SOLUTION SET TO ASSIGNMENT 2

9. The problem is to minimize the following functional

$$J(x) = \int_0^{t_f} [\ddot{x}(t)]\,\mathrm{d}t$$

subject to $x(0) = \dot{x}(1) = 1$, and the condition that t_f and $x(t_f)$ satisfy the constraint $x(t_f) = -t_f^2$, where t_f is free.

This is in the form of *Problem 6*, but with variable final time. By following the derivation given in class for the standard calculus of variations problem with variable end points, it is possible to derive transversality conditions for this problem. Here, we will take a more direct approach, because of the simplicity of the solution to the associated Euler-Lagrange equation, which has exactly the same form as the solution to *Problem 6*:

$$x^o(t) = a + bt + ct^2 + et^3$$

In the above, a and b can again be readily computed from the given initial conditions: $a = b = 1$. Now substitute the resulting extremal, $x^o(t)$, into $J(x)$, to obtain:

$$J(x^o) = \int_0^{t_f} \left(2c + 6et\right)^2 \mathrm{d}t = 4[c^2 t_f + 3ect_f^2 + 3e^2 t_f^3]$$

This is a function of three parameters, c, e, t_f, whose optimal values can be obtained by minimizing it subject to the terminal (target) set constraint, which can be re-written as (in terms of these parameters):

$$1 + t_f + (c+1)t_f^2 + et_f^3 = 0$$

This minimization problem can be solved by first solving for c in terms of t_f from the above equation, substituting this into $J(x^o)$, and then differentiating the resulting function of e and t_f with respect to these two parameters, and setting the derivatives equal to zero. Some manipulations lead to the unique solution:

$$t_f^\star = \frac{1+\sqrt{13}}{2} = 2.3028; \quad c = -\frac{11+\sqrt{13}}{6} = -2.43425, \quad e = \frac{1+5\sqrt{13}}{54} = 0.35236,$$

The optimal control sought is therefore

$$\boxed{u^\star(t) = \ddot{x}^o(t) = -4.8685 + 2.11419\,t}$$

10. First solve for y and z from the first equation:

$$x(t) + y(t) - z(t) = x(0)$$

and since $y(1) = 3$, $z(1) = 1$, this leads to the constraint

$$x(1) - x(0) = -2.$$

The second equation leads to (because the condition at $t = 1$ is automatically satisfied)

$$w(t) + 2z(t) = 0$$

$$\Rightarrow \quad w^o(t) = -2t, \; z^o(t) = t$$

as one possible solution, but there are clearly others; simply choose $z^o(t)$ to be any continuously differentiable function that vanishes at $t = 0$, and passes through $z = 1$ at $t = 1$. Then, $w^o(t) = -2z^o(t)$, and $y^o(t) = z^o(t) + x^o(0) - x^o(t)$ where x^o is now the solution to

$$\boxed{\min \int_0^1 [\dot{x}(t)]^2 \, \mathrm{d}t \quad \text{subject to} \quad -x(1) + x(0) = 2.}$$

This is precisely in the form of Problem 7, and admits a solution: $x^o(t) = -2\,t + c$, where c is arbitrary. In this case we can in fact say that the family $\{x(t) = -2\,t + c, \, c \in \mathbf{R}\}$ constitutes the complete class of minimizing solutions.

11. As far as the derivation of 1st order necessary conditions go, we can take $x^\circ \to x^\circ + \epsilon n_1$ where n_1 satisfies

$$\int_0^1 \left(\psi_{\dot{x}} \dot{n}_1 + \psi_x n_1\right) \mathrm{d}t = 0.$$

Let $J(x) = \displaystyle\int_0^1 \phi[t, x, \dot{x}, \ddot{x}] \, \mathrm{d}t$. Then,

$$J(x^\circ + \epsilon n_1) - J(x^\circ) = \epsilon \int_0^1 \left(\phi_x n_1 + \phi_{\dot{x}} \dot{n}_1 + \phi_{\ddot{x}} \ddot{n}_1\right) \mathrm{d}t + o(\epsilon) \geq 0$$

