite, $\delta\mathbf{u}$ can be completely arbitrary. Where the admissible control is bounded, $\delta\mathbf{u}$ cannot be completely arbitrary, and $\partial H/\partial\mathbf{u} = 0$ may not be the correct requirement. We will have more to say about this later. The solution we have obtained for this problem is a special case of the Pontryagin maximum principle.

It is also interesting to note that, since $H = \phi + \boldsymbol{\lambda}^T\mathbf{f}$, we may compute the total derivative with respect to time as

$$\frac{dH}{dt} = \frac{\partial\phi}{\partial t} + \dot{\mathbf{x}}^T\left[\frac{\partial\phi}{\partial\mathbf{x}} + \left(\frac{\partial\mathbf{f}^T}{\partial\mathbf{x}}\right)\boldsymbol{\lambda}\right] + \dot{\mathbf{u}}^T\left[\frac{\partial\phi}{\partial\mathbf{u}} + \left(\frac{\partial\mathbf{f}^T}{\partial\mathbf{u}}\right)\boldsymbol{\lambda}\right] + \dot{\boldsymbol{\lambda}}^T\mathbf{f} + \boldsymbol{\lambda}^T\frac{\partial\mathbf{f}}{\partial t} \tag{4.3-18}$$

but from Eqs. (4.3-9) and (4.3-4) we have

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial\mathbf{x}} = -\frac{\partial\phi}{\partial\mathbf{x}} - \left(\frac{\partial\mathbf{f}^T}{\partial\mathbf{x}}\right)\boldsymbol{\lambda} \tag{4.3-19}$$

and from Eq. (4.3-4)

$$\frac{\partial H}{\partial\mathbf{u}} = \frac{\partial\phi}{\partial\mathbf{u}} + \left(\frac{\partial\mathbf{f}^T}{\partial\mathbf{u}}\right)\boldsymbol{\lambda}. \tag{4.3-20}$$

Thus, since $\dot{\mathbf{x}}^T\dot{\boldsymbol{\lambda}} = \dot{\boldsymbol{\lambda}}^T\mathbf{f}$, Eq. (4.3-18) becomes

$$\frac{dH}{dt} = \frac{\partial\phi}{\partial t} + \boldsymbol{\lambda}^T\frac{\partial\mathbf{f}}{\partial t} + \dot{\mathbf{u}}^T\frac{\partial H}{\partial\mathbf{u}}. \tag{4.3-21}$$

We see that, if ϕ and $\mathbf{f}$ are not explicit functions of time, the Hamiltonian is constant along an optimal trajectory where $\partial H/\partial\mathbf{u} = \mathbf{0}$. It can be shown that this is always true along an optimal trajectory, even if we cannot require $\partial H/\partial\mathbf{u} = 0$. We will make use of this fact in a later development.

In order that J be a minimum, the second variation of J must be nonnegative along all trajectories such that Eq. (4.3-1) is satisfied. Therefore we need to compute the second variation of J in Eq. (4.3-6) and impose the requirement that the variation of Eq. (4.3-1) is zero, or that

$$\delta\dot{\mathbf{x}} - \left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\delta\mathbf{x} - \left(\frac{\partial\mathbf{f}}{\partial\mathbf{u}}\right)\delta\mathbf{u} = \mathbf{0}. \tag{4.3-22}$$

Applying this condition and taking the quadratic part of the Taylor series expansion of $J(\mathbf{x} + \delta\mathbf{x}, \mathbf{u} + \delta\mathbf{u}) - J(\mathbf{x}, \mathbf{u})$, we have for the second variation

$$\delta^2 J = \frac{1}{2}\left[\delta\mathbf{x}^T\frac{\partial^2\theta}{\partial\mathbf{x}^2}\delta\mathbf{x}\right]_{t=t_o}^{t=t_f} +$$
$$\frac{1}{2}\int_{t_o}^{t_f} [\delta\mathbf{x}^T\ \delta\mathbf{u}^T]\begin{bmatrix} \dfrac{\partial^2 H}{\partial\mathbf{x}^2} & \dfrac{\partial}{\partial\mathbf{u}}\dfrac{\partial H}{\partial\mathbf{x}} \\ \left[\dfrac{\partial}{\partial\mathbf{u}}\dfrac{\partial H}{\partial\mathbf{x}}\right]^T & \dfrac{\partial^2 H}{\partial\mathbf{u}^2} \end{bmatrix}\begin{bmatrix}\delta\mathbf{x} \\ \delta\mathbf{u}\end{bmatrix} dt \tag{4.3-23}$$

and this must be nonnegative for a minimum. This will be the case if the $n + m$ square matrix under the integral sign and $\partial^2\theta/\partial\mathbf{x}^2$ are nonnegative definite.

Example 4.3-1. We are given the differential system consisting of three cascaded integrators

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0$$
$$\dot{x}_2 = x_3 \qquad x_2(0) = 0$$
$$\dot{x}_3 = u \qquad x_3(0) = 0.$$

We wish to drive the system so that we reach the terminal manifold

$$x_1^2(1) + x_2^2(1) = 1$$

such that the cost function

$$J = \tfrac{1}{2}\int_0^1 u^2\, dt$$

is minimized. The solution to the problem proceeds as follows. We compute the Hamiltonian from Eq. (4.3-4) as

$$H = \tfrac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 u$$

and determine the coupling relation, Eq. (4.3-10),

$$\frac{\partial H}{\partial u} = 0 = u + \lambda_3$$

and the adjoint Eq. (4.3-9),

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial x_3} = -\lambda_2.$$

From Eqs. (4.3-14) and (4.3-17) we see that the transversality condition at the terminal time is

$$x_1^2(1) + x_2^2(1) = 1$$

$$\boldsymbol{\lambda}(1) = \frac{\partial \theta}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{N}^T}{\partial \mathbf{x}}\right)\mathbf{v}, \qquad t = t_f$$

where

$$N[\mathbf{x}(t_f), t_f] = x_1^2(t_f) + x_2^2(t_f) - 1 = 0, \qquad t_f = 1.$$

Thus

$$\boldsymbol{\lambda}(1) = \begin{bmatrix} \lambda_1(1) \\ \lambda_2(1) \\ \lambda_3(1) \end{bmatrix} = \begin{bmatrix} 2x_1(1)\nu \\ 2x_2(1)\nu \\ 0 \end{bmatrix}.$$

Thus the problem of finding the optimal control and associated trajectories for this example is completely resolved when we solve the two-point boundary value problem represented by

$$\begin{array}{ll} \dot{x}_1 = x_2 & x_1(0) = 0 \\ \dot{x}_2 = x_3 & x_2(0) = 0 \\ \dot{x}_3 = -\lambda_3 & x_3(0) = 0 \\ \dot{\lambda}_1 = 0 & \lambda_1(1) = 2x_1(1)\nu \\ \dot{\lambda}_2 = -\lambda_1 & \lambda_2(1) = 2x_2(1)\nu \\ \dot{\lambda}_3 = -\lambda_2 & \lambda_3(1) = 0 \end{array} \quad \left.\right\} x_1^2(1) + x_2^2(1) = 1$$

Although the six first-order differential equations represented above are perfectly linear and time invariant, the solution to this problem is complicated by the nonlinear nature of the terminal conditions. We shall discover various iterative schemes for overcoming this difficulty in Chapter 10.

4.3-2 *Continuous optimal control problems— fixed beginning and unspecified terminal times— no inequality constraints*

The material of the previous subsection may be easily extended to the case where the terminal manifold equation is a function of the terminal time,

and the terminal time is unspecified. For convenience we will assume that the initial time and the initial state vector are specified. Solution may then easily be obtained for the case where the initial time and initial state vector are unspecified. Therefore the problem becomes one of minimizing the cost function

$$J = \theta[\mathbf{x}(t_f), t_f] + \int_{t_o}^{t_f} \phi[\mathbf{x}(t), \mathbf{u}(t), t]\, dt \tag{4.3-24}$$

for the system described by

$$\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t], \qquad \mathbf{x}(t_o) = \mathbf{x}_o \tag{4.3-25}$$

where t_o is fixed and where, at the unspecified terminal time $t = t_f$, the q vector terminal manifold equation

$$\mathbf{N}[\mathbf{x}(t_f), t_f] = \mathbf{0} \tag{4.3-26}$$

is satisfied. It may be noted here that the terminal manifold line, $x(t_f) = c(t_f)$ of the previous chapter becomes $\mathbf{N}[\mathbf{x}(t_f), t_f] = \mathbf{0}$, which is more general. We adjoin the equality constraints to the cost function via Lagrange multipliers to obtain

$$J = \theta[\mathbf{x}(t_f), t_f] + \mathbf{v}^T\mathbf{N}[\mathbf{x}(t_f), t_f] + \int_{t_o}^{t_f} \{\phi[\mathbf{x}(t), \mathbf{u}(t), t] + \boldsymbol{\lambda}^T(t)\{\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] - \dot{\mathbf{x}}]\}\, dt. \tag{4.3-27}$$

