SOLUTION SET TO ASSIGNMENT 5

29. The optimal control problem is:

$$\dot{x} = -x + u - 1, \quad x(0) = 1$$

$$J(u) = [x(1)]^2 + \int_0^1 u^2(t)\,\mathrm{d}t \to \text{ minimize}$$

i) The HJB equation is

$$\begin{cases} -\dfrac{\partial V}{\partial t} = \min\limits_u \left\{ \dfrac{\partial V}{\partial x} \cdot (-x + u - 1) + u^2 \right\} \\ V(1, x) = x^2 \end{cases}$$

Minimizing the RHS leads to

$$\frac{\partial V}{\partial x} + 2u = 0 \Rightarrow u = -\frac{1}{2} \cdot \frac{\partial V}{\partial x}$$

Substitute this into the HJB equation:

$$\begin{cases} -\dfrac{\partial V}{\partial t} = -\dfrac{1}{4}\left(\dfrac{\partial V}{\partial x}\right)^2 - \dfrac{\partial V}{\partial x}(1 + x) \\ V(1, x) = x^2 \end{cases}$$

This admits the solution

$$V(t, x) = p(t)x^2 + 2k(t)x + m(t)$$

where

$$\begin{aligned} &\dot{p} - p^2 - 2p = 0, \quad p(1) = 1 \\ &\dot{k} - kp - p - k = 0, \quad k(1) = 0 \\ &\dot{m} - k^2 - 2k = 0, \quad m(1) = 0 \end{aligned}$$

$$\Rightarrow \quad \boxed{p(t) = \frac{2e^{2t}}{3e^2 - e^{2t}}; \quad k(t) = \frac{2e^t(e - e^t)}{3e^2 - e^{2t}}}$$

and

$$\boxed{u^*(t) = \mu^*(t, x) = -p(t)\,x - k(t)} \quad \text{the optimal feedback solution}$$

ii) Let $x^*(t)$, $0 \le t \le 1$, be the optimal trajectory. It solves:

$$\dot{x}^* = -(1 + p(t))x^* - k(t) - 1; \quad x^*(0) = 1$$

A "representation" of μ^* on this trajectory is

$$u(t) = \mu(t, x) = -p(t)x - k(t) + \beta(t)[x - x^*(t)],$$

which is an optimal controller for every $\beta(\cdot)$. In particular, choosing

$$\beta(t)x^*(t) = -k(t), \quad \forall\, t \text{ except when } x^*(t) = 0,$$

makes the controller depend linearly on x:

$$u(t) = [-p(t) + \beta(t)]\, x$$

Therefore,

$$\boxed{\alpha(t) = -p(t) - \frac{k(t)}{x^*(t)}} \quad \forall t,\ x^*(t) \neq 0,$$

where

$$x^*(t) = \frac{(6e^2 - e)e^{-t} + 1 + (-2 + e - 3e^2)e^t}{3e^2 - 1} + \frac{3e^2 e^{-t} - e^t}{e\sqrt{3}} \ln\left(\frac{\sqrt{3}e^{1-t} - 1}{\sqrt{3}e^{1-t} + 1} \cdot \frac{\sqrt{3}e + 1}{\sqrt{3}e - 1} \right)$$

Another approach would be to substitute for the control $u(t) = \alpha(t)x(t)$ into the state equation and the cost function:

$$\dot{x} = (\alpha - 1)x - 1\,, \quad x(0) = 1$$

$$J(u = \alpha x) =: F(\alpha) = [x(1)]^2 + \int_0^1 \alpha(t)^2 x(t)^2 dt$$

and then minimize F with respect to α. Note that this is another optimal control problem, but of the open-loop type, where now α is the control. We can use the minimum principle to solve this problem, which leads to exactly the same solution as above.

30. $\dot{x} = xu, \quad x(0) = 1; \quad J(u) = [x(1)]^2 + \int_0^1 [x(t)u(t)]^2 \, \mathrm{d}t.$

Letting $xu = v$, this becomes a standard LQ control problem in v; therefore the optimal solution for v is a linear function of $x \Rightarrow$ optimal u is just a time function.