$$\Rightarrow \quad \int_0^1 \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi_{\ddot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\dot{x}} + \phi_x\right) n_1(t) \, \mathrm{d}t + \dot{n}_1 \phi_{\ddot{x}}\big|_0^1 + \left(\phi_{\dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\ddot{x}}\right) n_1 \bigg|_0^1 \geq 0$$

$$\forall n_1 \text{ such that } \int_0^1 \left(\psi_{\dot{x}} \dot{n}_1 + \psi_x n_1\right) \mathrm{d}t = 0$$

$$\iff \quad \int_0^1 \left(\psi_x - \frac{\mathrm{d}}{\mathrm{d}t}\psi_{\dot{x}}\right) n_1 \, \mathrm{d}t + \psi_{\dot{x}} n_1\big|_0^1 = 0 \qquad (\star)$$

To obtain the first-order necessary condition(s), we can take the end points fixed. Then, under the assumption that at $x = x^\circ$, $\psi_x - \frac{\mathrm{d}}{\mathrm{d}t}\psi_{\dot{x}} \neq 0$ for at least one t (that is, the constraint is **regular** at $x = x^\circ$), there exists a constant λ such that

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi_{\ddot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\dot{x}} + \phi_x = -\lambda \left[\psi_x - \frac{\mathrm{d}}{\mathrm{d}t}\psi_{\dot{x}}\right] \text{ at } x = x^\circ.} \qquad (\star\star)$$

The argument used here is exactly the same as the one used in class in deriving the necessary conditions for the standard isoperimetric problem. Hence, $x^\circ(t)$ is the extremal for

$$\int_0^1 L(t, x, \dot{x}, \ddot{x}; \lambda) \, \mathrm{d}t \, ; \qquad L := \phi + \lambda\psi.$$

To determine the NBCs, use the necessary condition $(\star\star)$ in $(\star)$ to lead to [by also noting that n_1 admissible $\Rightarrow -n_1$ is also admissible]:

$$-\lambda \int_0^1 \left[\psi_x - \frac{\mathrm{d}}{\mathrm{d}t}\psi_{\dot{x}}\right] n_1(t) \, \mathrm{d}t + \dot{n}_1 \psi_{\ddot{x}}\bigg|_0^1 + \left(\phi_{\dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\ddot{x}}\right) n_1(t)\bigg|_0^1 = 0$$

$$\forall\, n_1 \text{ such that } \int_0^1 \left(\psi_x - \frac{\mathrm{d}}{\mathrm{d}t}\psi_{\dot{x}}\right) n_1(t) \, \mathrm{d}t + \psi_{\dot{x}} n_1(t)\bigg|_0^1 = 0.$$

Multiplying the $2nd$ line by λ, and using it in the first line, we obtain the equivalent condition:

$$\lambda \; \psi_{\dot{x}} n_1(t)\big|_0^1 + \left(\phi_{\dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\ddot{x}}\right) n_1(t)\bigg|_0^1 + \dot{n}_1 \phi_{\ddot{x}}\big|_0^1 = 0.$$

Since $n_1(0)$, $\dot{n}_1(0)$, $n_1(1)$, $\dot{n}_1(1)$ are arbitrary (if no end points are specified), we obtain as the NBCs:

$$\phi^\circ_{\dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\phi^\circ_{\ddot{x}} + \lambda\phi^\circ_{\dot{x}} = 0 \quad \text{at} \quad t = 0, 1$$
$$\phi^\circ_{\ddot{x}} = 0, \quad \text{at} \quad t = 0, 1.$$

These are clearly the ones associated with an unconstrained calculus of variations problem with integral kernel

$$L(t, x, \dot{x}, \ddot{x}, \lambda)$$

(see the solution of Problem 5).

