Paradigms for Robustness in Controller and Filter Designs

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OUTLINE

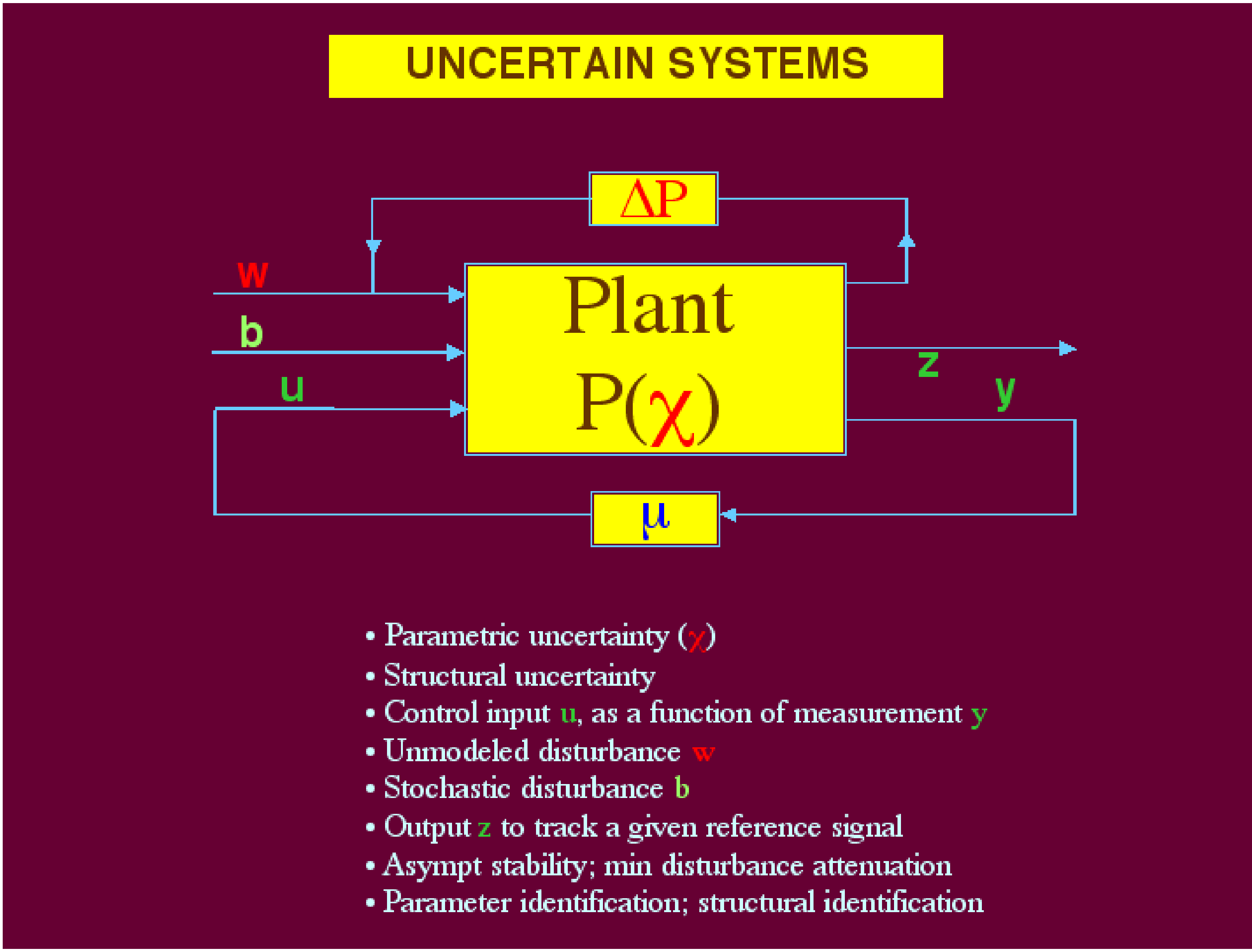
- **Various Paradigms for Robustness**
  - •• Minimax vs Risk sensitive
  - •• Disturbance attenuation; small gain; local vs global performance
  - •• Basic results on linear H^∞ control

- **RS Control and Connections with Worst-Case Designs**
  - •• An illustrative example
  - •• Bellman eqn; threshold on RS parameter
  - •• Relationship with stochastic DGs
  - •• Relationship with nonlinear H^∞ control

- **Robust Deterministic and RS Filtering**
  - •• Filter with deterministic uncertainty
  - •• RS filter with stochastic uncertainty
  - •• Relationship with stochastic DGs
  - •• Robustness to modeling uncertainty

- **Extensions of the Paradigm**
  - •• Piecewise deterministic systems
  - •• Robust adaptive control

- **What lies ahead?**



Error to be optimized

$$e(\chi, w, x_0; b, \xi; u)$$

Regulation	**Tracking**
Estimation	**Filtering**

Fundamental Issue : Drive error to zero

$$e(\chi, w, x_0; b, \xi; u)$$

1. **Hard-bounded uncertainty**

$$\min_{u} \max_{(\chi, w, x_0) \in \Omega} E_{b,\xi} \|e(\chi, w, x_0; b, \xi; u)\|^2$$

2. **Soft-bounded uncertainty**

disturbance attenuation, dissipation

$$E_{b,\xi} \|e(\chi, w, x_0; b, \xi; u^\circ)\|^2 \leq \gamma^2 \|w, \chi, x_0\|^2$$

$\Rightarrow$

$$\min_{u} \max_{(\chi, w, x_0)} \left\{ E_{b,\xi} \|e(\chi, w, x_0; b, \xi; u)\|^2 - \gamma^2 \|w, \chi, x_0\|^2 \right\}$$