As before, we define the Hamiltonian

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = \phi[\mathbf{x}(t), \mathbf{u}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]$$

and integrate a portion of the cost function, Eq. (4.3-27), to obtain

$$J = \theta[\mathbf{x}(t_f), t_f] + \mathbf{v}^T\mathbf{N}[\mathbf{x}(t_f), t_f] - \boldsymbol{\lambda}^T(t_f)\mathbf{x}(t_f) + \boldsymbol{\lambda}^T(t_o)\mathbf{x}(t_o) + \int_{t_o}^{t_f} \{H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] + \dot{\boldsymbol{\lambda}}^T\mathbf{x}(t)\}\, dt. \tag{4.3-28}$$

We again form the first variation by letting

$$\mathbf{x}(t) = \hat{\mathbf{x}}(t) + \mathbf{h}(t), \qquad \mathbf{u}(t) = \hat{\mathbf{u}}(t) + \delta\mathbf{u}(t), \qquad t_f = \hat{t}_f + \delta t_f \tag{4.3-29}$$

and then we form the difference $J[\mathbf{x}, \mathbf{u}, t_f] - J[\hat{\mathbf{x}}, \hat{\mathbf{u}}, \hat{t}_f]$ and retain only the linear terms. Thus we have, after dropping the ^ notation for convenience,

$$\delta J = \delta t_f\left\{H[\mathbf{x}(t_f), \mathbf{u}(t_f), \boldsymbol{\lambda}(t_f), t_f] + \frac{\partial \Theta}{\partial t_f}\right\} + \delta\mathbf{x}^T(t_f)\left\{\frac{\partial \Theta}{\partial \mathbf{x}} - \boldsymbol{\lambda}(t_f)\right\} + \int_{t_o}^{t_f} \left\{\mathbf{h}^T(t)\left[\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}\right] + \delta\mathbf{u}^T(t)\left[\frac{\partial H}{\partial \mathbf{u}}\right]\right\} dt \tag{4.3-30}$$

where

$$\Theta[\mathbf{x}(t_f), \mathbf{v}, t_f] = \theta[\mathbf{x}(t_f), t_f] + \mathbf{v}^T\mathbf{N}[\mathbf{x}(t_f), t_f]. \tag{4.3-31}$$

We must set this first variation equal to zero to obtain the necessary conditions for a minimum. Therefore, the equations which determine the optimal control and state vector are

$$H = \phi[\mathbf{x}(t), \mathbf{u}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] \tag{4.3-32}$$

$$\frac{\partial H}{\partial \boldsymbol{\lambda}} = \dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] \tag{4.3-33}$$

$$\frac{\partial H}{\partial \mathbf{x}} = -\dot{\boldsymbol{\lambda}} = \frac{\partial \mathbf{f}^T[\mathbf{x}(t), \mathbf{u}(t), t]}{\partial \mathbf{x}}\boldsymbol{\lambda}(t) + \frac{\partial \phi[\mathbf{x}(t), \mathbf{u}(t), t]}{\partial \mathbf{x}} \tag{4.3-34}$$

$$\frac{\partial H}{\partial \mathbf{u}} = 0 = \frac{\partial \phi[\mathbf{x}(t), \mathbf{u}(t), t]}{\partial \mathbf{u}} + \frac{\partial \mathbf{f}^T[\mathbf{x}(t), \mathbf{u}(t), t]}{\partial \mathbf{u}}\boldsymbol{\lambda}(t). \tag{4.3-35}$$

These represent the $2n$ differential equations for the two-point boundary value problems. The conditions at the initial time are

$$\mathbf{x}(t_o) = \mathbf{x}_o \tag{4.3-36}$$

whereas those at the final time are

$$\boldsymbol{\lambda}(t_f) = \frac{\partial \Theta}{\partial \mathbf{x}(t_f)} = \frac{\partial \theta}{\partial \mathbf{x}(t_f)} + \left[\frac{\partial \mathbf{N}^T}{\partial \mathbf{x}(t_f)}\right]\boldsymbol{\nu} \tag{4.3-37}$$

$$\mathbf{N}[\mathbf{x}(t_f), t_f] = \mathbf{0} \tag{4.3-38}$$

and

$$H[\mathbf{x}(t_f), \mathbf{u}(t_f), \boldsymbol{\lambda}(t_f), t_f] + \frac{\partial \theta}{\partial t_f} + \left(\frac{\partial \mathbf{N}^T}{\partial t_f}\right)\boldsymbol{\nu} = 0. \tag{4.3-39}$$

Equation (4.3-37) provides n conditions with q Lagrange multipliers to be determined. Equation (4.3-38) provides q equations to eliminate the Lagrange multipliers, and Eq. (4.3-39) provides the one additional equation which we must have to determine the unspecified terminal time.

Example 4.3-2. For the first-order single integration system

$$\dot{x} = u, \qquad x(0) = 1,$$

we desire to find the control $u(t)$ which makes $x(t_f) = 0$, where t_f is unspecified, such as to make, for specified values of α and β,

$$J = t_f^\alpha + \tfrac{1}{2}\beta \int_0^{t_f} u^2\, dt$$

a minimum. For this problem

$$N[x(t_f), t_f] = x(t_f) = 0, \qquad \phi = \tfrac{1}{2}\beta u^2$$

$$\theta = t_f^\alpha, \qquad H = \tfrac{1}{2}\beta u^2 + \lambda u.$$

The canonic equations are:

$$\dot{x} = u = -\frac{\lambda}{\beta}, \qquad \dot{\lambda} = 0$$

with the boundary conditions $x(0) = 1$, $x(t_f) = 0$, where we determine the final time by solving Eq. (4.3-39) which becomes, for this example,

$$-\frac{\lambda^2(t_f)}{2\beta} + \alpha t_f^{\alpha-1} = 0.$$

The solutions to the canonic equations are

$$x(t) = -\frac{\lambda(t_f)t}{\beta} + 1, \qquad \lambda(t) = \lambda(t_f).$$

But since $x(t_f) = 0$, $t_f = \beta\lambda^{-1}(t_f)$, and in the particular case where $\beta = \alpha = 1$, we can easily show from the foregoing that $\lambda(t_f) = +(2)^{1/2}$, which determines the solution to this example. The optimum control is $u(t) = -\lambda(t) = -2^{1/2}$; the corresponding trajectory is $x(t) = 1 - 2^{1/2}t$, with $t_f = 2^{-1/2}$.

Example 4.3-3. A problem which will be of considerable interest to us later will be the "minimum time" problem. In that case

$$\theta[\mathbf{x}(t_f), t_f] = t_f, \qquad \phi = 0,$$

and we specify the optimal control and corresponding trajectory by solving Eqs. (4.3-32) through (4.3-35), which become

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = \boldsymbol{\lambda}^T(t)\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]$$

$$\frac{\partial H}{\partial \boldsymbol{\lambda}} = \dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]$$

$$\frac{\partial H}{\partial \mathbf{x}} = -\dot{\boldsymbol{\lambda}} = \frac{\partial \mathbf{f}^T[\mathbf{x}(t), \mathbf{u}(t), t]}{\partial \mathbf{x}}\boldsymbol{\lambda}(t)$$

$$\frac{\partial H}{\partial \mathbf{u}} = 0 = \frac{\partial \mathbf{f}^T[\mathbf{x}(t), \mathbf{u}(t), t]}{\partial \mathbf{u}}\boldsymbol{\lambda}(t)$$

with the boundary conditions specified by Eqs. (4.3-36) through (4.3-39)

$$\mathbf{x}(t_o) = \mathbf{x}_o$$

$$\boldsymbol{\lambda}(t_f) = \frac{\partial \mathbf{N}^T}{\partial \mathbf{x}(t_f)}\nu$$

$$\mathbf{N}[\mathbf{x}(t_f), t_f] = \mathbf{0}$$

$$H[\mathbf{x}(t_f), \mathbf{u}(t_f), t_f] = -1 - \left(\frac{\partial \mathbf{N}^T}{\partial t_f}\right)\nu.$$

In many cases, the system is brought to rest at the unspecified time, and the terminal manifold is the origin, so that

$$\mathbf{N}[\mathbf{x}(t_f), t_f] = \mathbf{x}(t_f) = \mathbf{0}.$$

Then the foregoing expressions reduce to

$$\mathbf{x}(t_o) = \mathbf{x}_o, \qquad \mathbf{x}(t_f) = \mathbf{0}$$

$$H[\mathbf{x}(t_f), \mathbf{u}(t_f), \boldsymbol{\lambda}(t_f), t_f] = -1.$$

If the Hamiltonian is not an explicit function of time, Eq. (4.3-21) which applies here as well, yields $dH/dt = 0$; therefore, for this minimum time problem

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)] = -1.$$

It should be emphasized that we are not solving the usual minimum time problem since we have imposed no inequality constraints on the control (or state) variables. An alternate version of this problem would be to consider $\theta = 0$ and $\phi = 1$. This changes the Hamiltonian for this particular problem, but it certainly does not change the optimal control and state vector, as the reader can easily verify.

