ECE 553 **SPRING 2006**
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SOLUTION SET TO ASSIGNMENT 3

16. The Hamiltonian is

$$H(t,x,p,u) = \frac{1}{2}u^2 + p_1x_2 + p_2u$$

(a) First order necessary conditions are

$$\left.\begin{array}{lcl} \dot{p}_1 & = & 0 \\ \dot{p}_2 & = & -p_1 \end{array}\right\} \Rightarrow \begin{array}{lcl} p_1(t) & = & C_1 \\ p_2(t) & = & -C_1t + C_2 \end{array} \qquad \begin{array}{r}(1)\\(2)\end{array}$$

$$H_u = u + p_2 = 0 \quad \Rightarrow \quad u = C_1t - C_2 \tag{3}$$

Since t_f is free:

$$H\Big|_{t=t_f} + \frac{\partial q}{\partial t_f} = 0 \quad \Rightarrow \quad \frac{1}{2}(C_1t_f - C_2)^2 + p_1x_2\Big|_{t_f} + p_2u\Big|_{t_f} = -2t_f \tag{4}$$

Also,

$$\dot{x}_2 = C_1t - C_2; \quad x_2(0) = 0$$

$$\Rightarrow \quad x_2(t) = \frac{C_1}{2}t^2 - C_2t \tag{5}$$

$$\dot{x}_1 = x_2, x_1(0) = 10 \quad \Rightarrow \quad x_1(t) = \frac{C_1}{6}t^3 - \frac{C_2}{2}t^2 + 10 \tag{6}$$

Use $x_1(t_f) = x_2(t_f) = 0$ in (4), (5) and (6):

$$(6) \quad \rightarrow \quad \frac{C_1}{6}t_f^3 - \frac{C_2}{2}t_f^2 + 10 = 0 \tag{6$'$}$$

$$(5) \quad \rightarrow \quad \frac{C_1}{2}t_f^2 - C_2t_f = 0 \quad \Rightarrow \quad C_2 = \frac{C_1}{2}t_f \quad (\text{since } t_f \neq 0)$$

$$(6)' \quad \rightarrow \quad \frac{C_1}{12}t_f^3 = 10 \quad \Rightarrow \quad C_1t_f^3 = 120. \tag{6$''$}$$

$$(4) \quad \rightarrow \quad \frac{1}{2}\cdot\frac{1}{4}C_1^2t_f^2 - \frac{1}{4}C_1^2t_f^2 = -2t_f \quad \Rightarrow \quad C_1^2t_f = 16 \tag{4$'$}$$

From (6)$''$ and (4)$'$ $\Rightarrow t_f^5 = \dfrac{(120)^2}{16} \Rightarrow \mathrm{t_f} = (30)^{2/5} = 3.898$

$$C_1 = 2.026; \qquad C_2 = 3.949$$

$$\boxed{u^\star(t) = 2.026\,t - 3.949}\;; \quad \boxed{J_{(a)}(u^\star) = 25.32}$$

(b) Here the only difference is that, instead of $x_2(t_f) =$ specified, we have to use the natural boundary condition on $p_2 \Rightarrow p_2(t_f) = 0$. This leads to:

$$C_2 = C_1t_f \rightarrow t_f = \frac{C_2}{C_1} \tag{$\star$}$$

$$(4) \quad \rightarrow \qquad -C_1\frac{C_2}{2}t_f + 2t_f = 0 \quad \Rightarrow \quad C_1C_2 = 4 \qquad (4)''$$

From (6):

$$\frac{C_1}{6}t_f^3 - \frac{C_2}{2}t_f^2 + 10 = 0 \quad \Rightarrow \quad C_1t_f^3 = 30 \qquad (6)'''$$

Using $(\star)$, $(4)''$ and $(6)'''$, we obtain the unique solution:

$$\boxed{t_f = (15)^{2/5} = 2.954} \; ; \quad C_1 = 2(15)^{-1/5} = 1.164$$

$$C_2 = 2(15)^{1/5} = 3.438$$

$$\boxed{u^\star(t) = 1.164\,t - 3.438}$$

$$\boxed{J_{(b)}(u^\star) = 14.55}$$

Here $J_{(a)}(u^*_{(a)}) > J_{(b)}(u^*_{(b)})$ which is what we would have expected, since in (b) one of the end points (on x_2) is free.