3. **Risk sensitive**

emphasis on unfavorable sample paths

$$\min_{u} E_{b,\xi} \exp\left\{\theta \|e(0, 0, x_0; b, \xi; u)\|^2\right\}$$

- **Can we connect 2 and 3 ?**

Small Gain Theorem

Perturbed Plant $\dot{x} = Ax + Bu\,, \;\; y = Cx$

$$A = A_0 + \Delta A, \quad << \Delta A >> := \bar{\sigma}(\Delta A) \le a\,,$$
$$B = B_0 + \Delta B, \quad << \Delta B >> \le b\,.$$

$$\dot{x} = A_0 x + B_0 u + w\,, \;\; y = Cx$$

$$z = \begin{pmatrix} x \\ u \end{pmatrix}\,, \;\; w = (\Delta A \quad \Delta B)\, z \equiv \Delta P\, z$$

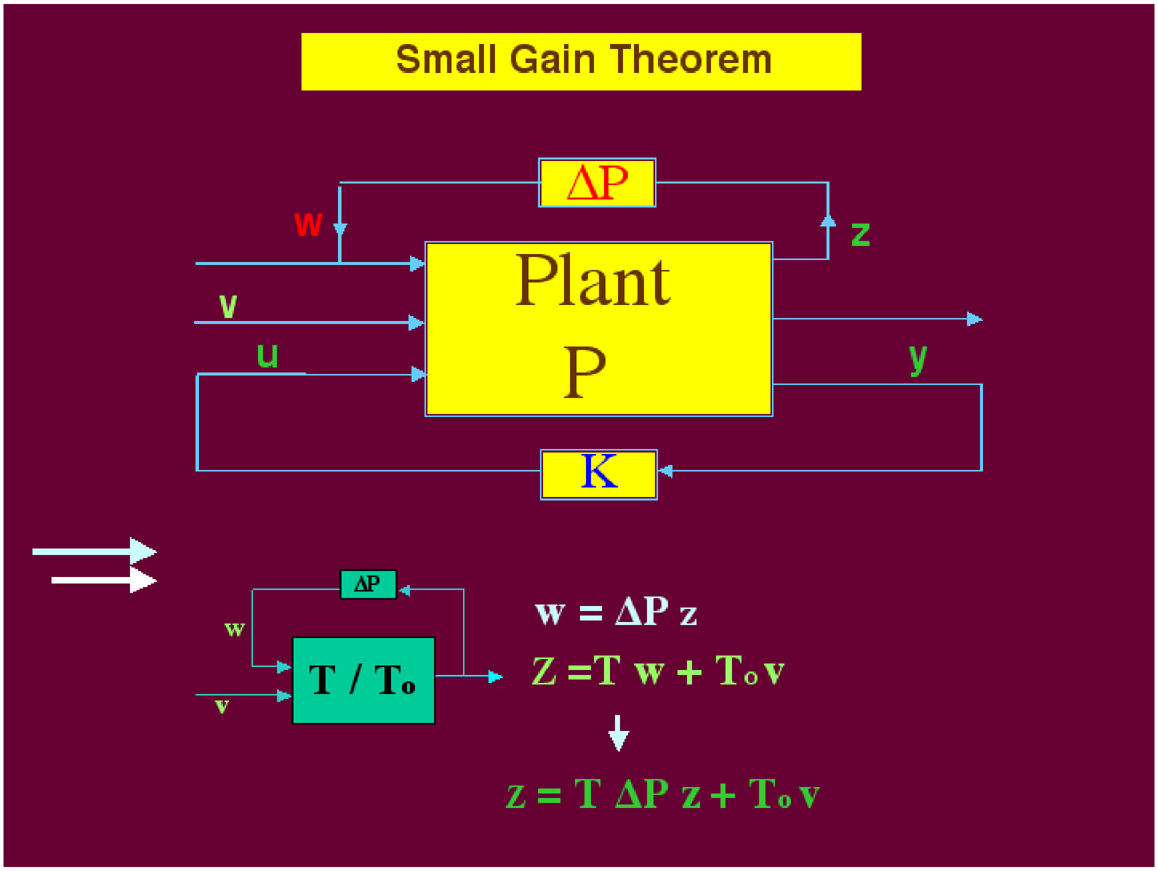
Control $u = Ky \Rightarrow$ **CL transfer function** T_K**,**
extra input $v \Rightarrow \;\; z = T_K\, w + T_0\, v \;\Rightarrow$

$$z = T_K \Delta P\, z + T_0 v \quad \text{or} \quad w = \Delta P\, T_K w + \Delta P\, T_0 v\,.$$

SMALL GAIN THEOREM.

Let T_K and ΔP be linear and stable.
If either $\|T_K\, \Delta P\|_\infty < 1$ or $\|\Delta P\, T_K\|_\infty < 1$, then the combined system is stable.

$\Rightarrow$ **If $<< \Delta P >> \le 1/\gamma$, choose K such that** $\|T_K\|_\infty \le \gamma$



Standard H$^\infty$ Control Problem

$$\dot{x} = Ax + Bu + Dw\,, \quad z = \begin{pmatrix} Hx \\ u \end{pmatrix}\,, \quad u = Kx$$

$\|z\| \leq \gamma \|w\| \quad \Rightarrow$

$$\|z\|^2 - \gamma^2\,\|w\|^2 = \int_0^\infty \{|Hx|^2 + |u|^2 - \gamma^2 |w|^2\} \leq 0$$

(A, B) stabilizable, (A, H) detectable $\quad \Rightarrow$

Unique minimal nnd solution $\bar{Z}$ for $\gamma > \bar{\gamma}$:

$$A'\bar{Z} + \bar{Z}A + H'H - \bar{Z}\,(\,BB' - \gamma^{-2}DD'\,)\,\bar{Z} = 0$$