4.4
The Bolza problem with inequality constraints

The last section treated the Bolza problem with differential equation equality constraint, Eq. (4.3-1). There were no inequality constraints on either the control function or the state trajectory. We now consider the Bolza problem subject to one or more inequality constraints on the control and/or state variable.

4.4-1
The maximum principle with control variable inequality constraints

We now consider the determination of an admissible control function which minimizes the criterion function of Eq. (4.3-2) subject to both the equality constraint, Eq. (4.3-1), and the control variable constraint,

$$\mathbf{u}(t) \in \mathbf{\mho},\ t \in [t_o, t_f], \tag{4.4-1}$$

where $\mathbf{\mho}$ is a given subset of R^m. We assume also that the initial state $\mathbf{x}(t_o) = \mathbf{x}_o$ and the initial time t_o are fixed and that the terminal time t_f is defined by the vector terminal manifold equation

$$\mathbf{N}[\mathbf{x}(t_f), t_f] = \mathbf{0}, \tag{4.4-2}$$

where $\mathbf{N}: R^{n+1} \rightarrow R^q$.

Equation (4.4-1) distinguishes the problem posed above from the problems we considered in Sec. 4.3. Such a restricting assumption has important modeling significance since the controls that can be applied to many physical systems must be constrained in magnitude or in the number of feasible control settings. For example, the thrust of a rocket has a maximum and a minimum value; also, there may be a finite set of possible thrust values. We now present necessary conditions for this important problem, leading to the celebrated maximum principle due to McShane [16], Pontryagin [2], and others. Proof of the result for a simplified case will then follow.

We assume $\mathbf{u}$, having corresponding trajectory $\mathbf{x}$, is optimal on $[t_o, t_f]$. Then, recalling the definition of the Hamiltonian from Eq. (4.3-4),

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] \le H[\mathbf{x}(t), \mathbf{v}, \boldsymbol{\lambda}(t), t], \quad \mathbf{v} \in \mathbf{\mho} \tag{4.4-3}$$

$$\frac{\partial H}{\partial \boldsymbol{\lambda}} = \dot{\mathbf{x}} \tag{4.4-4}$$

$$\frac{\partial H}{\partial \mathbf{x}} = -\dot{\boldsymbol{\lambda}} \tag{4.4-5}$$

subject to the two-point boundary conditions

$$\mathbf{x}(t_o) = \mathbf{x}_o$$

$$\mathbf{N}[\mathbf{x}(t_f), t_f] = \mathbf{0}$$

$$\frac{\partial \theta}{\partial t_f} + \left(\frac{\partial \mathbf{N}^T}{\partial t_f}\right)\mathbf{v} + H = 0, \quad \text{at} \quad t = t_f \tag{4.4-6}$$

$$\frac{\partial \theta}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{N}^T}{\partial \mathbf{x}}\right)\mathbf{v} - \boldsymbol{\lambda} = \mathbf{0}, \quad \text{at} \quad t = t_f. \tag{4.4-7}$$

We frequently wish to transfer the system to the origin in minimum time so that we have

$$\mathbf{N}[\mathbf{x}(t_f), t_f] = \mathbf{0} = \mathbf{x}(t_f)$$

$$\theta[\mathbf{x}(t_f), t_f] = t_f$$

$$\phi = 0.$$

In this particular case, the transversality conditions become

$$\mathbf{x}(t_o) = \mathbf{x}_o$$

$$\mathbf{x}(t_f) = \mathbf{0}$$

$$H = -1, \quad \text{at} \quad t = t_f.$$

Example 4.4-1. Let us consider briefly the time optimal control problem for a linear time-invariant system where the length of the control vector is constrained. We wish to minimize

$$J = t_f$$

for the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{x}(t_o) = \mathbf{x}_o$$

where $\mathbf{u}(t) \in \mathbf{\mho}$ means $\|\mathbf{u}(t)\| \le 1$.

The Hamiltonian becomes

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = \boldsymbol{\lambda}^T(t)[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)].$$

To make H as small as possible with respect to a choice of $\mathbf{u}(t)$, we must have

$$\mathbf{u}(t) = \frac{-\mathbf{B}^T\boldsymbol{\lambda}(t)}{\|\mathbf{B}^T\boldsymbol{\lambda}(t)\|}.$$

The canonic equations become

$$\frac{\partial H}{\partial \boldsymbol{\lambda}} = \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \frac{\partial H}{\partial \mathbf{x}} = -\dot{\boldsymbol{\lambda}} = \mathbf{A}^T\boldsymbol{\lambda}(t)$$

with the boundary conditions

$$\mathbf{x}(t_o) = \mathbf{x}_o, \qquad \mathbf{x}(t_f) = 0$$

where we determine t_f by solving

$$H[\mathbf{x}(t_f), \boldsymbol{\lambda}(t_f), \mathbf{u}(t_f)] = -1.$$

But, from Eq. (4.3-21) we see that $dH/dt = 0$ since the Hamiltonian does not depend explicitly on t. Thus the above equation becomes

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)] = -1 = \boldsymbol{\lambda}^T(t)[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)]$$

which is the additional relation needed to determine the terminal time.

In a manner similar to [21], we now sketch a proof of the above conditions for optimality under the assumptions that: t_f is fixed, $\mathbf{x}(t_f)$ is unspecified, $\theta = 0$, the problem is time-invariant, and the function $\mathbf{f}$ satisfies the (uniform Lipschitz) condition

$$\|\mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{y}, \mathbf{v})\| \leq M(\|\mathbf{x} - \mathbf{y}\| + \|\mathbf{u} - \mathbf{v}\|), \tag{4.4-8}$$

where M is a known constant and where the above norms are finite-dimensional. The proof requires preliminary establishment of the Lipschitz condition

$$\|\mathbf{x} - \mathbf{y}\|_X \leq k\|\mathbf{u} - \mathbf{v}\|_U, \tag{4.4-9}$$

where $\mathbf{y}$ is the unique solution of Eq. (4.3-1) for given control function $\mathbf{v}$,

$$\|\mathbf{x}\|_X = \max_{t_o \leq t \leq t_f} \|\mathbf{x}(t)\|, \tag{4.4-10}$$

and

$$\|\mathbf{u}\|_U = \int_{t_o}^{t_f} \|\mathbf{u}(s)\|\, ds. \tag{4.4-11}$$

This is shown as follows. We note that

$$\delta\dot{\mathbf{x}}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}[\mathbf{x}(t), \mathbf{u}(t)]\, \delta\mathbf{x}(t) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}[\mathbf{x}(t), \mathbf{u}(t)]\, \delta\mathbf{u}(t) \tag{4.4-12}$$

where $\delta\mathbf{x}(t_o) = 0$ since $\mathbf{x}(t_o)$ is fixed. Thus, there is a unique fundamental matrix $\boldsymbol{\Phi}(t, s)$ such that

$$\delta\mathbf{x}(t) = \int_{t_o}^{t} \boldsymbol{\Phi}(t, s) \frac{\partial \mathbf{f}}{\partial \mathbf{u}}[\mathbf{x}(s), \mathbf{u}(s)]\, \delta\mathbf{u}(s)\, ds \tag{4.4-13}$$

which can easily be shown to imply

$$\|\delta\mathbf{x}(t)\| \leq M e^{M(t_f - t_o)} \int_{t_o}^{t_f} \|\delta\mathbf{u}(s)\|\, ds = K\|\delta\mathbf{u}\|_U$$

for all $t \in [t_o, t_f]$, and the preliminary result follows directly.

We now show that for any admissible control function $\mathbf{v}$,

$$J(\mathbf{u}) - J(\mathbf{v}) = \int_{t_o}^{t_f} [H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) - H(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda})]\, dt + \text{H.O.T.}(\|\mathbf{u} - \mathbf{v}\|_U). \tag{4.4-14}$$

To prove this result, assume $\mathbf{y}$ is the trajectory associated with $\mathbf{v}$. Then,

$$\phi(\mathbf{x}, \mathbf{u}) - \phi(\mathbf{y}, \mathbf{v}) = \phi(\mathbf{x}, \mathbf{u}) - \phi(\mathbf{x}, \mathbf{v}) + \phi(\mathbf{x}, \mathbf{v}) - \phi(\mathbf{y}, \mathbf{v}) =$$

$$\phi(\mathbf{x}, \mathbf{u}) - \phi(\mathbf{x}, \mathbf{v}) + \left[\frac{\partial \phi(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}}\right]^T (\mathbf{x} - \mathbf{y}) + \left[\frac{\partial \phi(\mathbf{y}, \mathbf{v})}{\partial \mathbf{x}} - \frac{\partial \phi(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}}\right]^T (\mathbf{x} - \mathbf{y}) +$$

$$\text{H.O.T.}(\|\mathbf{x} - \mathbf{y}\|_X) = \phi(\mathbf{x}, \mathbf{u}) - \phi(\mathbf{x}, \mathbf{v}) +$$

$$\left[\frac{\partial \phi(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}}\right]^T (\mathbf{x} - \mathbf{y}) + \text{H.O.T.}(\|\mathbf{u} - \mathbf{v}\|_U).$$