Now let us obtain this solution directly from the associated HJB equation:

$$-\frac{\partial V}{\partial t} = \min_u \left\{ \frac{\partial V}{\partial x} \cdot xu + (xu)^2 \right\}; \quad V(1, x) = x^2.$$

The minimizing control is:

$$\frac{\partial V}{\partial x} \cdot x + 2x^2 u = 0 \Rightarrow u = -\frac{1}{2x} \cdot \frac{\partial V}{\partial x}$$

where we have assumed that $x \neq 0$ (which indeed can be shown to be the case). Then, the HJB equation becomes:

$$-\frac{\partial V}{\partial t} = -\frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2; \quad V(1, x) = x^2. \qquad (\star)$$

Choose $V(t, x) = p(t)x^2$, and use this in $(\star)$:

$$-\dot{p}x^2 = -p^2 x^2; \quad p(1) = 1$$

$$\Rightarrow \qquad \dot{p} = p^2; \quad p(1) = 1 \Rightarrow p(t) = \frac{1}{2 - t}.$$

Hence,

$$u^*(t) = -\frac{1}{2x}\frac{\partial V}{\partial x} = -\frac{2}{2(2 - t)}$$

$$\Rightarrow \qquad \boxed{u^*(t) = -\frac{1}{2 - t}, \quad 0 \leq t \leq 1}$$

31 We are given the affine system dynamics

$$\dot{x} = Ax + Bu + c(t)\,, \quad x(0) = x_0\,, \quad t \geq 0\,,$$

and generalized quadratic cost function (to be minimized)

$$J(u) = |x(t_f)|^2_{Q_f} + 2x^T(t_f)q_f + \int_0^{t_f} \left(|x(t)|^2_{Q(t)} + 2x(t)^T M(t)u(t) + 2x(t)^T q(t) + |u(t)|^2_{R(t)} \right) dt\,,$$

The associated HJB equation is

$$-V_t = \min_u \left\{ V_x \cdot \big(Ax + Bu + c\big) + |x(t)|^2_{Q(t)} + 2x(t)^T M(t)u(t) + 2x(t)^T q(t) + |u(t)|^2_{R(t)} \right\}$$

with boundary condition $V(x, t_f) = |x(t_f)|^2_{Q_f} + 2x^T(t_f)q_f$.

Since $R > 0$, the minimization on the RHS of the HJB equation leads to the unique solution (note that what is being minimized is a strictly convex quadratic function of u):

$$u = -R^{-1}\big[M^T x + \frac{1}{2}B^T V_x^T(x,t)\big]$$

It is fairly straightforward to show that the *optimum cost-to-go* function is of the form

$$V(x,t) = x^T P(t)x + 2k^T(t)x + 2m(t)$$

where P, k, and m are obtained as solutions of

$$\dot{P} + PA + A^T P - (PB + M)R^{-1}(PB + M)^T + Q = 0\,, \quad P(t_f) = Q_f$$

$$\dot{k} + \big(A^T - (PB + M)R^{-1}B^T\big)k + Pc + q = 0\,, \quad k(t_f) = q_f$$

$$\dot{m} + k^T c - \frac{1}{2}k^T BR^{-1}B^T k = 0\,, \quad m(t_f) = 0$$

Simply substitute the given general quadratic form into the HJB equation along with the expression for u above, and show that we have an identity in x when P, k, and m satisfy the differential equations above. Now, since $V_x(x,t) = 2x^T P(t) + 2k^T(t)$, the optimal feedback control is:

$$u(t) = \mu^*(x,t) = -R^{-1}\big[M^T x + B^T\big(P(t)x + k(t)\big)\big] \equiv -R^{-1}\big[\big(M^T + B^T P(t)\big)x + B^T k(t)\big]$$

32. In terms of the notation introduced in class:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ \beta \end{pmatrix}, \quad C^T = \begin{pmatrix} \rho \\ 1 \end{pmatrix}, \quad R = 1.$$

For $\alpha < 0$, A is stable, and hence (A, B) is stabilizable and (A, C) is detectable for all β, ρ.
For $\alpha \geq 0$, (A, B) is *stabilizable* if, and only if,

$$\text{rank}(A - \alpha I \mid B) = 2 \Leftrightarrow \text{rank}\begin{pmatrix} -1-\alpha & 1 & 1 \\ 1 & 0 & \beta \end{pmatrix} = 2 \quad \Leftrightarrow \quad \boxed{\beta \neq 0}$$

For $\alpha \geq 0$, (A, C) is *detectable* if, and only if,

$$\text{rank}(A^T - \alpha I \mid C^T) = 2 \Leftrightarrow \text{rank}\begin{pmatrix} -1-\alpha & 0 & \rho \\ 1 & 0 & 1 \end{pmatrix} = 2 \quad \Leftrightarrow \quad \boxed{\rho + 1 + \alpha \neq 0}.$$

i) From above, either $\alpha < 0$, or $\alpha \geq 0$ and $\beta \neq 0$, $\rho + \alpha + 1 \neq 0$.

ii) For the solution of ARE to be positive definite, we need (A, B) to be stabilizable and (A, C) observable. For the latter, we need

$$\text{rank}(C^T \mid A^T C^T) = 2 \Leftrightarrow \text{rank}\begin{pmatrix} \rho & -\rho \\ 1 & \rho+\alpha \end{pmatrix} = 2 \quad \Leftrightarrow \quad \rho+\alpha+1 \neq 0\,;\ \rho \neq 0$$