Optimal (DA) (stabilizing) controller:

$\mu^*(x) = K\,x = -B'\bar{Z}\,x \qquad (w^* = \gamma^{-2}D'\bar{Z}\,x)$

allows for perturbations $<< \Delta P >> \leq 1/\gamma$

Risk sensitivity and exponentiated loss

Loss function: $L(u, \xi)$; **Measurement:** y

Decision (control) rule: $u = \mu(y)$

Risk-neutral cost: $\min_\mu E_{\xi|y} L(\mu(y), \xi)$

Risk-sensitive cost:

$$J_\theta(\mu, y) = \frac{1}{\theta} \ln E_{\xi|y} \{\exp \theta L(\mu(y), \xi)\}$$

$\Rightarrow$ **minimize over decision rules** μ

θ **: risk sensitivity parameter**

Around $\theta = 0$**:**

$$J_\theta(\mu, y) \sim E_{\xi|y} L + \frac{\theta}{2} \mathrm{var} L + O(\theta^2)$$

$\theta = 0$ **: risk-neutral**

$\theta < 0$ **: risk-seeking (optimistic)**

$\theta > 0$ **: risk-averse (pessimistic)**

If (ξ, y) **is jointly Gaussian distributed :**

$$\theta \exp J_\theta \sim \int e^{\theta L(\mu(y), \xi)} e^{-\frac{1}{2\sigma^2}(\xi - \alpha y)^2} d\xi$$

For finiteness, we need :

$$\theta L(\mu(y), \xi) < \frac{1}{2\sigma^2} \xi^2 \quad \text{as } |\xi| \to \infty$$

An Example

$$L(u,\xi) = (u-\xi)^2 + u^2\,; \quad \xi \sim N(1,1)$$

No measurement. **Threshold on θ is $1/2$.**

$J_\theta(u)$ **unbounded for all u if $\theta \geq 1/2$**

For $\theta < 1/2$, unique minimum is:

$$u_\theta = \frac{1}{2(1-\theta)}\,; \quad J_\theta(u_\theta) = \frac{1}{2(1-\theta)} - \frac{\ln(1-2\theta)}{2\theta}$$

$\Rightarrow$ $J_\theta(u_\theta)$ **monotonically increasing in θ**

Consider the stochastic zero-sum game ($\theta > 0$):

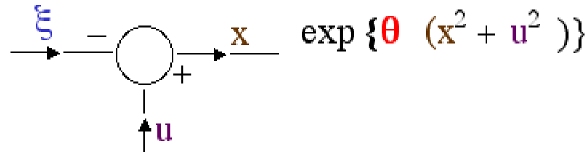
$$L_\theta(u,w,\xi) = (u+w-\xi)^2 + u^2 - \frac{1}{2\theta}w^2$$

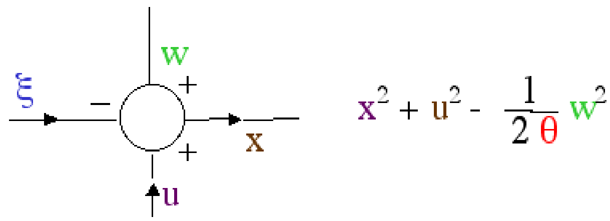
$E[L(u,w,\xi)]$**:** **Min wrt u and max wrt w**

$$\Rightarrow \quad u_\theta = \frac{1}{2(1-\theta)}\,; \quad w_\theta = -\frac{\theta}{1-\theta}\,; \qquad \theta < 1/2$$

$$\Rightarrow E\big[\,(u_\theta + w - \xi)^2 + {u_\theta}^2\,\big] \leq \frac{1}{2\theta}\,w^2 + \frac{3-2\theta}{2(1-\theta)} \qquad \forall\, w$$

THE TWO FORMULATIONS ARE EQUIVALENT FOR u_θ **!**





Now consider the case $\theta < 0$:

Risk-seeking

$$L_\theta(u, w, \xi) = (u + w - \xi)^2 + u^2 - \frac{1}{2\theta} w^2$$

$E[L(u, w, \xi)]$: **Minimize wrt u and w**

$$\Rightarrow \quad u_\theta = \frac{1}{2(1-\theta)}\,; \quad w_\theta = -\frac{\theta}{1-\theta}\,; \qquad \theta < 0$$

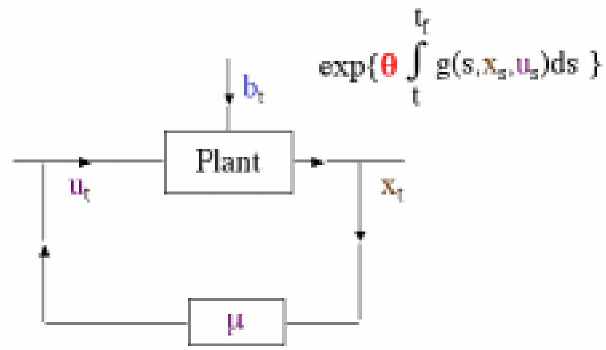
AGAIN THE TWO FORMULATIONS ARE

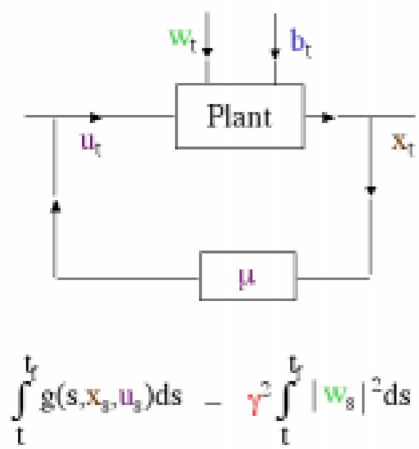
EQUIVALENT FOR u_θ

Risk-Neutral Case $\theta = 0$

$$u = \frac{1}{2}\,; \qquad w = 0$$

Natural extension to the dynamic case !!