Similarly,

$$\dot{\mathbf{x}} - \dot{\mathbf{y}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x}, \mathbf{v}) + \left[\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}}\right]^T (\mathbf{x} - \mathbf{y}) + \text{H.O.T.}(\|\mathbf{u} - \mathbf{v}\|_U).$$

Thus,

$$J(\mathbf{u}) - J(\mathbf{v}) = \int_{t_o}^{t_f} [H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) - H(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda})]\, dt + \int_{t_o}^{t_f} \left[\frac{\partial H(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}}\right]^T (\mathbf{x} - \mathbf{y})\, dt -$$

$$\int_{t_o}^{t_f} \boldsymbol{\lambda}^T(\dot{\mathbf{x}}, \dot{\mathbf{y}})\, dt + \text{H.O.T.}(\|\mathbf{u} - \mathbf{v}\|_U). \tag{4.4-15}$$

Integration by parts and use of Eq. (4.4-5) indicate that the last two integrals on the right-hand side of Eq. (4.4-15) equal

$$-\int_{t_o}^{t_f} \dot{\boldsymbol{\lambda}}^T(\mathbf{x} - \mathbf{y})\, dt + \int_{t_o}^{t_f} \dot{\boldsymbol{\lambda}}^T(\mathbf{x} - \mathbf{y})\, dt + \boldsymbol{\lambda}^T(\mathbf{x} - \mathbf{y})\Big|_{t_o}^{t_f} = 0,$$

and Eq. (4.4-14) is validated.

We now proceed to prove Eq. (4.4-3) by contradiction. Suppose there is a $\bar{t} \in [t_o, t_f]$ and a $\mathbf{w} \in \mathbf{U}$ such that $H[\mathbf{x}(\bar{t}), \mathbf{u}(\bar{t}), \boldsymbol{\lambda}(\bar{t})] > H[\mathbf{x}(\bar{t}), \mathbf{w}, \boldsymbol{\lambda}(\bar{t})]$. The piecewise continuity of $\mathbf{u}$ and the continuity of $\mathbf{x}$, $\boldsymbol{\lambda}$, $\mathbf{f}$, and ϕ imply the existence of an interval $[t_a, t_b] \in [t_o, t_f]$, $\bar{t} \in [t_a, t_b]$, and an $\epsilon > 0$ such that for all $t \in [t_a, t_b]$, $H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)] - H[\mathbf{x}(t), \mathbf{w}, \boldsymbol{\lambda}(t)] > \epsilon$. We choose $\mathbf{v}$ so that $\mathbf{v}(t) = \mathbf{u}(t)$ for $t \notin [t_a, t_b]$, and $\mathbf{v}(t) = \mathbf{w}$ for $t \in [t_a, t_b]$. Then,

$$J(\mathbf{u}) - J(\mathbf{v}) > \epsilon\,(t_b - t_a) + \text{H.O.T.}(\|\mathbf{u} - \mathbf{v}\|_U).$$

We note, however, that $\|\mathbf{u} - \mathbf{v}\|_U = (t_b - t_a)[\text{H.O.T.}(t_b - t_a)]$. Thus, selection of $t_b - t_a$ small enough implies $J(\mathbf{u}) - J(\mathbf{v})$ can be made positive, which contradicts the optimality of $\mathbf{u}$; hence, Eq. (4.4-3) is proved. Considerably more exotic derivations of the maximum principle may be presented. For proofs of some of the more general forms of the maximum principle, see [17], [2], [6], and the references contained therein.

Often, Eq. (4.4-1) can conveniently be described by a set of inequalities of the form

$$\mathbf{g}[\mathbf{u}(t), t] \geq 0, \tag{4.4-16}$$

where $\mathbf{g}: R^{m+1} \longrightarrow R^r$. We may convert this inequality constraint to an equality constraint by writing for each component of $\mathbf{g}$ either

$$(\dot{z}_i)^2 = g_i[\mathbf{u}(t), t], \qquad z_i(t_0) = 0, \qquad i = 1, 2, \ldots, r \tag{4.4-17}$$

or

$$(y_i)^2 = g_i[\mathbf{u}(t), t], \qquad i = 1, 2, \ldots, r. \tag{4.4-18}$$

It is apparent that either of these two equations force g_i to be greater than or equal to zero since $(\dot{z}_i)^2$ and $(y_i)^2$ must certainly be greater than or equal to zero. This technique was apparently first proposed by Valentine [8] and extended by Berkovitz [6]. It is quite similar to the penalty function technique of Kelly [9] as we shall see in our section concerning the gradient and second variation methods for the computation of optimal controls. The choice between Eqs. (4.4-17) and (4.4-18) will depend largely upon the particular computer [for an analog computer, Eq. (4.4-17) is generally easier to implement than Eq. (4.4-18)] and the particular computational algorithms used (for the quasilinearization method, Eq. (4.4-18) is considerably simpler to use than Eq. (4.4-17) and also results in less computer solution time).

Example 4.4-2. It is quite easy to see that the constraint used here includes, as a special case, that considered in Sec. 3.8. For example, if we require for a scalar control u, $u_{min} \leq u \leq u_{max}$, then we may write

$$g_1[u(t), t] = u_{max} - u \geq 0, \qquad g_2[u(t), t] = u - u_{min} \geq 0,$$

and we convert these inequality constraints to equality constraints by writing

$$(y_1)^2 = u_{max} - u, \qquad (y_2)^2 = u - u_{min}$$

for which

$$(y_1 y_2)^2 = (u_{max} - u)(u - u_{min})$$

which is precisely the constraint used in Sec. 3.8.

For the problem at hand we adjoin, via the Lagrange multiplier, constraints (4.3-1), (4.4-2), and (4.4-16) to Eq. (4.3-2) to obtain

$$J = \theta[\mathbf{x}(t_f), t_f] + \mathbf{v}^T\mathbf{N}[\mathbf{x}(t_f), t_f] + \int_{t_0}^{t_f} \Big\{H[\mathbf{x}(t), \dot{\mathbf{w}}(t), \boldsymbol{\lambda}(t), t] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}} - \boldsymbol{\Gamma}^T(t)\{\mathbf{g}[\dot{\mathbf{w}}(t), t] - \dot{\mathbf{z}}^2\}\Big\}\, dt$$

where

$$(\mathbf{z}^2)^T = [z_1^2, z_2^2, z_3^2, \ldots, z_r^2]$$

$$H[\mathbf{x}(t), \dot{\mathbf{w}}(t), \boldsymbol{\lambda}(t), t] = \phi[\mathbf{x}(t), \dot{\mathbf{w}}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{f}[\mathbf{x}(t), \dot{\mathbf{w}}(t), t]$$

$$\dot{\mathbf{w}} = \mathbf{u}(t), \qquad \mathbf{w}(t_0) = \mathbf{0}.$$

We may now apply the Euler-Lagrange equations to the above cost function or take a first variation of it in order to obtain the necessary conditions for a minimum. It is thus convenient to define a scalar function Φ, the Lagrangian, as

$$\Phi[\mathbf{x}(t), \dot{\mathbf{x}}(t), \dot{\mathbf{w}}(t), \boldsymbol{\lambda}(t), \boldsymbol{\Gamma}(t), \dot{\mathbf{z}}(t), t] = H[\mathbf{x}(t), \dot{\mathbf{w}}(t), \boldsymbol{\lambda}(t), t] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}} - \boldsymbol{\Gamma}^T(t)\{\mathbf{g}[\dot{\mathbf{w}}(t), t] - \dot{\mathbf{z}}^2\} \tag{4.4-19}$$

We will use the Euler-Lagrange Eqs. (3.5-3). Since there are no $\mathbf{w}(t)$ and $\mathbf{z}(t)$ terms in Eq. (4.4-19) we may write the Euler-Lagrange equations as

$$\frac{d}{dt}\frac{\partial \Phi}{\partial \dot{\mathbf{x}}} - \frac{\partial \Phi}{\partial \mathbf{x}} = 0 \tag{4.4-20}$$

$$\frac{d}{dt}\frac{\partial \Phi}{\partial \dot{\mathbf{w}}} = 0 \tag{4.4-21}$$

$$\frac{d}{dt}\frac{\partial \Phi}{\partial \dot{\mathbf{z}}} = 0. \tag{4.4-22}$$

Each piecewise continuously differentiable solution of the Euler-Lagrange equations (4.4-20), (4.4-21), and (4.4-22) will be called an extremal curve or an extremal trajectory of the associated variational problem. It can be shown that the function Φ need be only piecewise smooth, and thus the Euler-Lagrange equations require that every arc of the extremal trajectory on which the first derivatives of Φ have no discontinuities be a solution of the Euler-Lagrange equations. The corner condition will answer our questions concerning what happens at possible points of discontinuity of some of the derivatives of the state or control variables. This corner condition will ensure continuity of the state and control variables by forcing $\partial\Phi/\partial\dot{\mathbf{z}}$ to be zero everywhere since it is zero at the terminal time.