Hence, we have a unique positive definite solution to the ARE if, and only if,

$$\boxed{\rho \neq 0,\ \rho+\alpha+1 \neq 0 \text{ and either } \alpha < 0 \text{ or } \alpha \geq 0 \text{ and } \beta \neq 0.}$$

iii) For this we need (A, B) stabilizable, and no unobservable eigenvalues of A on the imaginary axis. For the latter it will suffice to check only the case $\alpha = 0$, and see whether $\lambda = 0$ is an unobservable eigenvalue. For it to be unobservable, from the Hautus test:

$$\text{rank}\begin{pmatrix} -1 & 0 & \rho \\ 1 & 0 & 1 \end{pmatrix} < 2 \Leftrightarrow \rho = -1.$$

Hence, a stabilizing control in the given form exists if, and only if,

$$\boxed{\text{either } \alpha < 0 \text{ or } \alpha = 0,\ \rho \neq -1,\ \beta \neq 0 \text{ or } \alpha > 0,\ \beta \neq 0.}$$

33. Let $v := u + x_1$, and view it as the new control variable. In terms of v:

$$\begin{aligned} J(v) &= \int_0^\infty (|x|^2_{\widetilde{Q}} + v^2)dt \\ \dot{x} &= \widetilde{A}x + Bv \end{aligned}$$

where

$$\widetilde{Q} = \begin{pmatrix} \rho^2 - 1 & \rho \\ \rho & 2 \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} -2 & 1 \\ \beta & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ \beta \end{pmatrix}.$$

We first need $\widetilde{Q} \geq 0 \Leftrightarrow \rho^2 \geq 2$.
We know from the discussion in class that $(\widetilde{A}, B)$ is stabilizable if, and only if, (A, B) is stabilizable, and hence here the results of Problem 34 apply. Therefore, we now turn to checking detectability (and observability).

For $\rho^2 > 2$, $\widetilde{Q} > 0$ and hence trivially $(\widetilde{A}, \widetilde{C})$ is observable for any $\widetilde{C}$ such that $\widetilde{C}^T\widetilde{C} = \widetilde{Q}$. For $\rho^2 = 2$, $\widetilde{Q}$ has rank 1, and hence can be written as $\widetilde{Q} = (1 \quad \sqrt{2})^T(1 \quad \sqrt{2})$, which says that

$$\widetilde{C} \;=\; (1 \quad \sqrt{2}).$$

$(\widetilde{A}, \widetilde{C})$ is observable $\Leftrightarrow$ rank $\begin{pmatrix} 1 & -2+\beta\sqrt{2} \\ \sqrt{2} & 1+\alpha\sqrt{2} \end{pmatrix} = 2 \quad \Leftrightarrow \quad 1 + 2\sqrt{2} + \alpha\sqrt{2} - 2\beta \neq 0.$

For detectability, we have to check the rank condition

$$\text{rank}(\widetilde{A}^T - \lambda I \mid \widetilde{C}^T) = 2$$

for any unstable eigenvalue λ of $\widetilde{A}$. This is equivalent to

$$\text{rank}\begin{pmatrix} -2-\lambda & \beta & 1 \\ 1 & \alpha-\lambda & \sqrt{2} \end{pmatrix} = 2.$$

If this does not hold, then the last column has to be linearly dependent on both the first and second columns. The former is possible if, and only if,

$$-\sqrt{2}(2+\lambda) = 1 \quad \Rightarrow \quad \lambda = -\frac{\sqrt{2}}{2} - 2$$

that is, only if an eigenvalue of $\widetilde{A}$ is negative. Hence, $(\widetilde{A}, \widetilde{C})$ is detectable for all range of parameters. In view of this, we have the following conditions for the three parts:

i) $\rho^2 \geq 2$ and either $\alpha < 0$ or $\alpha \geq 0$ and $\beta \neq 0$

ii) $\rho^2 \geq 2$, $1 + 2\sqrt{2} + \alpha\sqrt{2} - 2\beta \neq 0$, and either $\alpha < 0$ or $\alpha \geq 0$ and $\beta \neq 0$.

iii) Same as part (i).

