ECE 553 March 8, 2006

Correspondence # 12

SOLUTION SET TO ASSIGNMENT 4

22. $J(x) = \int_0^{3\pi/2} \left[\dot{x}^2(t) - x^2(t)\right] dt; \quad x(0) = 0,\, x(3\pi/2) = 1.$

The E-L equation is:

$$2\frac{d}{dt}\dot{x} = -2x \Rightarrow \ddot{x} = -x\,.$$

With the given boundary conditions, it admits the unique solution $\dot{x}(t) = -\sin t$ which is the only extremal.

i) Jacobi's equation is: $\ddot{\eta} + \eta = 0$. Taking $\eta(0) = 0$, it admits the parametrized solution $\eta(t) = A\sin t$, where A is a constant. Clearly, the point $t = \pi$ is conjugate to $t = 0$, since $A\sin\pi = 0$ for all A. Hence, the given extremal cannot be a solution to the problem. Since it was the only extremal, it follows that there is no solution to this CV problem.

ii) Since $J(x)$ is quadratic in x and $\dot{x}$,

$$J(x^\circ + \epsilon\eta) \underset{\overset{\uparrow}{exact}}{=} J(x^\circ) + \epsilon\delta J(x^\circ;\eta) + \frac{\epsilon^2}{2}\delta^2 J(x^\circ;\eta)$$

where the first variation δJ is *zero* since x° is the solution of the E-L equation. Hence, $\Delta J := J(x^\circ + \epsilon\eta) - J(x^\circ) = \frac{\epsilon^2}{2}\delta^2 J(x^\circ;\eta)$ and we had already derived an expression for the second variation, $\delta^2 J$, in class:

$$\delta^2 J(x^\circ;\eta) = 2\int_0^{3\pi/2} \left([\dot{\eta}(t)]^2 - [\eta(t)]^2\right) dt$$

With

$$\begin{aligned} \eta(t) &= \sin\tfrac{2t}{3},\ \dot{\eta}(t) = \tfrac{2}{3}\cos\tfrac{2t}{3} \\ \Rightarrow \qquad \delta^2 J &= 2\int_0^{3\pi/2} [(\frac{2}{3})^2\cos^2\frac{2t}{3} - \sin^2\frac{2t}{3}]dt \\ &= 2\int_0^{3\pi/2} [\frac{13}{9}\cos^2\frac{2t}{3} - 1]\,dt = -\frac{5\pi}{6}. \end{aligned}$$

Hence, $\boxed{\Delta J = -\frac{5\pi}{12}\epsilon^2}$ which shows that $J(x^\circ + \epsilon\eta) < J(x^\circ) \quad \forall\epsilon$.

– – Another indication that x°, the extremal, is not a minimizing solution.

23. The Hamiltonian is

$$H(x,u,p) = u^2 + 3u - 2x + p(x+u)$$

with the DE satisfied by the co-state, p, being

$$H_x = -2 + p = -\dot{p}\,,\ p(2) = 0 \quad \Rightarrow \quad p(t) = 2(1 - e^{2-t})\,,\ 0 \le t \le 2$$

If we had no constraints on u, then the unique minimizing control would be (unique, because of strict convexity of H in u):

$$\min_u H(x,u,p) \quad \Rightarrow \quad 2u + 3 + p = 0 \quad \Leftrightarrow \quad u(t) = e^{2-t} - \frac{5}{2} =: \alpha(t)$$

Now, the original constraint, $0 \le u(t) \le 2$, holds for

$\alpha(t) \ge 0 \quad \Leftrightarrow \quad t \le 2 - \ln\frac{5}{2} =: t_2 \quad$ and $\quad \alpha(t) \le 2 \quad \Leftrightarrow \quad t \ge 2 - \ln\frac{9}{2} =: t_1$

Since $t_1 > 0$, $t_2 < 2$, and $H_u < 0 \, \forall \, t > t_2$, $H_u > 0 \, \forall \, t < t_1$, the Hamiltonian-minimizing control would be at its upper bound, 2, for $t < t_1$, and at its lower bound, 0, for $t > t_2$.

This verifies the structural form of u^* as given in the statement of the problem, with α, t_1, t_2 as given above.

24. (i) The Hamiltonian is

$$H(x, u, p) = x + pu$$

with the DE satisfied by the co-state, p, being

$$H_x = 1 = -\dot{p}\,, \; p(2) = 0 \quad \Rightarrow \quad p(t) = 2 - t\,, \; 0 \le t \le 2$$

Furthermore,

$$\min_{|u| \le 1} H(x, u, p) = x + (2 - t) \min_{|u| \le 1} u \quad \Rightarrow \quad u = -1$$

where we have been able to take the multiplying factor $(2 - t)$ outside the min because it is positive for all $0 \le t < 2$. Hence, the optimal control is: $\quad u^*(t) = -1 \quad \forall \, t \in [0, 2] \quad \Rightarrow \quad x^*(t) = 1 - t\,,\quad$ and the optimum value of J is $\quad J(u^*) = 0$.