12. First eliminate the dependence of J on x and $\dot{x}$ by using the relationship $\dot{y} = x$. This leads to the equivalent cost function:

$$\tilde{J}(y) = \int_0^1 \left[\, \ddot{y}(t)^2 + \dot{y}(t)^2 + 2\dot{y}(t)y(t)\,\right] \mathrm{d}t \tag{1}$$

The E-L equation for this fixed end-point calculus of variations problem is (see the solution of Problem 5):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\ddot{y}) - \frac{\mathrm{d}}{\mathrm{d}t}(2\dot{y} + 2y) + 2\dot{y} = 0\,, \quad \dot{y}(0) = \dot{y}(1) = y(1) = 0,\ y(0) = 1. \tag{2}$$

Integrating (2) once, dividing throughout by 2, and substituting $\dot{y} = x$, yields the following equivalent E-L equation:

$$\ddot{x} = x + c_1$$

whose solution is:

$$\Rightarrow \qquad x(t) = A\cosh t + B\sinh t - c_1$$
$$x(0) = 0 \qquad \Rightarrow\ c_1 = A\cosh 0 = A$$
$$x(1) = 0 \qquad \Rightarrow\ A\cosh 1 + B\sinh 1 = c_1 = A\cosh 0 = A$$
$$\Rightarrow \qquad B = \left(\frac{1 - \cosh 1}{\sinh 1}\right) A. \tag{3}$$

$$y(t) = \int_0^t x(s)\,\mathrm{d}s + 1 = A\sinh t + B\cosh t - At - B + 1$$
$$y(1) = 0 \Rightarrow \qquad A\sinh 1 + B\cosh 1 - A + 1 - B = 0.$$

Now, substituting from (3) for B yields:

$$\left[\sinh 1 - 1 + \frac{2\cosh 1 - \cosh^2 1 - 1}{\sinh 1}\right] A = -1 \quad \Rightarrow$$

$$A = \frac{\sinh 1}{2(1 - \cosh 1) + \sinh 1} = 13.1986; \quad B = \left(\frac{1 - \cosh 1}{\sinh 1}\right) A = -6.099$$

$$x^\circ(t) = A\cosh t + B\sinh t - A$$
$$y^\circ(t) = A\sinh t + B\cosh t - At - B + 1$$

If J does not have the cross term, then the associated E-L equation reduces to the same differential equation ($\ddot{x} = x + c_1$), leading to exactly the same solution as above.Hence, what we have seen here is that two different cost functions could result in the same extremal.

13. We can convert this optimal control problem into a calculus of variations problem with a single variable, by substituting for u and $\dot{x}_2$ in terms of x_1 and its two derivatives. This leads to the equivalent cost function:

$$\tilde{J}(x_1) = t_f^2 + \frac{1}{2}\int_0^{t_f} \left[\ddot{x}_1\right]^2 \mathrm{d}t$$

For the moment let us assume that t_f is fixed. Then the E-L equation is:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} 2\ddot{x}_1 = 0 \quad \Rightarrow \quad \ddot{x}_1 = c_1 t + c_0 \quad \Rightarrow \quad x_1^\circ(t) = \frac{c_1}{6}t^3 + \frac{c_0}{2}t^2 + 10$$

where in arriving at the last expression I have made use of the given initial conditions: $x_1(0) = 10, \dot{x}_1(0) = 0$. Substitution of x_1° into $\tilde{J}$ yields:

$$\tilde{J}(x_1^\circ) = t_f^2 + \frac{1}{6c_1}\left[\,(c_1 t_f + c_0)^3 - c_0^3\,\right]$$

Note that this expression for the optimal cost depends on three parameters, t_f, c_1 and c_0. The best value for t_f will be the one that minimizes this expression; the values of c_1, c_0 will be obtained either from given terminal conditions (case (a) below), or by minimization if some terminal conditions are missing (case (b) below).