RS Problem

$$dx_t = f(t, x_t, u_t)\, dt + \sqrt{\epsilon}\, D\, db_t\, ; \qquad x_{t|t=0} = x_0$$

$b_t, t \geq 0$, **standard Wiener process;** $\epsilon > 0$**;**

$u_t \in U,\ t \geq 0$ **(state FB control law** $\mu \in \mathcal{M}$**)**

Objective : Choose μ **to minimize : (** $\theta > 0$ **)**

$$J(\mu; t, x_t) = \frac{2\epsilon}{\theta} \ln E\{\exp \frac{\theta}{2\epsilon} L(x_{[t,t_f]}, u_{[t,t_f]})\}$$

$$L(x_{[t,t_f]}, u_{[t,t_f]}) := q(x_{t_f}) + \int_t^{t_f} g(s, x_s, u_s) ds$$

R-S Stochastic Control

State dynamics :

$$dx_t = f(t, x_t, u_t)\,dt + \sqrt{\epsilon}\,D\,db_t\,; \qquad x_{t|t=0} = x_0$$

$b_t, t \geq 0$, **standard Wiener process;** $\epsilon > 0$**;**

$u_t \in U,\ t \geq 0$ **(state FB control law** $\mu \in \mathcal{M}$**)**

Objective : Choose μ **to minimize : (** $\theta > 0$ **)**

$$J(\mu; t, x_t) = \frac{2\epsilon}{\theta} \ln E\{\exp \frac{\theta}{2\epsilon} L(x_{[t,t_f]}, u_{[t,t_f]})\}$$

$$L(x_{[t,t_f]}, u_{[t,t_f]}) := q(x_{t_f}) + \int_t^{t_f} g(s, x_s, u_s)ds$$

$\psi(t; x)$ – **value function associated with**

$$E\{\exp \frac{\theta}{2\epsilon}\big[q(x_{t_f}) + \int_t^{t_f} g(s, x_s, u_s)\,ds\big]\}$$

$$\Rightarrow \qquad V(t; x) := \inf_{\mu \in \mathcal{M}} J(\mu; t, x) =: \frac{2\epsilon}{\theta} \ln \psi(t; x)\,,$$

DP and Itô differentiation rule $\Rightarrow$

$$-V_t(t; x) = \inf_{u \in U} \big\{V_x(t; x)\, f(t, x, u) + g(t, x, u)\big\} + \frac{1}{4\gamma^2}|DV_x'(t; x)|^2 + \frac{\epsilon}{2}\mathrm{Tr}\big[V_{xx}\,DD'\big]$$

$$V(t_f; x) \equiv q(x) \qquad \big(\gamma^{-2} := \theta\big)$$

If $U = \mathbf{R}^{m_1}$, f linear in u, and g quadratic in u :

$f(x,u) = f_0(t,x) + B(t,x)u\,;\ g(t,x,u) = g_0(t,x) + |u|^2$

Optimal control law:

$$u^*(t) = \mu^*(t,x) = -\frac{1}{2}\,B'(t,x)\,V_x'(t;x)\,, \quad 0 \le t \le t_f$$

$\Rightarrow$ **HJB equation :**

$$-V_t = V_x\,f_0(t,x) + g_0(t,x) - \frac{1}{4}\left[|BV_x'|^2 - \gamma^{-2}|DV_x'|^2\right]$$

$$+\frac{\epsilon}{2}\mathrm{Tr}\big[V_{xx}(t;x)DD'\big]\,; \qquad V(t_f;x) \equiv q(x)$$

A further special case : LEQG Problem

$$f_0(t,x) = A(t)x\,, \quad g_0(t,x) = \frac{1}{2}\,x'Qx\,,\ Q \ge 0$$

$q(x) = (1/2)\,x'Q_f x \quad \Rightarrow$ **Explicit solution:**

$$V(t;x) = \frac{1}{2}\,x'Z(t)x + \ell^\epsilon(t)\,, \quad t \ge 0$$

$$\dot{Z} + A'Z + ZA + Q - Z\,(\,BB' - \gamma^{-2}DD'\,)\,Z = 0$$

$$\ell^\epsilon(t) = \frac{\epsilon}{2}\int_t^{t_f} \mathrm{Tr}\,[\,Z(s)\,D(s)D'(s)\,]\,ds$$

$$\Rightarrow \quad u^*(t) = \mu^*(t,x) = -B'(t)\,Z(t)\,x\,, \quad 0 \le t \le t_f$$

A class of stochastic differential games

Two Players : **Player 1:** u_t ; **Player 2:** w_t

$$dx_t = f(x_t, u_t)\,dt + D(x_t) w_t\,dt + \sqrt{\epsilon}\,D\,db_t\,; \quad x_0$$

$$J(\mu, \nu; t, x_t) := E\Big\{q(x_{t_f}) + \int_t^{t_f} g(s, x_s, u_s)\,ds - \gamma^2 \int_t^{t_f} |w_s|^2\,ds\Big\}$$

Upper-Value (UV) Function :

$$\bar{W}(t; x) = \inf_{\mu} \sup_{\nu} J(\mu, \nu; t, x)$$

HJI UV equation :

$$\inf_{u \in U} \sup_{w \in \boldsymbol{R}^{m_2}} \Big\{\bar{W}_t + \bar{W}_x\,(\,f + Dw\,) + g - \gamma^2 |w|^2 + \frac{\epsilon}{2} \mathrm{Tr}\big[\bar{W}_{xx} D D'\big]\Big\} = 0$$

