SOLUTION SET TO ASSIGNMENT 1

1. i) $\|\lambda t(1-t)\|_s = |\lambda| \max_t\{|t(1-t)| + |1-2t|\}$

$= |\lambda| \max\left\{\max_{0\le t\le \frac{1}{2}}(1-t-t^2)\,,\ \max_{\frac{1}{2}\le t\le 1}(3t-t^2-1)\right\} = |\lambda|\max\{1,1\} = |\lambda| < \frac{1}{2}$

ii) $\|\lambda t(1-t)\|_w = |\lambda|\{\max_t |t(1-t)|\} < \frac{1}{2} \iff |\lambda| < 2.$

iii) If instead we had $\|x\|_s := \max_t |x(t)| + \max_t |\dot{x}(t)|$, then:

$\|\lambda t(1-t)\|_s = |\lambda|\{\max_t |t(1-t)| + \max_t |1-2t|\} = |\lambda|\{\frac{1}{4}+1\} = \frac{5}{4}|\lambda| < \frac{1}{2} \iff |\lambda| < \frac{2}{5}$

2. The Euler-Lagrange (E-L) equation associated with this problem is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}x^2) = -x(1-\dot{x}^2), \quad x(0) = x(\pi) = 0.$$

Clearly $x = x^\circ(t) \equiv 0$ is an extremal, and

$$J(x^\circ) = \int_0^\pi x^\circ(t)^2[1-\dot{x}^\circ(t)]\,\mathrm{d}t = 0.$$

We now show that there exist $n_o > 0, \epsilon_o > 0$, such that

$$x_{(n)} = (1/\sqrt{n})\sin nt \in N^w_{\epsilon_o}(x^\circ = 0), \quad \text{and} \quad J(x_{(n)}) < 0, \quad \text{for all } n > n_o.$$

This will then imply that $x \equiv 0$ cannot be a strong local minimum. Note that $x_{(n)}$ is admissible since $x_{(n)}(0) = x_{(n)}(\pi) = 0$.

First, $\|x_{(n)}\|_w := \max_t |x(t)| = 1/\sqrt{n}$, and hence for any given ϵ_o, $x_{(n)} \in N^w_{\epsilon_o}(0)$ for all $n \ge n_o := \lceil \frac{1}{\epsilon_o^2} \rceil$, where $\lceil \alpha \rceil$ denotes the smallest integer (strictly) larger than α.

Second,

$$\int_0^\pi x^2_{(n)}(t)[1-\dot{x}^2_{(n)}(t)]\,\mathrm{d}t = \frac{1}{n}\int_0^\pi \sin^2 nt\,\mathrm{d}t - \int_0^\pi \sin^2 nt\cos^2 nt\,\mathrm{d}t.$$

The second term above is bounded away from zero (uniformly) for all $n > 0$, whereas the first term can be made arbitrarily close to zero by choosing n sufficiently large. Hence, there exists an n'_o such that for all $n > n'_o$ the second term dominates and thus $J(x_{(n)}) < 0$. Now choosing ϵ_o such that $n'_o = n_o$, completes the proof. Note that we have not proven that x° is not a weak local minimum, but only that it cannot be a strong local minimum.

3. The Euler-Lagrange equation is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x\dot{x}}{\sqrt{1+\dot{x}^2}}\right) = \sqrt{1+\dot{x}^2}, \quad x(0) = 1,\ x(\ln 2) = \frac{5}{4}$$

$$\iff \quad (\dot{x}^2 + x\ddot{x})(1+\dot{x}^2) - x\dot{x}^2\ddot{x} = (1+\dot{x}^2)^2 \iff -\dot{x}^2 + x\ddot{x} - 1 = 0.$$

The unique solution to this differential equation, with the given initial and final conditions, is

$$x^\circ(t) = \cosh t.$$

(Note that $x^\circ(0) = 1, x^\circ(\ln 2) = (2+\frac{1}{2})/2 = \frac{5}{4}$, and hence the initial conditions are satisfied.)