(a) Use $x_1(t_f) = \dot{x}_1(t_f) = 0$ in the expression for x°; this leads to::

$$\frac{c_1}{6}t_f^3 + \frac{c_0}{2}t_f^2 + 10 = 0 \;\text{ and }\; \frac{c_1}{2}t_f^2 + c_0 t_f = 0 \quad \Rightarrow \quad c_0 = -\frac{c_1}{2}t_f \quad (\text{since } t_f \neq 0)$$

It then readily follows that

$$c_1 = 120\, t_f^{-3}, \quad c_0 = -60\, t_f^{-2}$$

Using these expressions for c_1 and c_0 in $\tilde{J}(x_1^\circ)$, and minimizing the resulting expression with respect to t_f (by differentiation) yields the unique value : $t_f = (30)^{2/5} = 3.898$, and hence $c_1 = 2.026\,;\;\; c_0 = -3.949$. This leads to:

$$\boxed{u^\star(t) = 2.026\, t - 3.949} \qquad \boxed{J_{(a)}(u^\star) = 25.32}$$

(b) Here the only difference is that, instead of $\dot{x}_1(t_f)$ specified, we have to obtain an optimal value for it. We have a single relationship between the three unknowns t_f, c_1 and c_0:

$$\frac{c_1}{6}t_f^3 + \frac{c_0}{2}t_f^2 + 10 = 0$$

from which we can solve for c_1, and arrive at an expression for $\tilde{J}(x_1^\circ)$ that depends on two independent variables t_f and c_0. Minimization of this expression with respect to these two variables (by simple differentiation) leads to the unique solution:

$$\boxed{t_f = (15)^{2/5} = 2.954} \qquad c_1 = 2(15)^{-1/5} = 1.164$$

$$c_0 = -2(15)^{1/5} = -3.438$$

$$\boxed{u^\star(t) = 1.164\, t - 3.438}$$

$$\boxed{J_{(b)}(u^\star) = 14.55}$$

Note that $J_{(a)}(u^*_{(a)}) > J_{(b)}(u^*_{(b)})$, which is what we would have expected since in (b) one of the end points (on x_2, or equivalently, on $\dot{x}_1$) is free.
Another approach (that avoids minimization with respect to c_0) is to use the natural boundary condition at $\dot{x}_1(t_f)$ (for fixed t_f), which follows from *solution to Problem 5* as

$$\phi_{\ddot{x}}|_{t=t_f} = 0 \qquad \Rightarrow \qquad c_1 t_f + c_0 = 0\,.$$

Using this extra condition (in place of minimization) leads to the same solution as above.

14.

$$J = \int_0^{t_f} [\dot{x}_2(t)]^2\,\mathrm{d}t \to \min \quad \text{such that} \qquad \begin{array}{l} \dot{x}_1 = x_2, \quad x_1(0) = x_2(0) = 1 \\ x_1(t_f) = -t_f^2, \quad t_f \text{ is free} \end{array}$$

$$\text{The Lagrangian is :} \qquad L = \dot{x}_2^2 + \lambda(\dot{x}_1 - x_2)$$

E-L equations:

$$\left.\begin{array}{rcl} 2\ddot{x}_2 &=& -\lambda \\ \dot{\lambda} &=& 0 \end{array}\right\} \to \lambda(t) = c_1$$

$$x_2(t) = -\frac{1}{4}c_1 t^2 + c_2 t + 1$$

$$x_1(t) = -\frac{1}{12}c_1 t^3 + \frac{1}{2}c_2 t^2 + t + 1.$$

Transversality condition:

$$\lambda(t_f)\,[-2t_f - x_2(t_f)] + \dot{x}_2(t_f)^2 = 0$$

$$\iff \qquad c_1\left[2t_f + 1 + c_2 t_f - \frac{1}{4}c_1 t_f^2\right] = \left(c_2 - \frac{1}{2}c_1 t_f\right)^2.$$