The transversality conditions for this problem are obtained in the usual fashion as explained in Chapter 3 and the previous three sections. For this problem, they are easily shown to be Eq. (4.4-2), the initial conditions, and

$$\frac{\partial \theta}{\partial t_f} + \left(\frac{\partial \mathbf{N}^T}{\partial t_f}\right)\mathbf{v} + \phi - \dot{\mathbf{x}}^T\frac{\partial \Phi}{\partial \dot{\mathbf{x}}} = 0, \qquad \text{for} \quad t = t_f$$

$$\frac{\partial \theta}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{N}^T}{\partial \mathbf{x}}\right)\mathbf{v} - \boldsymbol{\lambda} = \mathbf{0}, \qquad \text{for} \quad t = t_f.$$

Also, we have for the final transversality condition

$$\delta\mathbf{z}^T(t_f)\left[\frac{\partial \Phi}{\partial \dot{\mathbf{z}}}\right] = \delta\mathbf{z}^T(t_f)\begin{bmatrix} 2\Gamma_1\dot{z}_1 \\ 2\Gamma_2\dot{z}_2 \\ \cdot \\ \cdot \\ 2\Gamma_r\dot{z}_r \end{bmatrix} = 0, \qquad \text{for} \quad t = t_f$$

which allows us to write, because of Eq. (4.4-21),

$$\frac{\partial \Phi}{\partial \dot{\mathbf{z}}} = 0, \qquad \forall t \in [t_o, t_f].$$

Since when $\Gamma_i \neq 0$, $\dot{z}_i = 0 = g_i$, and when $z_i \neq 0, \Gamma_i = 0$,

$$\Gamma_i z_i = 0, \qquad i = 1, 2, \ldots, r, \qquad \forall t \in [t_o, t_f].$$

Also, with similar reasoning, we have

$$\frac{\partial \Phi}{\partial \dot{\mathbf{w}}} = 0, \qquad \forall t \in [t_o, t_f].$$

4.4-2 *The maximum principle with state (and control) variable inequality constraints*

We now wish to extend the work of Sec. 4.4-1 to include inequality constraints on some or all of the state variables. We will represent this inequality constraint by the s vector equation

$$\mathbf{h}[\mathbf{x}(t), t] \geq 0 \qquad (4.4\text{-}23)$$

where each component of $\mathbf{h}$ is assumed to be continuously differentiable in state space. There are several methods whereby we may convert Eq. (4.4-23) to an equality constraint. We may define a new variable x_{n+1} by

$$\dot{x}_{n+1} = f_{n+1} = [h_1(\mathbf{x}, t)]^2 \mathrm{H}(h_1) + [h_2(\mathbf{x}, t)]^2 \mathrm{H}(h_2) + \cdots + [h_s(\mathbf{x}, t)]^2 \mathrm{H}(h_s) \qquad (4.4\text{-}24)$$

where $\mathrm{H}[h_s(\mathbf{x}, t)]$ is a modified Heaviside step defined such that

$$\mathrm{H}[h_s(\mathbf{x}, t)] = \begin{cases} 0, & \text{if } h_s(\mathbf{x}, t) \geq 0 \\ K_s, & \text{if } h_s(\mathbf{x}, t) < 0 \end{cases}$$

$$K_s > 0, \qquad s = 1, 2, \ldots, s \qquad (4.4\text{-}25)$$

and where the initial condition is

$$x_{n+1}(t_o) = 0.$$

Thus we see that $x_{n+1}(t_f)$ is a direct measure of penetration of the state variable inequality constraint

$$x_{n+1}(t_f) = \int_{t_o}^{t_f} \dot{x}_{n+1}(t)\, dt = \int_{t_o}^{t_f} \{[h_1(\mathbf{x}, t)]^2 \mathrm{H}(h_1) + \cdots + [h_s(\mathbf{x}, t)]^2 \mathrm{H}(h_s)\}\, dt.$$

We will require that the final value of $x_{n+1}(t_f)$ is zero,

$$x_{n+1}(t_f) = 0,$$

which will impose the restriction that we do not violate the inequality constraint. This approach is a modification by McGill [10] of a similar procedure by Kelley [9] which converts the s inequality constraints to s equality constraints of the form

$$\begin{aligned} \dot{x}_{n+1} &= [h_1(\mathbf{x}, t)]^2 \mathrm{H}(h_1), \qquad & x_{n+1}(t_o) &= 0 \\ \dot{x}_{n+2} &= [h_2(\mathbf{x}, t)]^2 \mathrm{H}(h_2), \qquad & x_{n+2}(t_o) &= 0 \\ &\vdots & &\vdots \\ \dot{x}_{n+s} &= [h_s(\mathbf{x}, t)]^2 \mathrm{H}(h_s), \qquad & x_{n+s}(t_o) &= 0 \end{aligned}$$

which are then added to the cost function to obtain

$$J_{\text{modified}} = J_{\text{original}} + \sum_{j=1}^{s} x_{n+j}(t_f).$$

The multipliers K_s are thus the penalty functions, and J_{modified} is minimized such that the constraint region is entered only slightly, if at all. If we require $x_{n+j}(t_f) = 0$ for $j = 1, 2, \ldots, s$, the constraint is, of course, not exceeded at all.

A slight modification of the penalty-function approach can be obtained if we define s new state variables

$$\begin{aligned} (\dot{x}_{n+1})^2 &= K_1 h_1(\mathbf{x}, t), \qquad & x_{n+1}(t_o) &= 0 \\ (\dot{x}_{n+2})^2 &= K_2 h_2(\mathbf{x}, t), \qquad & x_{n+2}(t_o) &= 0 \\ &\vdots & &\vdots \\ (\dot{x}_{n+s})^2 &= K_s h_s(\mathbf{x}, t), \qquad & x_{n+s}(t_o) &= 0. \end{aligned}$$

Berkovitz [7] suggests yet another method for converting the inequality constraint to an equality constraint. For the case of a scalar constraint, a variable

$$\gamma(\mathbf{x}, \eta, t) = \begin{cases} \eta^4 - h(\mathbf{x}, t) & \text{if } \eta > 0 \\ h(\mathbf{x}, t) & \text{if } \eta < 0 \end{cases}$$

is introduced, and we convert the inequality constraint $h(\mathbf{x}, t) \geq 0$ to an equality constraint by writing

$$\frac{\partial \gamma}{\partial \eta}\frac{d\eta}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt}$$

which satisfies the constraint if we have the end conditions

$$\gamma[\mathbf{x}(t_o), \eta(t_o), t_o] = \gamma[\mathbf{x}(t_f), \eta(t_f), t_f] = 0.$$

The Euler-Lagrange equations can, of course, be used to determine the differential equations for an extremum, and the associated transversality conditions can be used to specify the two-point boundary values. If inequality constraints on the control variables are present, we must of necessity incorporate these into our problem formulation. The Hamiltonian formulation may also be used. These methods provide us with necessary conditions only.

From Eq. (4.4-19) it follows that the Lagrangian for the problem at hand is

$$\tilde{\Phi} = \Phi + \lambda_{n+1}[f_{n+1} - \dot{x}_{n+1}]$$
$$\tilde{\Phi} = H - \boldsymbol{\lambda}^T\dot{\mathbf{x}} - \Gamma^T[\mathbf{g} - \dot{\mathbf{z}}^2] + \lambda_{n+1}[f_{n+1} - \dot{x}_{n+1}] \tag{4.4-26}$$

where Φ is the Lagrangian for no inequality state constraint. We are using the equality constraint method of Eqs. (4.4-24) and (4.4-25). The Euler-Lagrange equations yield,

$$\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\mathbf{x}}} - \frac{\partial\Phi}{\partial\mathbf{x}} - \frac{\partial f_{n+1}}{\partial\mathbf{x}}\lambda_{n+1} = 0 \tag{4.4-27}$$
$$\frac{\partial\Phi}{\partial\mathbf{u}} = \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\mathbf{w}}} = 0$$
$$\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\mathbf{z}}} = 0$$

which are, except for the f_{n+1} term, exactly the same as Eqs. (4.4-20), (4.4-21), and (4.4-22). Also, we see that

$$\frac{d}{dt}\lambda_{n+1}(t) = 0$$

with the transversality conditions exactly as before and, in addition,

$$x_{n+1}(t_o) = x_{n+1}(t_f) = 0.$$

It is desirable to reinterpret these results in terms of the Hamiltonian, just as we have done for the case of control variable constraints only. We can do this easily by combining Eqs. (4.4-26) and (4.4-27) which yields

$$\dot{\boldsymbol{\lambda}} = \frac{d\boldsymbol{\lambda}(t)}{dt} = -\frac{\partial H}{\partial\mathbf{x}} - \frac{\partial f_{n+1}[\mathbf{x}(t), t]}{\partial\mathbf{x}}\lambda_{n+1}$$
$$\dot{\mathbf{x}} = \frac{d\mathbf{x}(t)}{dt} = \frac{\partial H}{\partial\boldsymbol{\lambda}}$$
$$\dot{x}_{n+1} = \frac{dx_{n+1}(t)}{dt} = f_{n+1} = [h_1(\mathbf{x}, t)]^2\mathrm{H}(h_1) + \cdots + [h_s(\mathbf{x}, t)]^2\mathrm{H}(h_s)$$
$$\dot{\lambda}_{n+1} = \frac{d\lambda_{n+1}(t)}{dt} = 0$$

where

$$H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] = \phi[\mathbf{x}(t), \mathbf{u}(t), t] + \boldsymbol{\lambda}^T(t)\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]$$
$$H[\mathbf{x}(t), \hat{\mathbf{u}}(t), \boldsymbol{\lambda}(t), t] \leq H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t], \quad \mathbf{u} \in \mathbf{U}$$

with the two-point boundary conditions (transversality conditions)

$$\mathbf{x}(t_o) = \mathbf{x}_o$$
$$\mathbf{N}[\mathbf{x}(t_f), t_f] = 0$$
$$\left.\begin{aligned}\frac{\partial\theta}{\partial t_f} + \left(\frac{\partial\mathbf{N}^T}{\partial t_f}\right)\boldsymbol{\nu} + H = 0\\ \frac{\partial\theta}{\partial\mathbf{x}} + \left(\frac{\partial N^T}{\partial\mathbf{x}}\right)\boldsymbol{\nu} - \boldsymbol{\lambda} = 0\end{aligned}\right\} \quad \text{at} \quad t = t_f$$
$$x_{n+1}(t_o) = x_{n+1}(t_f) = 0.$$

Further discussion of the state-space constraint problem can be found in [18].