Isaacs condition holds $\Rightarrow$ Value Function :

$$-W_t(t; x) = \inf_{u \in U} \big\{W_x(t; x)\,f(t, x, u) + g(t, x, u)\big\} + \frac{1}{4\gamma^2} |D W_x'(t; x)|^2 + \frac{\epsilon}{2}\,\mathrm{Tr}\big[W_{xx}(t; x) D D'\big]\,;$$

$$W(t_f; x) \equiv q(x)$$

- IDENTICAL with V for all permissible ϵ, γ

IMPLICATION

Original stochastic dynamics

$$dx_t = f(x_t, u_t)\, dt + \sqrt{\epsilon}\, D\, db_t\,; \qquad x_0$$

Optimum RS FB control and value:

$$u_t = \mu^*(t, x_t)\,; \qquad V(t; x_t; t_f)\,, \quad t \geq t_0$$

Under μ^*, and for the perturbed dynamics

$$d\tilde{x}_t = f(\tilde{x}_t, u_t)\, dt + D(\tilde{x}_t) w_t\, dt \; + \sqrt{\epsilon}\, D\, db_t$$

$$E\Big\{q(\tilde{x}_{t_f}) + \int_t^{t_f} g(s, \tilde{x}_s, \mu^*(\tilde{x}_s, s))\, ds\Big\}$$

$$\leq \gamma^2 \int_t^{t_f} |w_s|^2\, ds \; + V(t; \tilde{x}_t; t_f)$$

In particular, if $\tilde{z}$ is controlled output:

$$\frac{1}{t_f} \int_0^{t_f} E\{|\tilde{z}_s|^2\} \leq \frac{\gamma^2}{t_f} \int_0^{t_f} |w_s|^2\, ds \; + \frac{1}{t_f} V(0; x_0; t_f)$$

A STOCHASTIC DISSIPATION INEQUALITY !

ROBUSTNESS TO MODELLING ERROR !!

Roles of ϵ, γ, and t_f

- In the LQ SDG, there exists a critical value of $\gamma > 0$, γ^*, such that for all $\gamma > \gamma^*$, there exists a unique pair of ϵ-independent saddle-point policies:
  $\mu(t,x) = -B'Z(t)x\,; \quad \nu(t,x) = \gamma^{-2}D'Z(t)x$
  For all $\gamma < \gamma^*$, upper value is unbounded
  $\Rightarrow$ **H^∞ control**
- For the general SDG, as $\epsilon \to 0$ for fixed $\gamma > 0$,
  $\Rightarrow$ well-defined deterministic DG $\Rightarrow$

$$-W_t(t;x) = \inf_{u\in U}\big\{W_x(t;x)\,f(t,x,u) + g(t,x,u)\big\} + \frac{1}{4\gamma^2}|D\,W_x'(t;x)|^2\,; \quad W(t_f;x) \equiv q(x)$$

"Viscosity solution"

Also large deviation limit for risk-sensitive SCP

- For the infinite-horizon RS SCP, appropriate cost:

$$J(\mu) = \lim_{t_f\to\infty} \frac{2\epsilon}{t_f\theta}\,\ln E\,\{\exp\frac{\theta}{2\epsilon}\int_0^{t_f} g(x_s,u_s)\,ds\}$$

Infimum J^*:

$$J^* = \inf_{u\in U}\big\{\tilde{V}_x(x)\,f(x,u) + g(x,u)\big\} + \frac{1}{4\gamma^2}|D\tilde{V}_x'(x)|^2 + \frac{\epsilon}{2}\mathrm{Tr}\big[\tilde{V}_{xx}\,DD'\big]$$

For the time-invariant LQ RS SCP,

(A, B) stabilizable, (A, Q) detectable $\Rightarrow$

$$J^* = \frac{\epsilon}{2} \operatorname{Tr}[\bar{Z} DD']; \quad \tilde{V}(x) = \frac{1}{2} x' \bar{Z} x$$

$$A'\bar{Z} + \bar{Z}A + Q - \bar{Z}(BB' - \gamma^{-2}DD')\bar{Z} = 0$$

Unique minimal nnd solution $\bar{Z}$ for $\gamma > \bar{\gamma}$.

Unique optimal stationary (stabilizing) controller:

$$\mu^*(x) = -B'\bar{Z}x$$

Again ϵ -independent $\Rightarrow$ well-defined as $\epsilon \to 0$

$\Rightarrow$ **An H^∞ controller for each $\gamma > \bar{\gamma}$.**

The SDG may not have a SP, but a finite UV achieved by μ^*

Deterministic worst-case (H^∞) filtering

$$\dot{x} = f(x) + \sigma(x)\, w \qquad y = h(x) + v$$

$$\hat{x}(t) = \delta_t(y_{[0,t)})$$

Cost function :

$$L_\gamma(\delta; x_0, w, v; t) = \int_0^t \{|x(\tau) - \delta_\tau(y_{[0,\tau)})|^2 - \gamma^2|w(\tau)|^2 - \gamma^2|v(\tau)|^2\} d\tau - \gamma^2 q_0(x_0)$$

Worst-case conditional cost :

$$W(\delta; t, x; y_{[0,t)}) := \sup_{w,v|x(t)=x,y_{[0,t)}} \quad \longrightarrow \quad \inf_\delta \sup_x$$

Forward dynamic programming $\Rightarrow$

$$\sup_{w\in \boldsymbol{R}^{m_2}} \{ -W_t - W_x\,(f + \sigma w) + |x-\delta|^2 - \gamma^2|w|^2 - \gamma^2|y-h|^2\} = 0\,, \quad W(0,x) = -\gamma^2 q_0(x)$$

$$\min_\delta \max_x W(\delta; t, x; y_{[0,t)}) \quad \longrightarrow \quad \delta_t = \hat{x}(t)\,;$$

$$W(\hat{x}; t, \hat{x}; y_{[0,t)}) = \max_x W(\hat{x}; t, x; y_{[0,t)})$$

$$\dot{\hat{x}} = f(\hat{x}) - 2W_{xx}^{-1}\, h_x \cdot (y - h(\hat{x}))\,, \quad \hat{x}(0) = \arg\min q_0(x)$$