(ii) Since x has to be nonnegative, clearly the optimal (globally minimizing) control is the one that drives x from 1 to 0 as quickly as possible, and then maintains the value of 0. This is achieved uniquely by the control

$$u^*(t) = -1 \quad \text{for } t < 1 =: t_1\,, \quad \text{and} \quad 0 \quad \text{for } 1 \le t \le 2 \quad \Rightarrow \quad J(u^*) = \frac{1}{2}$$

To write down the minimum principle for this problem, we have to introduce a Lagrange multiplier function, $\lambda(t)$, associated with the state positivity constraint. The corresponding Hamiltonian is:

$$H(x, u, p, \lambda) = (1 - \lambda)x + pu$$

The co-state equation is: $\quad H_x = (1 - \lambda) = -\dot{p}\,, \; p(2) = 0\,$, with the unique solution being

$$p(t) = 2 - t - \int_t^2 \lambda(s) ds$$

The Lagrange multiplier is obtained from the conditions $\lambda(t) \ge 0\,, \; \lambda(t)\, x^*(t) = 0$, which lead to (also using the property that for u^* as given above to minimize the Hamiltonian, $p(t)$ has to be zero in the interval $[1, 2]$, and positive in $[0, 1)$):

$$\lambda(t) = 1 \quad \text{for } 1 \le t \le 2\,, \quad \text{and} \quad 0 \quad \text{for } 0 \le t < 1$$

and

$$p(t) = 1 - t \quad \text{for } 0 \le t < 1\,, \quad \text{and} \quad 0 \quad \text{for } 1 \le t \le 2$$

25. The state equation associated with this minimum time problem is:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & 2 \\ 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u$$

The eigenvalues of A are -1, -2, which are real; hence by *Lemma 7* of *Correspondence # 13*, there can be at most 1 switch.

The switching curve is a curve in the 2-D space, which will be followed by an optimal trajectory toward the origin. If the initial states are on the switching curve, then there will be no switches, and control will be either +1 or -1.

Let us solve for the system trajectory, assuming $u = 1$:

$$\begin{aligned} x_2(t) &= 1 + (x_{20} - 1)e^{-t} \\ x_1(t) &= e^{-2t}[x_{10} + e^{2t_f} - 1 + 2(e^{t_f} - 1)(x_{20} - 1)] \end{aligned}$$

To determine the part of the switching curve that corresponds to $u = 1$, we set $x_2(t_f) = x_1(t_f) = 0$ above, and eliminate t_f:

$$e^{t_f} = 1 - x_{20} \qquad (\text{clearly } x_{20} < 0)$$

Substituting this in the second equation above:

$$\Rightarrow \quad x_{10} = (x_{20})^2 \qquad \begin{array}{l}(\text{This is the locus of intitial points, with } x_{20} < 0, \\ \text{ from which origin can be reached by } u = 1.)\end{array}$$

Similarly, for $u = -1$ we obtain

$$x_{10} = -(x_{20})^2 \qquad (x_{20} \geq 0)$$

Hence, the switching curve is

$$x_{10} + x_{20}|x_{20}| = 0$$

or generally

$$\boxed{x_1 + x_2|x_2| = 0}$$

26.

$$\dot{y} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}_{A'} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}_{B'} u$$

The eigenvalues of A' are -1, -2 — again at most one switch.

Follow the same procedure as in the solution to *Problem 25* above:

$$u = 1: \qquad \begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -2y_1 - 3y_2 + 2 \end{aligned}$$

Since the eigenvalues of A and A' are the same, there is a similarity transformation that relates A to A':

$$A' = \underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}}_{\pi} \underbrace{\begin{pmatrix} -2 & 2 \\ 0 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1/2 \end{pmatrix}}_{\pi^{-1}}$$

Note also that $B' = \pi B$. Hence,

$$y = \pi x,$$

and we can use the solution (for x) obtained in *Problem 25*:

$$\begin{aligned} y_1(t) &= \\ y_2(t) &= \end{aligned} \left\{ \pi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right. \begin{aligned} &= x_1(t) \\ &= 2(x_2(t) - x_1(t)) \end{aligned}$$

$$\begin{aligned} y_1(t) &= e^{-2t}[y_{10} + e^{2t_f} - 1 + 2(e^{t_f} - 1)(\frac{y_{20}}{2} + y_{10} - 1)] \\ y_2(t) &= 1 + \left(\frac{y_{20}}{2} + y_{10} - 1\right) e^{-t} \end{aligned}$$

Again solving for e^{t_f} from $y_2(t_f) = 0$ and substituting it into $y_1(t_f) = 0$, we obtain

$$y_{10} = \left(\frac{y_{20}}{2} + y_{10}\right)^2 \qquad\qquad \left(\frac{y_{20}}{2} < -y_{10}\right)$$

Similarly, for $u = -1$:

$$y_{10} = -\left(\frac{y_{20}}{2} + y_{10}\right)^2 \qquad\qquad \left(\frac{y_{20}}{2} \geq -y_{10}\right)$$

In general

$$\boxed{y_1 + \tfrac{1}{4}|y_2 + 2y_1|(y_2 + 2y_1) = 0}$$

Switching curve.