Example 4.4-3. As an example of optimization with a state variable constraint, we consider the brachistochrone problem previously treated by McGill [10] and Dreyfus [11]. A particle is falling for a specified time, $t_f - t_o$, under the influence of a constant gravitational acceleration g. The particle has initial velocity $x_3(t_o) = x_{3o}$. We wish to find the path that maximizes the final value of the horizontal coordinate $x_2(t_f)$. The final value of the vertical coordinate $x_2(t_f)$ and the velocity $x_3(t_f)$ are unspecified. The path is constrained by a line $h[x_1, x_2] \geq 0$ in the x_1x_2 plane, where it is known that the unconstrained solution intersects the line. The system dynamics are described by

$$\dot{x}_1 = x_3 \cos u, \qquad x_1(t_o) = x_{1o}$$
$$\dot{x}_2 = x_3 \sin u, \qquad x_2(t_o) = x_{2o}$$
$$\dot{x}_3 = g \sin u, \qquad x_3(t_o) = x_{3o}$$

where the control u is the slope of the path. The cost function is

$$J = -x_1(t_f)$$

with no specified endpoint equality constraints, and the state vector inequality constraint,

$$h(x_1x_2) = ax_1 + b - x_2 \geq 0,$$

which is converted to the equality constraint

$$\dot{x}_4 = f_4 = [h(x_1, x_2)]^2\mathrm{H}(h).$$

We can easily compute the requisite nonlinear two-point boundary value problem by direct application of the maximum principle given in this section. The equations for this TPBVP are

$$\dot{x}_1 = x_3^2\lambda_1[(\lambda_1x_3)^2 + (\lambda_2x_3 + \lambda_3g)^2]^{-1/2}, \qquad x_1(t_0) = x_{1o}$$
$$\dot{x}_2 = x_3(\lambda_2x_3 + \lambda_3g)[(\lambda_1x_3)^2 + (\lambda_2x_3 + \lambda_3g)^2]^{-1/2}, \qquad x_2(t_o) = x_{2o}$$
$$\dot{x}_3 = g(\lambda_2x_3 + \lambda_3g)[(\lambda_1x_3)^2 + (\lambda_2x_3 + \lambda_3g)^2]^{-1/2}, \qquad x_3(t_o) = x_{3o}$$
$$\dot{x}_4 = h(x_1, x_2)\mathrm{H}(h), \qquad x_4(t_o) = 0$$
$$\dot{\lambda}_1 = -2a\lambda_4h(x_1, x_2)\mathrm{H}(h), \qquad \lambda_1(t_o) = -1$$

$$\dot{\lambda}_2 = 2\lambda_4 h(x_1, x_2)\mathrm{H}(h), \qquad \lambda_2(t_f) = 0$$

$$\dot{\lambda}_3 = -\lambda_1^2 x_3[(\lambda_1 x_3)^2 + (\lambda_2 x_3 + \lambda_3 g)^2]^{-1/2}$$
$$-\lambda_2(\lambda_2 x_3 + \lambda_3 g)[(\lambda_1 x_3)^2 + (\lambda_2 x_3 + \lambda_3 g)^2]^{-1/2}, \qquad \lambda_3(t_f) = 0$$

$$\dot{\lambda}_4 = 0, \qquad x_4(t_f) = 0.$$

The solution of this set of nonlinear differential equations with the associated boundary conditions establishes the optimal trajectory and optimal control.

4.5 Hamilton-Jacobi equation and continuous time dynamic programming

We now return to the problem formulated in Subsection 4.4-1 and rederive the necessary condition Eq. (4.4-3) using an argument based on the interchange of the minimization and integration operations. This approach to the development of conditions for optimality is often called the *Principle of Optimality* or *dynamic programming*, the difference of which is elucidated in detail in [19] and [20].

We define

$$V[\mathbf{x}(t), t] = \min_{\mathbf{u}_t} \left\{\theta[\mathbf{x}(t_f), t_f] + \int_t^{t_f} \phi[\mathbf{x}(s), \mathbf{u}(s), s]\, ds\right\} \tag{4.5-1}$$

where $\mathrm{U}_t = \{\mathbf{u}(s), t \le s \le t_f\}$, and $\mathbf{x}(s), t \le s \le t_f$, is the trajectory associated with an optimal control function over the interval $[t, t_f]$, given $\mathbf{x}(t)$. The function V is therefore the optimal cost to be accrued over the interval $[t, t_f]$, given initial condition $\mathbf{x}(t)$. Clearly, V must satisfy the boundary condition

$$V[\mathbf{x}(t_f), t_f] = \theta[\mathbf{x}(t_f), t_f] \tag{4.5-2}$$

for all pairs $(\mathbf{x}(t_f), t_f)$ satisfying Eq. (4.4-2). We assume that V exists, is continuous, and has continuous first- and second-partial derivatives for all points of interest in R^{n+1}. It follows from Eq. (4.3-1) that $\mathbf{u}(t_1), t \le t_1$, does not affect $\mathbf{x}(s), s \le t$; thus, we may interchange the appropriate minimization and integration operations to produce

$$V[\mathbf{x}(t), t] = \min_{\mathbf{u}(\tau)} \Bigg\{\int_t^{t+\Delta t} \phi[\mathbf{x}(\tau), \mathbf{u}(\tau), \tau]\, d\tau +$$
$$\min_{\mathbf{u}_{t+\Delta t}} \left\{\theta[\mathbf{x}(t_f), t_f] + \int_{t+\Delta t}^{t_f} \phi[\mathbf{x}(s), \mathbf{u}(s), s]\, ds\right\}\Bigg\} \simeq$$
$$\min_{\mathbf{u}(\tau)} \{\phi[\mathbf{x}(t), \mathbf{u}(t), t]\,\Delta t + V[\mathbf{x}(t + \Delta t), t + \Delta t]\} =$$
$$\min_{\mathbf{u}(\tau)} \Bigg\{\phi[\mathbf{x}(t), \mathbf{u}(t), t]\,\Delta t + V[\mathbf{x}(t), t] + \frac{\partial V[\mathbf{x}(t), t]}{\partial t}\Delta t +$$
$$\left[\frac{\partial V[\mathbf{x}(t), t]}{\partial \mathbf{x}}\right]^T \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]\,\Delta t + \mathrm{H.O.T.}\,(\Delta t)\Bigg\}, \tag{4.5-3}$$

where for the last equality, $V[\mathbf{x}(t + \Delta t), t + \Delta t]$ is expanded in a Taylor series and the minimization is taken over $\tau \in [t, t + \Delta t]$. The smoothness assumptions and the interchange of the minimization and limit operations as $\Delta t \rightarrow 0$ (justification of which can be found in [6] and [11]) imply that V must necessarily satisfy the partial differential equation

$$\frac{-\partial V[\mathbf{x}(t), t]}{\partial t} = \min_{\mathbf{v} \in \mathbf{U}} \phi[\mathbf{x}(t), \mathbf{v}, t)] + \left[\frac{\partial V[\mathbf{x}(t), t]}{\partial \mathbf{x}}\right]^T \mathbf{f}[\mathbf{x}(t), \mathbf{v}, t] \tag{4.5-4}$$

called the Hamilton-Jacobi or Hamilton-Jacobi-Bellman equation [11–14]. A repetition of the above argument for given optimal control function $\mathbf{u}$ produces

$$\frac{-\partial V[\mathbf{x}(t), t]}{\partial t} = \phi[\mathbf{x}(t), \mathbf{u}(t), t] + \left[\frac{\partial V[\mathbf{x}(t), t]}{\partial \mathbf{x}}\right]^T \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]; \tag{4.5-5}$$

hence, $\mathbf{u}$ must necessarily satisfy Eq. (4.4-3) for all $t \in [t_0, t_f]$, where we assume

$$\boldsymbol{\lambda}(t) = \frac{\partial V[\mathbf{x}(t), t]}{\partial \mathbf{x}}. \tag{4.5-6}$$

Although the Hamilton-Jacobi equation is quite difficult to solve in general, when it can be solved, a candidate for an optimal control function is found as a function of the state trajectory—a highly desirable feedback form.