$\Rightarrow$

$$\|x - \hat{x}\|_t^2 \leq \gamma^2\|w\|_t^2 + \gamma^2\|v\|_t^2 + \gamma^2 q_0(x_0)\ \forall\,(w, v, x_0)$$

Deterministic worst-case (H$^\infty$) filtering

Linear-Quadratic case

$$\dot{x} = A\,x + D\,w \qquad y = C\,x + E\,w$$

$$L_\gamma(\delta; x_0, w; t) = \int_0^t \left\{ |x(\tau) - \hat{x}(\tau)|_Q^2 - \gamma^2 |w(\tau)|^2 \right\} d\tau$$

$$-\gamma^2 |x_0 - \bar{x}_0|_{Q_0}^2$$

$\Rightarrow$

$$\dot{\hat{x}} = A\,\hat{x} + \big(\Sigma C' + L\big) N^{-1} \left(y - C\hat{x}\right), \quad \hat{x}(0) = \bar{x}_0$$

$$\dot{\Sigma} = \tilde{A}\Sigma + \Sigma\tilde{A}' - \Sigma\big(C'N^{-1}C - \gamma^{-2}Q\big)\Sigma + DD' - LN^{-1}L'$$

$$\Sigma(0) = Q_0^{-1}$$

$$L := DE'\,; \quad N = EE' > 0\,; \quad \tilde{A} := A - LN^{-1}C$$

$$\|x - \hat{x}\|_{Q,t}^2 \le \gamma^2 \|w\|_t^2 + \gamma^2 |x_0 - \bar{x}_0|_{Q_0}^2 \;\; \forall\,(w, x_0)$$

Linear filtering under exponentiated cost

$$dx_t = A\,x_t\,dt + D\,db_t \qquad \text{signal model}$$

$$dy_t = C\,x_t\,dt + E\,db_t \qquad \text{measurement}$$

$$J = \frac{2}{\theta}\ln E\{\exp\frac{\theta}{2}\int_0^{t_f}|x_t - \hat{x}_t|_Q^2\,dt\} \quad \to \quad \min$$

$\Rightarrow$

$$d\hat{x}_t = A\,\hat{x}_t\,dt + \big(\Sigma C' + L\big)N^{-1}\,(dy_t - C\hat{x}_t\,dt)\,,$$

$$\hat{x}(0) = \bar{x}_0 = E[x_0]$$

$$\dot{\Sigma} = \tilde{A}\Sigma + \Sigma\tilde{A}' - \Sigma\big(C'N^{-1}C - \theta Q\big)\Sigma + DD' - LN^{-1}L'$$

$$\Sigma(0) = \Sigma_0 := \mathrm{cov}(\mathrm{x}_0)$$

$$L := DE'\,; \quad N = EE' > 0\,; \quad \tilde{A} := A - LN^{-1}C$$

SAME STRUCTURE AS IN H^∞ FILTERING

Linear filtering under exponentiated cost

Uncorrelated noises

$$dx_t = A\,x_t\,dt + D\,db_t \qquad \text{signal model}$$

$$dy_t = C\,x_t\,dt + G\,dv_t \qquad \text{measurement}$$

$$J = \frac{2}{\theta}\ln E\{\exp\frac{\theta}{2}\int_0^{t_f}|x_t - \hat{x}_t|_Q^2\,dt\} \quad \rightarrow \quad \text{min}$$

$\Rightarrow$

$$d\hat{x}_t = A\,\hat{x}_t\,dt + \Sigma C' N^{-1}\,(dy_t - C\hat{x}_t\,dt)\,,$$

$$\hat{x}(0) = \bar{x}_0 = E[x_0]$$

$$\dot{\Sigma} = \tilde{A}\Sigma + \Sigma\tilde{A}' - \Sigma\big(C'N^{-1}C - \theta Q\big)\Sigma + DD'$$

$$\Sigma(0) = \Sigma_0 := \mathrm{cov(x_0)}$$

Unique solution $\Sigma \geq 0$ for $\theta < \theta^*$

An equivalent stochastic differential game

$$d\zeta_t = A\,\zeta_t\,dt + H w_t dt + D\,db_t \qquad DD' > 0$$

perturbed signal model

$$d\tilde{y}_t = C\,\zeta_t\,dt + G\,dv_t \qquad \text{measurement}$$

$$\tilde{J}(\hat{\zeta}, w) = E\big\{|\zeta_{t_f} - \hat{\zeta}_{t_f}|^2_{\tilde{Q}_f} + \|\zeta_t - \hat{\zeta}_t\|^2_{\tilde{Q}} - \gamma^2\|w_t\|^2\big\}$$

$$\tilde{Q}_f = \Sigma^{-1}(t_f)\,,\ \tilde{Q}(t) = \tfrac{1}{2}\theta Q + C'N^{-1}C + \Sigma^{-1}DD'\Sigma^{-1}$$

$$\theta < \theta^*, \quad H = \gamma\sqrt{\tfrac{\theta}{2}}\Sigma(t)Q^{\frac{1}{2}}(t) \quad \Rightarrow$$

$$\left(\hat{\zeta}^*_t = \hat{x}^*_t,\ w^*_t = \frac{1}{\gamma}\sqrt{\frac{\theta}{2}}Q^{\frac{1}{2}}(\zeta_t - \hat{x}^*_t)\right) \text{ in SP eqm}$$