For the last part of the problem, substitute $y_1 = x_1$, $y_2 = 2x_2 - 2x_1$ above, and recover the switching curve of *Problem 25*.

27.

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}\right\}, \quad |u| \leq 1, \ t_f \text{ is free}$$

a) Time-optimal control, with $x_1(t_f) = 1$, $x_2(t_f)$ free:

$$H = p_1 x_2 + p_2 u$$

$$\left.\begin{aligned} \dot{p}_1 = 0, &\quad p_1(t_f) \text{ free} \\ \dot{p}_2 = -p_1, &\quad p_2(t_f) = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} p_1 &= c \quad \text{(a constant)} \\ p_2 &= c(t_f - t) \end{aligned}$$

Optimal Control: $u^\circ(t) = -sgn(p_2(t))$.
Since p_2 is never zero (except at $t = t_f$), there is no switch, and hence given any initial condition we have either $u = 1$ or $u = -1$. To determine the corresponding regions in the space of initial conditions (states), we also have to use the time-optimality property:

$$H|_{t=t_f} = -1 \quad \Rightarrow \quad cx_2(t_f) = -1.$$

Now,

$$c < 0 \Rightarrow u^o = 1 \Rightarrow \left\{ \begin{aligned} x_2 &= t + x_{20} \\ x_1 &= \tfrac{t^2}{2} + x_{20}t + x_{10} \end{aligned} \right.$$

By eliminationg the variable t, we obtain the following curve in the state space:

$$2x_1 = x_2^2 - x_{20}^2 + 2x_{10}.$$

A plot of this in view of the fact $\dot{x}_2 = 1 > 0$, and the terminal condition $x_1(t_f) = 1$, leads to the conclusion that $u = 1$ is optimal for

$$\boxed{x_{10} < 1.} \qquad\qquad (u^\circ = 1).$$

Likewise, $c > 0 \Rightarrow u = -1$

$$\Rightarrow 2x_1 + x_2^2 = 2x_{10} + x_{20}^2$$

$$\Rightarrow \boxed{x_{10} > 1} \qquad\qquad (u^\circ = -1)$$

b) If the terminal condition is changed to

$$x_2(t_f) = 0, \quad x_1(t_f) \text{ free },$$

using the same Hamiltonian, the co-state equations become

$$\left.\begin{array}{ll} \dot{p}_1 = 0, & p_1(t_f) = 0 \\ \dot{p}_2 = -p_1, & p_2(t_f) \text{ free} \end{array}\right\} \Rightarrow \begin{array}{lcll} p_1 & \equiv & 0 & \\ p_2 & = & c & \text{(a constant)} \end{array}$$

$$H|_{t=t_f} = -1 \quad \Rightarrow \quad -c\, sgn(c) = -1 \Rightarrow c = \pm 1$$

$$c < 0 \Rightarrow u^o = 1 \Rightarrow x_2 = t + x_{20}$$

and

$$x_2(t_f) = 0 \Rightarrow t_f + x_{20} = 0$$

Since $t_f \geq 0$, we need $x_{20} < 0$.
Likewise, $c > 0 \Rightarrow u^o = -1 \Rightarrow x_{20} > 0$

$$\therefore \boxed{u^\circ(t) = -sgn(x_{20})}$$

28.

$$\left.\begin{array}{l} \dot{x}_1 = x_2 + u_1 \\ \dot{x}_2 = -x_1 + u_2 \end{array}\right\} \quad |u_1| \leq 1, \quad |u_2| \leq 1$$

$$H = p_1(x_2 + u_1) + p_2(-x_1 + u_2)$$

$$\Rightarrow \left.\begin{array}{lcl} \dot{p}_1 & = & p_2 \\ \dot{p}_2 & = & -p_1 \end{array}\right\} \Rightarrow \begin{array}{lcl} p_1 & = & A\sin t + B\cos t \\ p_2 & = & \dot{p}_1 = A\cos t - B\sin t \end{array}$$

$$u_1^\circ = -sgn(p_1); \qquad u_2^\circ = -sgn(p_2).$$

The minimum time condition:

$$H|_{t=f_f} = -1 \qquad \underset{\substack{\uparrow \\ \text{since } x_1(t_f) = x_2(t_f) = 0}}{\Rightarrow} \qquad p_1(t_f) + p_2(t_f) = +1$$

$$\Leftrightarrow (A - B)\sin t_f + (A + B)\cos t_f = 1.$$

Since p_1 and p_2 are out of phase by $\frac{\pi}{2}$, u_1° and u_2° will not switch simultaneously, but at alternate times, $\frac{\pi}{2}$ apart. The control pair (u_1, u_2) takes four possible values: $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$. For each such choice, the trajectories (that make up the phase diagram) are circular:

$$(x_2 + u_1)^2 + (x_1 - u_2)^2 = c^2$$

where c^2 is determined by the initial conditions.

The complete picture is as in Fig. 5.3-1 (p. 108) of *Sage and White* (Correspondence 14), but with switching curves not only on the x_1 axis but also on the x_2 axis. There is no easy upper bound on the number of switches.

◇ ◇ ◇