4. The equivalent formulation in the calculus of variations is:

$$\min_{x_1,x_2} \tilde{J}, \quad \tilde{J} := \frac{1}{2}\int_0^2 [(\dot{x}_1 - x_2)^2 + (\dot{x}_2)^2]\,\mathrm{d}t$$

subject to the boundary conditions $\quad x_1(0) = x_2(0) = 1,\ x_1(2) = 0,\ x_2(2) = \text{free}.$

The E-L equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}_1 - x_2) = 0 \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}_2) = x_2 - \dot{x}_1 \tag{2}$$

which need to be solved under the three boundary conditions given above, and the natural boundary condition (NBC)

$$\phi_{\dot{x}_2}\,|_{t=2} = 0 \iff \dot{x}_2(2) = 0.$$

$$(1) \to \quad \dot{x}_1 - x_2 = c_1 \quad \text{a constant}$$

$$(2) \to \quad \frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}_2) = x_2 - \dot{x}_1 = -c_1 \quad \Rightarrow \quad \dot{x}_2(t) = -c_1 t + c_2$$

Hence,

$$x_2(t) = -\frac{1}{2}c_1 t^2 + c_2 t + 1; \quad x_1(t) = -\frac{1}{6}c_1 t^3 + \frac{1}{2}c_2 t^2 + (1 + c_1)t + 1.$$

Using the additional two boundary conditions, we can solve (uniquely) for the constants c_1 and c_2, to obtain $c_1 = -9/14$, $c_2 = -9/7$. Hence an extremal (in fact the only one) for the calculus of variations problem is

$$x_1^\circ(t) = \frac{3}{28}t^3 - \frac{9}{14}t^2 + \frac{5}{14}t + 1; \quad x_2^\circ(t) = \frac{9}{28}t^2 - \frac{9}{7}t + 1.$$

The candidate optimal controls are therefore:

$$u_1^*(t) = \dot{x}_1^\circ - x_2^\circ = c_1 = -\frac{9}{14}; \quad u_2^*(t) = \dot{x}_2^\circ = \frac{9}{14}t - \frac{9}{7}.$$

Sufficient conditions for calculus of variations problems with multiple variables are rather complicated (notationally cumbersome), and since they are generally "overly sufficient" they are not very useful. In this particular problem, however, since the performance index is a quadratic function of the variables and their derivatives, there are other ways of proving the optimality of a given extremal, as we shall see in the coming weeks. The solution obtained above is in fact the unique (open-loop) minimizing control.

5. Let x° be an extremal. For a given ϵ_o, every $x \in N^w_{\epsilon_o}(x^\circ)$ can be written as

$$x(t) = x^\circ(t) + \epsilon\eta(t), \quad 0 \le \epsilon < \epsilon_o,$$

where $\eta : [0,1] \mapsto \mathbf{R}$ is twice continuously differentiable, and $\|\eta\|_s = 1$. Denote the class of all such (admissible) normalized variations by $\mathcal{N}_n$.

Condition 1: ϕ is twice continuously differentiable in its arguments.

Then, following the procedure developed in class:

$$J(x^\circ) - J(x^\circ + \epsilon\eta) = J(x^\circ) - J(x^\circ) - \epsilon\delta J(x^\circ;\eta) + o(\epsilon) \le 0, \quad \forall \eta \in \mathcal{N},\ 0 \le \epsilon < \epsilon_o, \tag{1}$$

where

$$\delta J(x^\circ;\eta) := \int_0^1 [\phi_x\eta + \phi_{\dot{x}}\dot{\eta} + \phi_{\ddot{x}}\ddot{\eta}]\,\mathrm{d}t \equiv \int_0^1 [\phi_x\eta + (\phi_{\dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\ddot{x}})\dot{\eta}]\,\mathrm{d}t + \dot{\eta}\phi_{\ddot{x}}\,|_0^1\,.$$

Since $\eta \in \mathcal{N}_n \Rightarrow -\eta \in \mathcal{N}_n$, inequality (1) is equivalent to:

$$\delta J(x^\circ;\eta) = 0 \quad \forall \eta \in \mathcal{N}_n. \tag{2}$$

Since we can take $\dot{\eta}(0) = \dot{\eta}(1) = 0$, (2) is further equivalent to:

$$\int_0^1 [\phi_x \eta + (\phi_{\dot{x}} \dot{-} \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\ddot{x}})\dot{\eta}]\,\mathrm{d}t = 0 \quad \forall \eta \in \mathcal{N}_n.$$

Furthermore, since we are allowed to take $\eta(0) = \eta(1) = 0$, it follows from *Lemma 1* of *Handout 2* that, for some constant c:

$$\phi_{\dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\ddot{x}} = \int_0^t \phi_x(s, x(s), \dot{x}(s), \ddot{x}(s))\,\mathrm{d}s + c, \quad \text{at } x = x^\circ(t) \tag{3}$$

which is the Euler-Lagrange equation associated with this problem (in partial integral form). In arriving at this, we have assumed, in addition to *Condition 1*:

Condition 2 : ϕ is thrice continuously differentiable in its arguments.