$$d\hat{x}^*_t = A\,\hat{x}^*_t\,dt + \Sigma C'N^{-1}\,(d\tilde{y}_t - C\hat{x}^*_t\,dt)\,,$$
$$\hat{x}^*_t = \bar{x}_0 = E[x_0]$$

$$\dot{\Sigma} = \tilde{A}\Sigma + \Sigma\tilde{A}' - \Sigma\big(C'N^{-1}C - \theta Q\big)\Sigma + DD'$$
$$\Sigma(0) = \Sigma_0 := \mathrm{cov}(\mathrm{x}_0)$$

$$\tilde{J}(\hat{x}^*, w^*_t) = n + \int_0^{t_f} \mathrm{Tr}(\Sigma^{-1}(DD' + \Sigma C'N^{-1}C\Sigma))\,dt$$

Player 1's problem: *Kalman filter for the signal*

$$d\zeta_t = (A + \frac{\theta}{2}\Sigma Q)\,\zeta_t\,dt - \frac{\theta}{2}\Sigma Q\,\hat{x}^*_t\,dt + D\,db_t$$

Conditional distribution: $\mathrm{N}\big(\hat{\zeta}_t, \tilde{\Sigma}\big)$

$\Rightarrow \quad \tilde{\Sigma} = \Sigma\,;\hat{\zeta}_t = \hat{x}^*_t$

Player 2's problem:

$$\max_w E\big\{|\epsilon_{t_f}|^2_{\tilde{Q}_f} + \|\epsilon\|^2_{\tilde{Q}} - \gamma^2\|w\|^2\big\}$$

$$d\epsilon_t = (A - \Sigma C' N^{-1} C)\,\epsilon + +Hw_t dt + D\,db_t \\ - \Sigma C' N^{-1} + G\,dv_t$$

$\Rightarrow \quad w^*_t = \frac{1}{\gamma}H'S\,\epsilon_t$ **and** $S = \Sigma^{-1}$

Implication. *Stochastic dissipation inequality*

$$E\big\{|\hat{x}^*_{t_f} - \zeta_{t_f}|^2_{\tilde{Q}_f} + \|\hat{x}^* - \zeta\|^2_{\tilde{Q}}\big\} \leq \gamma^2\|w\|^2 + \tilde{k}$$

Since $\tilde{Q} > (\theta/2)Q\,, \quad \tilde{Q}_f > 0,$

$$E\big\{\|\hat{x}^* - \zeta\|^2_Q\big\} \leq \frac{2\gamma^2}{\theta}\|w\|^2 + \tilde{k}$$

Another stochastic differential game

$$d\zeta_t = A\,\zeta_t\,dt + H w_t dt + D\,db_t \qquad DD' > 0$$

perturbed signal model

$$d\tilde{y}_t = C\,\zeta_t\,dt + G\,p_t\,dt + G\,dv_t$$

perturbed measurement

$$\mathcal{J}(\hat{\zeta}, w) = E\{|\zeta_{t_f} - \hat{\zeta}_{t_f}|^2_{\tilde{Q}_f} + \|\zeta_t - \hat{\zeta}_t\|^2_{\mathcal{Q}} - \gamma^2 \|w_t, p_t\|^2\}$$