17. Multiplying J with $\frac{1}{2}$, without any loss of generality, the Hamiltonian is

$$H = \frac{1}{2}u^2 + p\,(2x + 3u)$$

The Minimum Principle gives:

$$\dot{p} = -H_x = -2p\,, \;\; p(4) = 2x(4) \;\Rightarrow\; p(t) = ke^{-2t} \quad \text{where k is a constant}, k = 2e^8x(4)$$

$$\dot{x} = 2x + 3u\,, \;\; x(0) = 1$$

$$H_u = 0 \;\;\Rightarrow\;\; u = -3p = -3ke^{-2t}$$

Substitute this expression for u into the state equation, and solve for x as a function of k:

$$\dot{x} = 2x - 9ke^{-2t}\,, \;\; x(0) = 1 \;\Rightarrow\; x(t) = \big[1 - \frac{9}{4}k + \frac{9}{4}ke^{-4t}\big]e^{2t}$$

Hence,

$$k = 2e^8x(4) = 2\big[1 - \frac{9}{4}k + \frac{9}{4}ke^{-16}\big] \;\Rightarrow\; k = \frac{4}{9 - 7e^{-16}}\,,$$

$$\Rightarrow \quad \boxed{p(t) = \frac{4}{9 - 7e^{-16}}\,e^{-2t} \approx \frac{4}{9}\,e^{-2t}} \qquad \boxed{u^o(t) = -\frac{12}{9 - 7e^{-16}}\,e^{-2t} \approx \frac{4}{3}\,e^{-2t}}$$

18. The feedback control is $u = -3P(t)x$, where P solves the RDE: $\dot{P} + 4P - 9P^2 = 0\,,\; P(4) = 2$.

To solve the RDE, let $S = 1/P$, assuming that $P(t)$ is positive over the entire interval $[0, 4]$.

S satisfies, $\dot{S} = 4S - 9\,,\; S(4) = \frac{1}{2} \;\Rightarrow\; S(t) = -\frac{7}{4}e^{4(t-4)} + \frac{9}{4} \;\Rightarrow\; P(t) = \frac{4}{9 - 7e^{4(t-4)}}$

Hence, the optimal feedback control is $\boxed{u = \mu(x,t) = -\frac{12}{9 - 7e^{4(t-4)}}\,x}$

If we substitute this into the state equation, it becomes

$\dot{x} = -\dfrac{18 + 14e^{4(t-4)}}{9 - 7e^{4(t-4)}}\, x\,,\ x(0) = 1 \quad \Rightarrow \quad x^o(t) = \dfrac{9e^{-4t} - 7e^{-16}}{9 - 7e^{-16}} e^{2t}$ which is the same trajectory as in Problem 17 above (as it should be). And using this in the boxed expression for the feedback control leads to the optimal open-loop control, which is precisely the one arrived at in Problem 17 above (as it should be). Note also that $p(t) = P(t)x^o(t)$

19. $\dot{x}_1 = x_2,\ \dot{x}_2 = u; \quad x_1(0) = x_{10},\ x_2(0) = x_{20}$

$$J = \frac{1}{2}x^T(t_f)\begin{bmatrix} s_{11} & 0 \\ 0 & s_{22} \end{bmatrix} x(t_f) + \frac{1}{2}\int_0^{t_f} u^2(t)\,\mathrm{d}t$$

i) For the finite-horizon version the solution is unique, and is given by

$$u^*(t) = \mu^*(t, x) = -B^T P x$$

$$B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \dot{P} + A^T P + PA - PBB^T P = 0; \quad P(t_f) = \begin{bmatrix} s_{11} & 0 \\ 0 & s_{22} \end{bmatrix}$$

Let $S = P^{-1}$, which satisfies

$$\begin{aligned} \dot{S} &= SA^T + AS - BB^T; \quad S(t_f) = \begin{bmatrix} \frac{1}{s_{11}} & 0 \\ 0 & \dfrac{1}{s_{22}} \end{bmatrix} \\ A &:= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad BB^T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

If

$$\begin{aligned} S &= \begin{pmatrix} S_1 & S_2 \\ S_2 & S_4 \end{pmatrix}, \quad \textit{then} \\ & \left\{ \begin{array}{ll} \dot{S}_1 = 2S_2; & S_1(t_f) = 1/s_{11} \\ \dot{S}_2 = S_4; & S_2(t_f) = 0 \\ \dot{S}_4 = -1; & S_4(t_f) = 1/s_{22} \end{array} \right. \\ \Rightarrow & \\ & S_4 = -(t - t_f) + 1/s_{22} \\ & S_2 = -\frac{1}{2}(t - t_f)^2 + \frac{1}{s_{22}}(t - t_f) + \frac{1}{2}t_f^2 \\ & S_1 = -\frac{1}{3}(t - t_f)^3 + \frac{1}{s_{22}}(t - t_f)^2 + \frac{1}{s_{11}} \\ P &= \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_4 \end{pmatrix}^{-1} = \frac{1}{\underbrace{S_1 S_4 - S_2^2}_{D}} \begin{pmatrix} S_4 & -S_2 \\ -S_2 & S_1 \end{pmatrix} \\ -B^T P &= \frac{1}{S_1 S_4 - S_2^2}(S_2 \quad -S_1) \\ D &:= S_1 S_4 - S_2^2 = \left[\frac{1}{s_{11}} + \frac{1}{3}(t_f - t)^3\right]\left[\frac{1}{s_{22}} + t_f - t\right] - \frac{1}{4}(t_f - t)^4 \end{aligned}$$