If, furthermore,

Condition 3 : ϕ is four times continuously differentiable in its arguments,

then (3) can be written in the equivalent differential equation form:

$$\frac{d^2}{\mathrm{d}t^2}\phi_{\ddot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\dot{x}} + \phi_x = 0, \quad \text{at } x = x^\circ(t).$$

Using this differential equation form of the E-L equation, we can rewrite (2) as follows:

$$\delta J(x^\circ; \eta) = \left[(\phi_{\dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\ddot{x}})\eta + \phi_{\ddot{x}}\dot{\eta}\right] |_{t=0}^{t=1} = 0,$$

from which the following natural boundary conditions follow (since the end points are independently free):

$$\phi_{\ddot{x}}^\circ\, |_{t=1 \text{ and } 0} = 0 \qquad \phi_{\dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\ddot{x}}^\circ\, |_{t=1 \text{ and } 0} = 0.$$

6. The Euler-Lagrange equation is the same as the one derived in the solution to *Problem 5*; however, the NBCs will not be the same as now $x(t_f)$ and $\dot{x}(t_f)$ are not free but are constrained to lie on a given curve. Following the derivation of NBCs in the solution to *Problem 5*, steps leading to the condition

$$\delta J(x^o; \eta) = 0$$

are still valid here, and hence we have the NBC:

$$\left[(\phi_{\dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\ddot{x}})\eta + \phi_{\ddot{x}}\dot{\eta}\right]_{t=t_f} = 0 \tag{$\circ$}$$

For the general case, we cannot simplify this any further, however, because we need to know the relationship between $\eta(t_f)$ and $\dot{\eta}(t_f)$, which cannot be known unless we are given a specific terminal constraint – which we are given for the specific problem at hand. Note that in *Problem 5*, $\eta(t_f)$ and $\dot{\eta}(t_f)$ were allowed to vary freely at $t = t_f$, which is why we were able to go a step further there and obtain two simpler (more explicit) conditions.

Now returning to the specific problem at hand, the cost function to be minimized is

$$J(x) = \int_0^1 [\ddot{x}(t)]^2\,\mathrm{d}t$$

with the end-point constraint being

$$x^2(1) + \dot{x}^2(1) = 1 \tag{$*$}$$

If x^o is a minimizing solution, then it should satisfy the Euler-Lagrange equation (from solution to *Problem 5*):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ddot{x} = 0 \quad \Rightarrow \quad x^o(t) = a + bt + ct^2 + et^3$$

From the given initial conditions, $x^o(0) = -2\sqrt{2}$, $\dot{x}^o(0) = 5\sqrt{2}$, the coefficients a and b can readily be determined:

$$a = -2\sqrt{2}\,, \quad b = 5\sqrt{2}$$

To determine the remaining coefficients, c and e, we need two independent equations at $t = 1$. One of them is the end-point constraint (*circle*) given by $(*)$. The other one will be obtained from the NBC.

For any admissible (first-order) variation η, we have

$$\frac{d}{d\epsilon}\left[\left[x^o(1) + \epsilon\eta(1)\right]^2 + \left[\dot{x}^o(1) + \epsilon\dot{\eta}(1)\right]^2\right]_{\epsilon=0} = 0 \quad \Rightarrow \quad x^o(1)\eta(1) + \dot{x}^o(1)\dot{\eta}(1) = 0$$

which gives us the desired relationship between η and its derivative at $t = 1$. Using this relationship in the NBC $(\circ)$ (substitute for $\dot{\eta}(1)$ in terms of $\eta(1)$), and then requiring the resulting expression to hold for all $\eta(1)$, leads to the specific NBC:

$$\dot{x}^o(1)\frac{\mathrm{d}}{\mathrm{d}t}\ddot{x}^o(1) + \ddot{x}^o(1)x^o(1) = 0$$