34. i) Let us first check stabilizability of (A, B). This is true if, and only if, we can find an $r \times n$ matrix K such that $A + BK$ is stable. Write K in the structural form $K = [K_1 \; K_2]$, where K_1 is $r \times n_1$ and K_2 is $r \times n_2$. Then,

$$A + BK = \begin{pmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ 0 & A_{22} \end{pmatrix}$$

which is stable if, and only if, $A_{11} + B_1 K_1$ and A_{22} are both stable. But this is indeed the case because A_{22} is given to be stable, and (A_{11}, B_1) is given to be controllable (which implies that there exists a K_1 such that $A_{11} + B_1 K_1$ is stable).
Next, let us check observability of (A, Q). But, this is clearly the case since Q is of rank n (since $Q = I_n$). Hence, (A, B) is stabilizable and (A, Q) is observable. This then implies that the ARE admits a unique positive-definite solution, $\bar{P}$, and $A - BR^{-1}B^T\bar{P} = A - BB^T\bar{P}$ is stable.

ii) The ARE is

$$A^T P + PA - PBB^T P + I = 0$$

Let its unique positive-definite solution be $\bar{P}$, and note that for the given structural form, we have:

$$\bar{P}A = \begin{pmatrix} \bar{P}_{11} A_{11} & \bar{P}_{11} A_{12} + \bar{P}_{12} A_{22} \\ \bar{P}_{12}^T A_{11} & \bar{P}_{12}^T A_{12} + \bar{P}_{22} A_{22} \end{pmatrix} \quad \text{and} \quad \bar{P}B = \begin{pmatrix} \bar{P}_{11} B_1 \\ \bar{P}_{12}^T B_1 \end{pmatrix},$$

In view of this, the ARE can be written equivalently as follows:

$$A_{11}^T \bar{P}_{11} + \bar{P}_{11} A_{11} - \bar{P}_{11} B_1 B_1^T \bar{P}_{11} + I = 0$$
$$\left(A_{11} - B_1 B_1^T \bar{P}_{11}\right)^T \bar{P}_{12} + \bar{P}_{12} A_{22} = -\bar{P}_{11} A_{12}$$
$$A_{22}^T \bar{P}_{22} + \bar{P}_{22} A_{22} = \bar{P}_{12}^T B_1 B_1^T \bar{P}_{12} - I - A_{12}^T \bar{P}_{12} - \bar{P}_{12}^T A_{12}$$

The first of these is an ARE, which admits a unique positive-definite solution, $\bar{P}_{11}$ (unique in the class of nonnegative-definite matrices), since (A_{11}, B_1) is stabilizable and (A_{11}, I) is observable. Furthermore, the matrix $A_{11} - B_1 B_1^T \bar{P}_{11}$ is stable. The second one is a linear equation in $\bar{P}_{12}$, which again admits a unique solution since the two matrices that multiply $\bar{P}_{12}$ are both stable (and hence invertible). Finally, the last equation is again a linear equation, in $\bar{P}_{22}$, which is in fact a Lyapunov equation. Since A_{22} is stable, this Lyapunov equation also admits a unique solution. We cannot conclude that it is positive definite immediately from this equation, without using the solution $\bar{P}_{12}$, since we need to show that the symmetric matrix $\bar{P}_{12}^T B_1 B_1^T \bar{P}_{12} - I - A_{12}^T \bar{P}_{12} - \bar{P}_{12}^T A_{12}$ is negative definite (WHY ?). We already know, however, from part (i) that $\bar{P}_{22}$ is indeed positive-definite, because $\bar{P}$ is. This then implies that the matrix $\bar{P}_{12}^T B_1 B_1^T \bar{P}_{12} - I - A_{12}^T \bar{P}_{12} - \bar{P}_{12}^T A_{12}$ is negative definite.

35. (A, B) is controllable, and (A, Q) is observable, and hence the ARE admits a unique positive-definite solution, $\bar{P}$, and the optimum system feedback matrix $A - BB^T\bar{P}$ is stable. Writing out the ARE in terms of the elements of the P matrix, we have:

$$-(p_{12})^2 + q = 0$$
$$p_{11} - 10p_{12} - p_{12}p_{22} = 0$$
$$2(p_{12} - 10p_{22}) - (p_{22})^2 = 0$$

where p_{ij} is the ij-th entry of the matrix P. These will have to be solved subject to the additional condition that P is positive definite, which translates to:

$$p_{11} > 0\,, \quad p_{11}p_{22} > (p_{12})^2$$

The solution is unique, and is given by:

$$p_{12} = \sqrt{q}\,, \quad p_{22} = -10\left(1 - \sqrt{1 + \frac{1}{50}\sqrt{q}}\right)\,, \quad p_{11} = (10 + p_{22})\sqrt{q}$$

Then, the optimal controller is:

$$\mu^*(x) = -B^T P x = -[p_{12} \;\; p_{22}]x = -\begin{bmatrix} \sqrt{q} & \sqrt{100 + 2\sqrt{q}} - 10 \end{bmatrix} x$$

Furthermore,

$$\det[sI - (A - BB^T\bar{P})] = s^2 + \left(\sqrt{100 + 2\sqrt{q}}\right) s + \sqrt{q}$$

which is the characteristic equation of the optimum system feedback matrix.

◇ ◇ ◇