Hence,

$$\boxed{\begin{aligned} u^* &= \mu^*(t,x) = -\begin{pmatrix} \dfrac{K_1}{D} & \dfrac{K_2}{D} \end{pmatrix} x \\ K_1 &= -S_2 = \frac{1}{2}(t-t_f)^2 - \frac{1}{s_{22}}(t-t_f) \\ K_2 &= S_1 = \frac{1}{3}(t_f-t)^3 + \frac{1}{s_{22}}(t_f-t)^2 + \frac{1}{s_{11}} \end{aligned}}$$

ii)

$$s_{22} \to 0: \qquad u^*(t) = \mu^*(x) = -\left(\frac{t_f - t}{\dfrac{1}{s_{11}} + \dfrac{1}{3}(t_f-t)^3}, \quad \frac{(t_f-t)^2}{\dfrac{1}{s_{11}} + \dfrac{1}{3}(t_f-t)^3} \right) x$$

$$s_{11} \to 0: \qquad u^*(t) = \mu^*(x) = -\left(0, \quad \frac{1}{\dfrac{1}{s_{22}} + t_f - t} \right) x$$

$$t_f \to \infty: \qquad \begin{aligned} u^* &= 0 \\ &\uparrow \text{this is to be expected since there is} \\ &\quad \text{no intermediate cost on state.} \end{aligned}$$

The optimum cost for $t_f < \infty$ is

$$J^*_{t_f} = \frac{1}{2} x^T(0) P(0,t_f) x(0),$$

where $P(0,t_f)$ is the solution of the Riccati equation, as given above, at $t = 0$, and with t_f considered as a variable. It is not difficult to see that for every fixed s_{11} and s_{22}, $P(0,t_f)$ converges to the *zero* matrix, so that

$$\lim_{t_f \to \infty} J^*_{t_f} = 0.$$

Note, however, that if the limiting control $u^* = 0$ is used, then the corresponding cost would diverge as $t_f \to \infty$, since the open-loop system is not stable:

$$J^* \to \infty.$$

20. Clearly $J(x) \geq 0$, and it is *zero* if either $\dot{x}(t) = 0$ or $\dot{x}(t) = 1$ or both. Integrating these, we have

$$x_1(t) = c_1; \quad x_2(t) = t + c_2.$$

Two possible solutions that satisfy the given boundary conditions are

$$x^\circ(t) = \begin{cases} t; & 0 \leq t \leq \frac{2}{3} \\ \frac{2}{3}; & \frac{2}{3} < t \leq 1 \end{cases} ; \qquad x^\star(t) = \begin{cases} 0; & 0 \leq t \leq \frac{1}{3} \\ t - \frac{1}{3}; & \frac{1}{3} < t \leq 1 \end{cases}$$

each with a single corner, at $t = \frac{2}{3}$ and $t = \frac{1}{3}$, respectively.

The W-E corner conditions are:

i) $\phi_{\dot{x}} = 2\dot{x}[1-\dot{x}]^2 - 2\dot{x}^2[1-\dot{x}]\,|_{x^\circ} = 0$ also $|_{x^\star} = 0$ first condition holds trivially

ii) $\phi - \dot{x}\phi_{\dot{x}} = \dot{x}^2[1-\dot{x}]\,\{1-\dot{x}-2+\dot{x}\} \overset{?}{=} 0$ which again holds trivially for both $x = x^\circ$ and $x = x^\star$.

21.

i) Construct the extremal piecewise over the sub-intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, while making sure that it remains continuous at the point $t = \frac{1}{2}$. The Euler-Lagrange equation (which applies to both sub-intervals) is

$$\frac{d}{dt}[2(\dot{x}_1 - \frac{1}{3})] = 0 \quad \Rightarrow \quad \dot{x}_1 = c_1 \quad \Rightarrow \quad x_1(t) = c_1 t + c_2$$

Over the first sub-interval, the given boundary conditions lead to: $c_2 = 0, c_1 = 2$, and over the second sub-interval, they lead to: $c_1 = 0, c_2 = 1$. Hence, the extremal for x_1 is:

$$x_1^\circ(t) = \begin{cases} 2t, & \text{if } t \in [0, \frac{1}{2}] \\ 1, & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

Since $\phi_{\dot{x}\dot{x}} = 2 > 0$, the strengthened Legendre condition is satisfied.

ii) Because of the mid-point constraint, which was forced on the solution, we cannot expect the Weierstrass-Erdmann corner conditions to be satisfied at the point $t = \frac{1}{2}$ where the extremal is not continuously differentiable. In fact, in the present case they are not satisfied.

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