Example 4.5-1. Let us consider the linear constant differential system described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \qquad \mathbf{x}(0) = \mathbf{x}_o$$

where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{b}$ is an n vector. Any $u(t)$ is assumed to be admissible. We wish to find $u(t)$ as a function of $\mathbf{x}(t)$ such that

$$J = \tfrac{1}{2}\int_0^\infty [\mathbf{x}^T\mathbf{Q}\mathbf{x} + ru^2]\, dt$$

is a minimum. $\mathbf{Q}$ is a positive constant semidefinite matrix, and r is positive. The Hamiltonian for the problem is

$$H(\mathbf{x}, u, \boldsymbol{\lambda}, t) = \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \tfrac{1}{2}ru^2 + \boldsymbol{\lambda}^T\mathbf{A}\mathbf{x} + \boldsymbol{\lambda}^T\mathbf{b}u.$$

We need to find the control u which minimizes the Hamiltonian. This is

$$\frac{\partial H}{\partial u} = 0 = ru + \mathbf{b}^T\boldsymbol{\lambda}$$

so

$$u = -\mathbf{b}^T\boldsymbol{\lambda}r^{-1}$$

and the Hamiltonian becomes

$$H(\mathbf{x}, \boldsymbol{\lambda}, t) = \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \boldsymbol{\lambda}^T\mathbf{A}\mathbf{x} - \tfrac{1}{2}\boldsymbol{\lambda}^T\mathbf{b}\mathbf{b}^T\boldsymbol{\lambda}r^{-1}.$$

Since the system and the $\mathbf{Q}$ and r terms are time invariant and since the optimization is for a process of infinite duration, it follows that $V(\mathbf{x}, t)$ will depend only upon the initial state $\mathbf{x}$. This implies that

$$\frac{\partial V(\mathbf{x}, t)}{\partial t} = 0.$$

Therefore, since $\boldsymbol{\lambda} = \partial V/\partial \mathbf{x}$, the Hamilton-Jacobi equation becomes

$$\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{A}\mathbf{x} - \frac{1}{2}\left[\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{b}\right]^2 r^{-1} = 0.$$

If we assume a solution

$$V(\mathbf{x}, t) = \tfrac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x},$$

we see that

$$\frac{\partial V}{\partial \mathbf{x}} = \mathbf{P}\mathbf{x}$$

and the Hamilton-Jacobi equation may be written as

$$\mathbf{x}^T[\tfrac{1}{2}\mathbf{Q} + \tfrac{1}{2}\mathbf{P}\mathbf{A} + \tfrac{1}{2}\mathbf{A}^T\mathbf{P} - \tfrac{1}{2}\mathbf{P}\mathbf{b}\mathbf{b}^T\mathbf{P}r^{-1}]\mathbf{x} = 0$$

which says that, for any nonzero $\mathbf{x}(t)$, the matrix $\mathbf{P}$ must satisfy the $n(n+1)/2$ algebraic equations (the $\mathbf{P}$ matrix is symmetric)

$$\mathbf{Q} + \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{b}\mathbf{b}^T\mathbf{P}r^{-1} = 0.$$

This equation is solved for $\mathbf{P}$, and then the control is computed from

$$u = -\mathbf{b}^T\boldsymbol{\lambda} r^{-1} = -\mathbf{b}^T r^{-1}\left(\frac{\partial V}{\partial \mathbf{x}}\right) = -\mathbf{b}^T\mathbf{P}\mathbf{x}r^{-1}.$$

If we further consider the system

$$\dot{x}_1 = x_2, \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = u, \qquad x_2(0) = x_{20},$$

and the cost function

$$J = \tfrac{1}{2}\int_0^\infty (4x_1^2 + u^2)\, dt,$$

it is easy for us to show that the optimum control is given by

$$u = -2x_1 - 2x_2.$$

Example 4.5-2. Consider the system

$$\dot{x} = -x^3 + u, \qquad x(0) = x_0$$

with cost function

$$J = \tfrac{1}{2}\int_0^{t_f} (x^2 + u^2)\, dt$$

where it is desired to determine the optimal feedback control. We accomplish this by forming the Hamiltonian

$$H(x, u, \lambda, t) = \tfrac{1}{2}x^2 + \tfrac{1}{2}u^2 + \lambda u - \lambda x^3.$$

We then set $\partial H/\partial u = 0$ and note that $\lambda = \partial V/\partial x$ to obtain $u = -\lambda$; then

$$H\left(x, \frac{\partial V}{\partial x}\right) = \frac{1}{2}x^2 - \frac{1}{2}\left[\frac{\partial V(x, t)}{\partial x}\right]^2 - \left[\frac{\partial V(x, t)}{\partial x}\right]x^3.$$

The Hamilton-Jacobi equation is

$$\frac{\partial V(x, t)}{\partial t} - \frac{1}{2}\left[\frac{\partial V(x, t)}{\partial x}\right]^2 - \left[\frac{\partial V(x, t)}{\partial x}\right]x^3 + \frac{1}{2}x^2 = 0$$

with $V[x(t_f), t_f] = 0$.

If the optimization interval is infinite, then $\partial V/\partial t = 0$, and we need to solve the differential equation

$$\left[\frac{dV(x)}{dx}\right]^2 + 2\left[\frac{dV(x)}{dx}\right]x^3 - x^2 = 0$$

with $V(0) = 0$ as the initial condition. We may approximate the solution to this ordinary differential equation by a series expansion

$$V(x) = p_0 + p_1 x + \frac{1}{2}p_2 x^2 + \frac{1}{3!}p_3 x^3 + \frac{1}{4!}p_4 x^4 + \cdots.$$

If we terminate the series after the fourth-order term, substitute the assumed solution into the differential equation, and equate like powers of x (up to x^4), we obtain $p_0 = p_1 = p_3 = 0$, $p_2 = 1$, $p_4 = -6$. Thus the approximate closed-loop control is

$$u = -\lambda = -\frac{dV}{dx} = -x + x^3.$$

We naturally may question the stability of the approximate control. However, with u as obtained, the system differential equation becomes

$$\dot{x} = -x^3 + u = -x$$

which is certainly stable.

A similar procedure to this could have been used to obtain an approximate solution to the nonlinear partial differential equation that is the Hamilton-Jacobi equation for this example. In this case, the p's would be functions of time, and we would obtain matrix Riccati-type equations [15]. This approach has many attractive features. In particular, only initial condition problems need be solved. However, there are two disadvantages: There is no assurance of system stability; the number of matrix Riccati differential equations which must be solved increases greatly with the order of the differential system and the order of the polynomial in $\mathbf{x}$ for the approximate solution to $V(\mathbf{x}, t)$. If an expansion in $\mathbf{x}$ to the N order is used for an n vector differential system, the number of distinct Riccati-type differential equations is

$$E = \sum_{j=1}^{N} \frac{(n - 1 + j)!}{(n - 1)!\, j!}$$

for an assumed solution of the form

$$V(\mathbf{x}, t) = \sum_{j=1}^{n} p_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} p_{jk}x_j x_k + \frac{1}{6}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} p_{jkl}x_j x_k x_l + \cdots.$$

If, for example, the solution to a four-vector differential system is approximated by terms up to and including the fourth power in $\mathbf{x}$, we need to solve 69 differential equations to obtain the closed-loop control.

Our discussion of the second variation technique, the invariant imbedding procedure, and specific optimal control using the quasilinearization approach will point out many interesting interconnections with the approach used to obtain the solution to this example.

In our development thus far, we have assumed that the terminal time, t_f, is fixed. It is possible to remove this restriction with the result that the

Hamilton-Jacobi equation is still applicable. The initial condition for the Hamilton-Jacobi equation is still Eq. (4.5-2) and, in addition, the terminal time is determined by the relation

$$H\left(\mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}, t\right) + 1 = 0, \qquad \text{at} \quad t = t_f$$

which holds if the problem is a minimum time problem such that

$$V(\mathbf{x}, t) = t_f - t.$$

If, further, the differential system is time invariant, the Hamiltonian is equal to -1 at all times along the optimal trajectory.

We may formally obtain the Pontryagin maximum principle by taking appropriate partial derivatives of the Hamilton-Jacobi equations (Problem 9). However, the resulting maximum principle is not applicable to as broad a class of problems as is possible. The reason for this is that it is necessary that $V(\mathbf{x}, t)$ be smooth or, in other words, twice continuously differentiable with respect to $\mathbf{x}$ in order to obtain the Hamilton canonic equations of the maximum principle. We shall illustrate these difficulties with a simple example.

Example 4.5-3. A second-order example will now be discussed to illustrate that the assumption of the differentiability of $V(\mathbf{x}, t)$ does not hold in some of the simplest cases. We will consider the problem of transferring the system represented by the differential equations

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u$$

from an initial state $\mathbf{x}_o$ to the origin in minimum time. We assume that the admissible set for the scalar control is described by $|u(t)| \leq 1$.