Solving this equation on the given circle $(*)$, we obtain the following unique values for c and e:

$$c = -3\sqrt{2}\,, \quad e = \frac{\sqrt{2}}{2}$$

Hence, the solution to the problem is:

$$\boxed{\begin{array}{lcl} x^\circ(t) & = & +\dfrac{1}{\sqrt{2}}\,t^3 - 3\sqrt{2}\,t^2 + 5\sqrt{2}\,t - 2\sqrt{2} \\ \dot{x}^\circ(t) & = & +\dfrac{3\sqrt{2}}{2}\,t^2 - 6\sqrt{2}\,t + 5\sqrt{2} \\ u^\star(t) & = & \ddot{x}_2^\circ(t) = 3\sqrt{2}\,t - 6\sqrt{2} \\ \uparrow & & \\ \text{optimal control.} & & \end{array}}$$

7. Let $g[x, \dot{x}] := \frac{\mathrm{d}}{\mathrm{d}t}\psi(x(t)) = \psi_x\dot{x}$. Then,

$$J(x) = \int_{t_0}^{t_f} \phi[t, x, \dot{x}]\,\mathrm{d}t + \int_{t_0}^{t_f} g[x, \dot{x}]\,\mathrm{d}t + \psi(x(t_0)).$$

Let η be an admissible variation, with $\|\eta\|_s = 1$. Then, $x^\circ \to x^\circ + \epsilon\eta$ leads to

$$\begin{array}{rcl} J(x^\circ + \epsilon\eta) - J(x^\circ) & = & \epsilon\displaystyle\int_{t_0}^{t_f}\left[f_x - \frac{\mathrm{d}}{\mathrm{d}t}f_{\dot{x}}\right]\eta(t)\,\mathrm{d}t + \epsilon f_{\dot{x}}\eta\big|_{t_0}^{t_f} + \epsilon\psi_x(x(t_0))\eta(t_0) + o(\epsilon) \geq 0 \\ f & := & \phi + g. \end{array} \qquad (\star)$$

Following the standard arguments, we arrive at the Euler-Lagrange equation

$$f_x - \frac{\mathrm{d}}{\mathrm{d}t}f_{\dot{x}} = 0 \Longleftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\dot{x}} + \frac{\mathrm{d}}{\mathrm{d}t}\psi_x = \phi_x + \dot{x}\psi_{xx}$$

Hence, $\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\dot{x}} = \phi_x, \text{ at } x = x^\circ}$ $\leftarrow$ the standard Euler-Lagrange equation.

Now $(\star)$ becomes

$$\epsilon \left[f_{\dot{x}}(t_f)\eta(t_f) + \left[\psi_x(x(t_0)) - f_{\dot{x}}(t_0)\right]\eta(t_0)\right] + o(\epsilon) \geq 0 \quad \forall \text{ admissible } \eta.$$

From the given boundary condition:

$$\eta(t_0) + 2\eta(t_f) = 0 \quad \Rightarrow \quad \eta(t_f) = -\frac{\eta(t_0)}{2}.$$

Since $\eta(t_0)$ can be both positive and negative, we need

$$\frac{1}{2}f_{\dot{x}}(t_f) = \psi_x(x(t_0)) - f_{\dot{x}}(t_0) \equiv -\phi_{\dot{x}}\left[t_0, x(t_0), \dot{x}(t_0)\right]$$

Hence,

$\boxed{\left[\phi^\circ_{\dot{x}} + \psi^\circ_x\right]\big|_{t=t_f} = -2\, \phi^\circ_{\dot{x}}\big|_{t=t_0}}$ along with $\boxed{x^\circ(t_0) + 2x^\circ(t_f) = 1.}$

8. The Euler-Lagrange equation is:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ddot{x} = x$$

which admits the general solution

$$x(t) = A\cosh t + B\sinh t + C\sin t + D\cos t$$

Using the given four boundary conditions, unique values can be obtained for A, B, C, and D:

$$A = \frac{(1 - \sinh\frac{\pi}{2})\sinh\frac{\pi}{2}}{\cosh^2\frac{\pi}{2}}, \quad B = \tanh\frac{\pi}{2}, \quad C = -B, \quad D = 1 - A$$

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