The NBC for x_2:

$$2\dot{x}_2(t_f) = 0 \quad \Rightarrow \quad \boxed{c_2 = \frac{1}{2}c_1 t_f} \tag{1}$$

$$2t_f + 1 + \frac{1}{2}c_1 t_f^2 - \frac{1}{4}c_1 t_f^2 = 0 \quad \Rightarrow \quad \boxed{c_1 t_f^2 = 4(1 + 2t_f).} \tag{2}$$

We also have

$$x_1(t_f) = -t_f^2 \quad \Rightarrow \quad 1 + t_f + \frac{1}{4}c_1 t_f^3 - \frac{1}{12}c_1 t_f^3 = -t_f^2$$

$$\Leftrightarrow \quad \boxed{\frac{1}{6}c_1 t_f^3 + t_f^2 + 1 + t_f = 0} \tag{3}$$

The unique solution to (1)-(3) is

$$t_f^\star = \frac{1+\sqrt{13}}{2} = 2.3028; \quad c_1 = -\frac{30+6\sqrt{13}}{5+2\sqrt{13}} = -4.22839, \quad c_2 = -\frac{27+9\sqrt{13}}{5+2\sqrt{13}} = -4.8685$$

and using these in the expression for control:

$$\boxed{u^\star(t) = c_2 - \frac{1}{2}c_1 t = -4.8685 + 2.11419\,t}$$

15. The Lagrangian is: $L(x, y, \dot{x}, \dot{y}, \lambda) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + 2xy) + \lambda(\dot{y} - x)$.
The E-L equations are:

$$\frac{\mathrm{d}}{\mathrm{d}t}L_{\dot{x}} = L_x \quad \Rightarrow \quad \ddot{x} = y - \lambda, \quad x(0) = x(1) = 0 \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}L_{\dot{y}} = L_y \quad \Rightarrow \quad \ddot{y} + \dot{\lambda} = x, \quad y(1) = 0,\ y(0) = 1. \tag{2}$$

From (1), $\quad \dot{\lambda} = \dot{y} - x^{(iii)}$, and using this in (2):

$$\ddot{y} + \dot{y} - x^{(iii)} = x.$$

But, $\quad \dot{y} = x, \quad \ddot{y} = \dot{x} \qquad \Rightarrow$

$$-x^{(iii)} + x + \dot{x} = x \quad \Rightarrow \quad x^{(iii)} = \dot{x}$$

Integrating once:

$$\ddot{x} = x + c_1$$
$$\Rightarrow \qquad x(t) = A\cosh t + B\sinh t - c_1$$
$$x(0) = 0 \qquad \Rightarrow c_1 = A\cosh 0 = A$$
$$x(1) = 0 \qquad \Rightarrow A\cosh 1 + B\sinh 1 = c_1 = A\cosh 0 = A$$
$$\Rightarrow \qquad B = \left(\frac{1 - \cosh 1}{\sinh 1}\right)A. \tag{3}$$
$$y(t) = \int_0^t x(s)\,\mathrm{d}s + 1 = A\sinh t + B\cosh t - At - B + 1$$
$$y(1) = 0 \Rightarrow \qquad A\sinh 1 + B\cosh 1 - A + 1 - B = 0.$$

Now, substituting from (3) for B yields:

$$\left[\sinh 1 - 1 + \frac{2\cosh 1 - \cosh^2 1 - 1}{\sinh 1}\right]A = -1 \quad \Rightarrow$$

$$\boxed{A = \frac{\sinh 1}{2(1 - \cosh 1) + \sinh 1} = 13.1986; \quad B = \left(\frac{1 - \cosh 1}{\sinh 1}\right)A = -6.099}$$

$$\boxed{\begin{aligned} x^\circ(t) &= A\cosh t + B\sinh t - A \\ y^\circ(t) &= A\sinh t + B\cosh t - At - B + 1 \end{aligned}}$$

If J does not have the cross term, then the associated E-L equation reduces to the same differential equation ($\ddot{x} = x$), leading to the same solution as above.

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