This problem can be solved by the Pontryagin maximum principle. In the time optimal problem

$$J = \int_{t_o}^{t_f} (1)\, dt.$$

Therefore, the Hamiltonian is

$$H[\mathbf{x}, u, \boldsymbol{\lambda}, t] = 1 + \lambda_1 x_2 + \lambda_2 u.$$

The adjoint equations are

$$\dot{\lambda}_1 = 0, \qquad \dot{\lambda}_2 = -\lambda_1.$$

The solutions to these equations are

$$\lambda_1 = C_1, \qquad \lambda_2 = C_2 - C_1 t$$

where C_j is the initial condition on λ_j. The control which minimizes the Hamiltonian subject to $|u| \leq 1$ is

$$u = -\text{sign}\, \lambda_2 = -\text{sign}\,(C_2 - C_1 t).$$

The initial conditions C_1 and C_2 are not arbitrary but must be such that $\mathbf{x}(t_f) = 0$ since it is desired to transfer the system $\mathbf{x}_o$ to the origin in minimum time.

When $u = +1$, the solution to the differential system equation is

$$x_2 = t + x_2(0)$$

$$x_1 = \frac{t^2}{2} + x_2(0)t + x_1(0).$$

If t is eliminated from the foregoing, we obtain

$$x_1 = \frac{x_2^2}{2} + x_1(0) - \frac{x_2^2(0)}{2}.$$

When $u = -1$, the solution to the differential system equations is

$$x_2 = -t + x_2'(0)$$

$$x_1 = \frac{-t^2}{2} + x_2'(0)t + x_1'(0)$$

and if t is eliminated in the foregoing, we obtain

$$x_1 = \frac{-x_2^2}{2} + x_1'(0) + \frac{x_2'^2(0)}{2}.$$

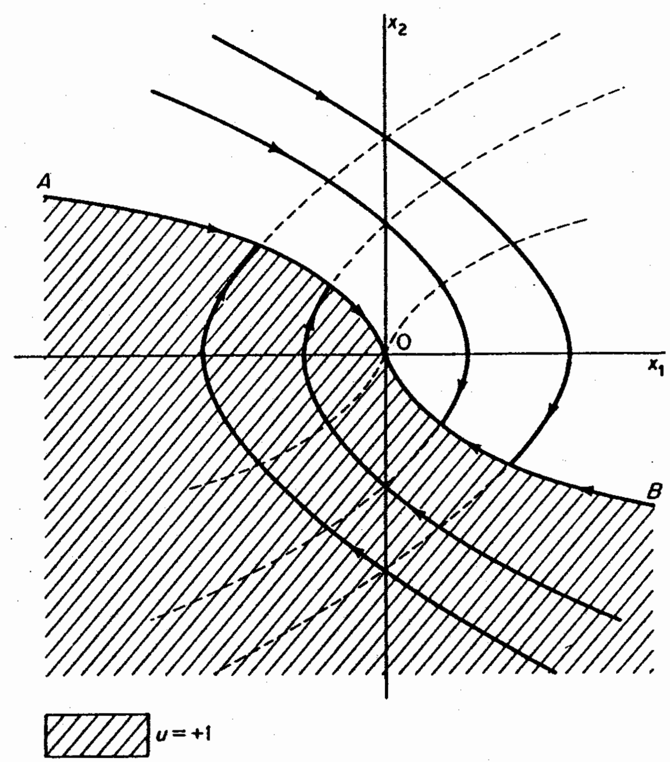

Fig. 4.5-1 Switching curve and trajectories for minimum time, Example (4.5-3).

By determining the constants C_1 and C_2 in terms of x_1 and x_2, it is a straightforward task for us to show that the control law is

$$u = -\text{sign}\,[x_1(t) + \tfrac{1}{2}x_2(t)\,|x_2(t)|].$$

These equations represent the optimal control and trajectories for $u = -1$ and $u = +1$, respectively, and they indicate that these trajectories are segments of parabolas. Figure 4.5-1 is a plot of some of these parabolas.

The segment of the parabola which is not an optimal trajectory has been represented by a broken line. The optimal control can be determined from Fig. 4.5-1 and a knowledge of the state of the system. The curve AOB represents the switching curve. When $\mathbf{x}$ lies below AOB, $u = +1$ until the system state reaches the curve AO, at which time the control switches to -1. If $\mathbf{x}$ lies above AOB, $u = -1$ until it reaches BO, where it switches to $+1$.

The optimal transition time $T(\mathbf{x})$, which is the cost function J or $V(\mathbf{x}, t)$, can be determined from the solutions for x_1 and x_2. Figure 4.5-2 is a plot of $T(\mathbf{x})$, the minimum time to transfer to the origin for the case in which the initial x_2 is held constant ($x_{2_0} = -2$), and x_{1_0} is varied about the switching line.

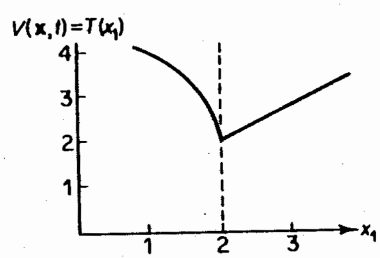

Fig. 4.5-2 Minimum time to origin for fixed x_{2_0}, Example (4.5-3).

From the graph it can be seen that $\partial T(\mathbf{x})/\partial x_1$ has a discontinuity at the switching curve. It can be shown analytically that $\partial T(\mathbf{x})/\partial x_1$ "blows up" as x_1 approaches $+2$ from the left. Hence the Hamilton-Jacobi equation would not be applicable in examples of this type. This example indicates the loss of generality which results from deriving the maximum principle from the Hamilton-Jacobi-Bellman equations.

References

1. GELFAND, I.M., and FOMIN, S.V., *Calculus of Variations*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963.

2. PONTRYAGIN, L.S., et al., *The Mathematical Theory of Optimal Processes*. Wiley, New York, 1962.

3. ROZONOÉR, L.I., "L.S. Pontryagin's Maximum Principle in the Theory of Optimum Systems I." *Automatica i Telemekhanika*, 20, no. 10, (October 1959), 1320–34.

4. ROZONOÉR, L.I., "L.S. Pontryagin's Maximum Principle in the Theory of Optimum Systems II." *Automatica i Telemekhanika*, 20, no. 11, (November 1959), 1441–58.

5. ROZONOÉR, L.I., "L.S. Pontryagin's Maximum Principle in the Theory of Optimum Systems III." *Automatica i Telemekhanika*, 20, no. 12, (December 1959), 1561–78.

6. BERKOVITZ, L.D., "Variational Methods in Problems of Control and Programming." *J. Math. Anal. Appl.*, 3, (1961), 145–69.

7. BERKOVITZ, L.D., "On Control Problems with Bounded State Variables." *J. Math. Anal. Appl.*, 5, (1962), 488–98.

8. VALENTINE, F.A., "The Problem of Lagrange with Differential Inequalities as Added Side Conditions." *Contributions to the Calculus of Variations 1933–1937*, University of Chicago Press, Chicago, 1937.

9. KELLEY, H.J., "Methods of Gradients," G. Leitman, ed., *Optimization Techniques*. Academic Press, New York, 1962, chapter 6.

10. MCGILL, R., "Optimal Control, Inequality State Constraints, and the Generalized Newton-Raphson Algorithm." *SIAM J. Control*, 3, (1965), 291–98.

11. DREYFUS, S.E., *Dynamic Programming and the Calculus of Variations*. Academic Press, New York, 1965.

12. KALMAN, R.E., "Contributions to the Theory of Optimal Control." *Bol. Soc. Mat. Mex.*, 5, (1960), 102–19.

13. BELLMAN, R., ed., *Mathematical Optimization Techniques*. University of California Press, Berkeley, 1963.

14. BRYSON, A.E., Jr., "Optimal Programming and Control." *Proceedings IBM Scientific Computing Symposium on Control Theory and Applications*, IBM Publication 320-1939, 1966.

15. MERRIAM, C.W. III, *Optimization Theory and the Design of Feedback Control Systems*. McGraw-Hill Book Co., New York, 1964.

16. MCSHANE, E.J., "On Multipliers for Lagrange Problems." *American J. Math.* 61, (1939), 809–19.

17. BAUM, R.F., and L. CESARI, "On a Recent Proof of Pontryagin's Necessary Conditions," *SIAM J. Control*, 10, (1972), 56–75.

18. FUNK, J.E., and E.G. GILBERT, "Some Sufficient Conditions for Optimality in Control Problems with State Space Constraints," *SIAM J. Control*, 8, (1970), 498–504.

19. HINDERER, K., *Foundations of Non-Stationary Dynamic Programming with Discrete Time Parameter*. Springer-Verlag, New York, 1970.

20. PORTEUS, E., "An Informal look at the Principle of Optimality." *Management Science*, 21, (1975), 1346–48.

21. LUENBERGER, D.G., *Optimization by Vector Space Methods*. Wiley, New York, 1969.