$$\mathcal{Q} := \tilde{Q} - \gamma^{-2} C' N^{-1} C\,, \quad \gamma > 1$$

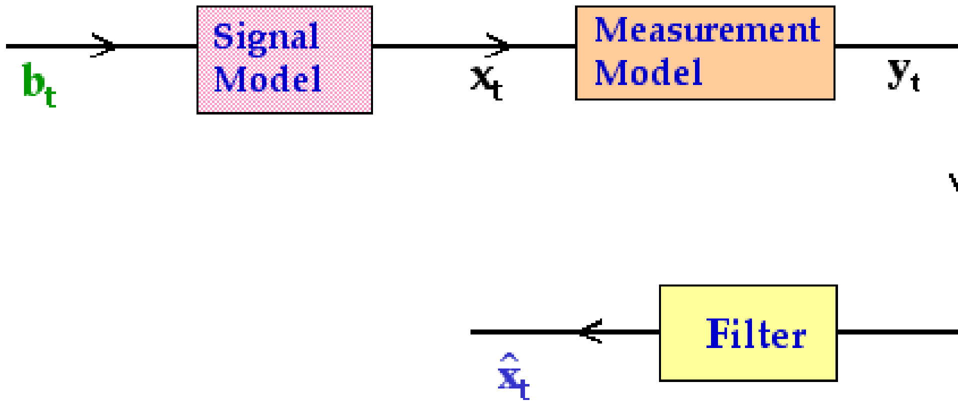
The same filter leads to

$$E\{\|\hat{x}^* - \zeta\|^2_Q\} \leq \frac{2\gamma^2}{\theta}\|w, p\|^2 + k$$

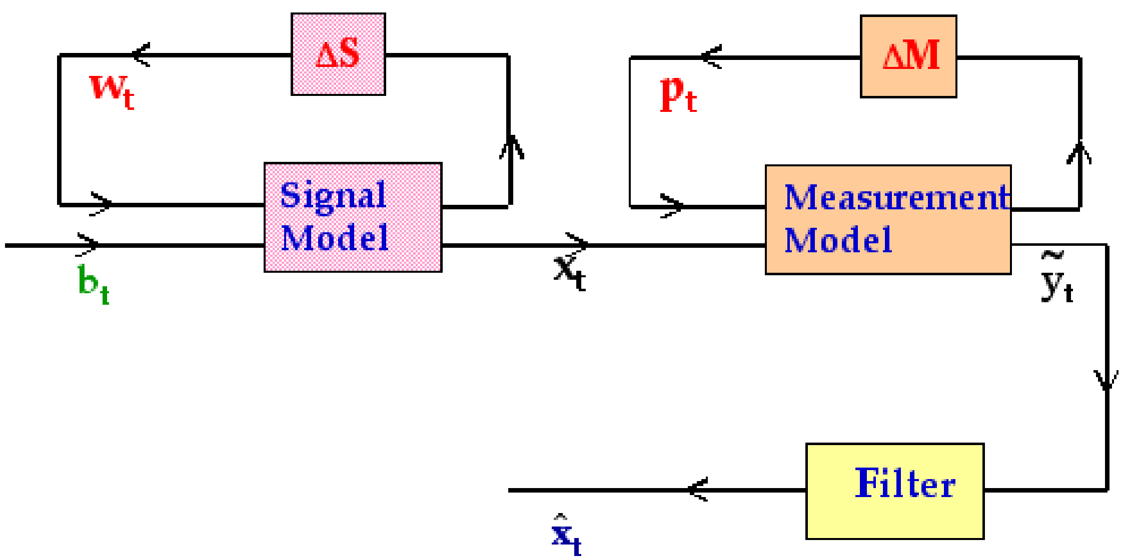
Worst-case perturbations:

$$\tilde{p}_t^* = -\gamma^{-2} G' N^{-1} C(\zeta_t - \hat{x}_t^*),\ w_t^* = \frac{1}{\gamma}\sqrt{\frac{\theta}{2}} Q^{\frac{1}{2}}(\zeta_t - \hat{x}_t^*)$$

Interpretation for signal uncertainty models



$e_t := x_t - \hat{x}_t$ $\exp\{\theta \, \| e \|^2\}$



$$\| e \|^2 - \gamma^2 \| w \|^2 - \gamma^2 \| p \|^2$$

Piecewise Deterministic (Switching) Systems

$$\dot{x} = f(t, x, u, w; \chi)$$

$\chi(t), t \geq 0,$ **a Markov jump process with**

$$\text{Prob}\{\chi(t+h) = j | \chi(t) = i\}$$

$$= \begin{cases} \lambda_{ij} h + o(h), & j \neq i \\ 1 + \lambda_{ii} h + o(h), & j = i \end{cases}$$

Performance measure:

$$\sup_{w} \left\{ \frac{E_\chi\{\int_0^{t_f} g(t, x, u; ; \chi) dt\}}{E_\chi\{\|w\|^2\}} \right\}$$

Under full state information, HJI equation:

$$-V_t^i(t, x) = \inf_u \sup_w \{V_x^i(t, x) f(t, x, u, w; \chi) + g(t, x, u; ; \chi) - \gamma^2 |w|^2 + \sum_j \lambda_{ij} V^j(t, x)\}$$

$$V^i(t_f, x) = 0$$

- **Solution generally in the viscosity sense**
- **Stochastic dissipation ineq; small gain**
- **LQ: coupled GRDEs & GAREs**

Robust Adaptive Control

$$\dot{x} = f(x, \theta) + G(x, \theta)\, u + \sigma(x)\, w\,, \quad x(0) = x_0$$

r-dimensional control : $u(t) = \mu(t, x_{[0,t]})$,

m-dimensional output : $z = h(x)$

Performance index (to be minimized by μ) :

$$\mathcal{I}_t(\mu) = \sup_{w,\theta,x_0} \frac{\|z - z_r\|_t^2 + \tilde{\ell}_t(x_{[0,t]}; u_{[0,t]})}{\|w\|_t^2 + |\theta - \bar{\theta}|_{Q_0}^2 + \ell(x_0, \theta - \bar{\theta})}$$

$$\tilde{\ell}_t := \int_0^t \ell(x_{[0,t]}; u(\tau), \tau)\, d\tau$$

$$\inf_{\mu} \mathcal{I}_t(\mu) =: \gamma_t^{*2}$$

Dissipation inequality for each $\gamma > \gamma_t^*$:

$$J_\gamma^t(\mu; \omega) := \|z - z_r\|_t^2 + \tilde{\ell}_t(x_{[0,t]}; u_{[0,t]}) - \gamma^2 \|w\|_t^2 - \gamma^2 |\theta - \bar{\theta}|_{Q_0}^2 - \gamma^2 \ell(x_0, \theta - \bar{\theta}) \leq 0$$

$$\Rightarrow \qquad \inf_{\mu} \sup_{\omega} J_\gamma^t(\mu; \omega)$$

Sequential decomposition of supremum into two separate suprema :

$$\sup_{\omega} J_\gamma^t(\mu;\omega) = \sup_{\theta, x_{[0,t]}} \; \sup_{(w_{[0,\infty)} | x_{[0,t]}, \theta)} J_\gamma^t(\mu;\omega)$$

Inner maximization: worst-case identification

Outer maximization: controller design

Partial noisy measurements

$$y(t) = h(x,\theta) + n(x)\, w$$

$\Rightarrow$ sequential decomposition

$$\sup_{\omega} J_\gamma^t(\mu;\omega) = \sup_{\theta, x_t, y_{[0,t]}} \; \sup_{(x_0, w_{[0,\infty)} | y_{[0,t]}, \theta, x(t)=x_t)} J_\gamma^t(\mu;\omega)$$

Auxiliary Dynamics : $\dot\theta = 0 \quad \theta = \theta_0$

◇ A nonlinear H$^\infty$ control problem with partial state measurements

What lies ahead ?

- **Applications of the available theory**
- **Embedding uncertainty in economics models within the robustness framework**
- **Extending the robustness framework to multiple decision maker environments**
- **Tradeoffs between conservatism inherent to robustness and learning/adaptation**
- **Computational tools**
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