Paradigms for Robustness in Controller and Filter Designs*

Tamer Başar

Coordinated Science Laboratory
University of Illinois
1308 West Main Street
Urbana, Illinois 61801-2307 / USA
tbasar@decision.csl.uiuc.edu
Fax: 217/244-1653; Tel: 217/333-3607

Abstract

We provide an overview of robustness issues that arise in controller and filter designs for dynamic systems subject to various types of uncertainty. Both linear and nonlinear systems are treated, with the nominal system being deterministic, stochastic or piecewise deterministic. The performance indices adopted cover a broad spectrum of scenarios, including risk-neutral, risk-sensitive, and worst-case formulations.

Keywords

Robust designs; dynamic systems; risk sensitivity; worst-case controller; worst-case filter.

1 Introduction

Robustness is an issue that pervades and cuts across many disciplines, from social sciences to physical sciences and engineering. It is considered today to be an indispensable requirement in any design, be it an incentive design mechanism in economics, a feedback controller design for jet engines, or routing and congestion control in high speed networks. Every design is based on a *model* that the designer adopts, but models by their nature provide only an approximate description of the phenomenon they represent. Hence, an important and a relevant question that comes up is: How good is a particular design if the model on which it is based is not accurate? In different words: Can a design survive the inaccuracies of the model based on which it was developed? Designs which respond positively to these inquiries within the confines of some specifications, we call "robust".

*August 26, 1999. Article prepared for *Macroeconomic Dynamics*. Research supported in part by the Department of Energy under Grant DEFG-02-97-ER-13939, and in part by the National Science Foundation.

In general terms, a design is *robust* if its nominal features (that is the features that it delivers for a nominal model), such as stability and performance, are by and large preserved in the presence of unmodeled or partially modeled 'reasonable' uncertainty.

Hence, to be able to talk about robustness, we have to have first a *nominal* model, and second a class of allowable perturbations around the nominal model. What we mean by a *model* here is a mathematical construct in the form of an input-output description, with identifiable input and output variables along with the spaces or constraint sets where they belong. A nominal model could be *deterministic*, in which case the output is uniquely determined by the input; or *stochastic*, in which case each input results in an output which is a stochastic process, though uniquely defined as such; or the intermediate case of *piecewise deterministic*, which means that except at a finite or countably infinite number of points on the time interval, the output of the system is deterministic, and at those finite or countably infinite points the system output exhibits nondeterministic behavior due to stochastic 'jumps' that occur at these times.

Perturbations around a nominal model of any of the three types above would also be of these three types: For example, a deterministic model could have deterministic (but unknown) perturbations, or stochastic perturbations, or perturbations that result in jumps in the system structure (such as failures occurring in a manufacturing system, or unexpected and unplanned events occurring in an economic system).

How does one deal with perturbations or uncertainty around a nominal model? First we have to have a quantity whose optimization in some sense is consistent with the designer's objectives. This will depend on the system output(s) and input(s), as well as the stochastic or deterministic uncertainty. The output, in turn, will depend on the (possibly unknown) initial state of the system and stochastic quantities (if any) that govern it along with the input. Let us denote this quantity (to be optimized) by e ('e' for *error*, as it could for example stand in a macroeconomic application for the difference (error) between an output variable and a preset target value this variable is desired to track), and write it as

$$e(\chi, w, x_0; b, \xi; u)$$

where the triple $(\chi, w, x_0) =: \omega$ stands for the unmodeled uncertainty, with χ being discrete and static (such as parametric uncertainty), w being continuous and dynamic (such as disturbance or unmodeled system dynamics), and x_0 being the initial state; the pair (b, ξ) stands for the stochastic variable associated with the nominal model, with b standing for a continuous stochastic process (such as Brownian motion) and ξ standing for a static random vector; and u denotes the input, which is generally vector-valued. Let $\|e(\chi, w, x_0; b, \xi; u)\|$ denote the norm of e in some appropriate space (this could, for example, be a Hilbert space norm), and $E_{b,\xi}$ denote the expectation operation under the statistics of b and ξ. If we take

$$E_{b,\xi}\|e(\chi, w, x_0; b, \xi; u)\|^2 \tag{1}$$

as the deterministic quantity of interest to minimize, then two ways of handling the uncertainty triple $\omega = (\chi, w, x_0)$ are the following:

(i) Hard-bounded uncertainty. Here it is assumed that the triple ω belongs to a given constraint set Ω and the maximum of (1) over this set is minimized by the input u, that is

$$\inf_u \sup_{\omega \in \Omega} E_{b,\xi}\|e(\omega; b, \xi; u)\|^2. \tag{2}$$

(ii) Soft-bounded uncertainty. Here the input is designed in such a way that in terms of ω the quantity (1) does not grow faster than the square of its norm; more precisely, for a $u = u^\circ$, for some $\gamma > 0$ and for all unconstrained ω belonging to some appropriate space,

$$E_{b,\xi}\|e(\omega; b, \xi; u^\circ)\|^2 \quad \leq \quad \gamma^2\|\omega\|^2. \tag{3}$$

This is known as a *disturbance attenuation* inequality, since through the design of an input, one is attempting to attenuate the effect of the disturbance ω. The parameter γ is the disturbance attenuation parameter, and the goal is generally to make the inequality (3) hold for the smallest possible value of γ, which would then make the performance minimally sensitive to the presence of unmodeled uncertainty. Note that an alternative way of writing (3) would be

$$\sup_\omega \left\{ E_{b,\xi}\|e(\omega; b, \xi; u^\circ)\|^2 - \gamma^2\|\omega\|^2 \right\} \quad \leq \quad 0$$

or better

$$\inf_u \sup_\omega \left\{ E_{b,\xi}\|e(\omega; b, \xi; u)\|^2 - \gamma^2\|\omega\|^2 \right\} \quad = \quad 0 \tag{4}$$

where the inequality has been replaced by an equality because by picking ω to be zero, the quantity above can always be made nonnegative.

Either formulation above, that is (2) or (4), can be interpreted as a zero-sum game played between the designer and the uncertainty, and the quantities of interest are the upper values of these games (that is the uncertainty is allowed the upper hand — to observe the designer's choice for the input).

Yet a third approach to robustness involves a completely stochastic formulation, which we introduce next, as a third alternative.

(iii) Risk-sensitive design. By eliminating the dependence of e on χ and w, and by making x_0 stochastic and absorbing it in ξ, we have as the minimization problem

$$\inf_u E_{b,\xi} \exp\{\theta \|e(b, \xi; u)\|^2\} \tag{5}$$

where $\theta > 0$ is a "risk-sensitivity" parameter. By exponentiating $\|e\|^2$ after scaling it by θ, one is placing more emphasis on unfavorable sample paths (for the decision maker), and hence an input design that results from (5) provides a safeguard against unfavorable events within the nominal model. Such designs are called *risk-sensitive*, and carry some appealing robustness features, as we will later see, along with connections with the criterion of (4).

During this decade, there have been substantial advances in our understanding of the solutions to the robust design problems formulated above, and in particular under the second and third criteria; as representative works on this line of research, see *Başar and Bernhard (1995); Green and Limebeer (1995); Hassibi, Sayed, and Kailath (1999); Whittle (1990); Fleming and McEneaney (1992, 1995); Van der Schaft (1992, 1993); Ball, Helton, and Walker (1993); James, Baras, and Elliott (1994); Runolfsson (1994); Başar, T. (1995, 1999a); Isidori and Kang (1995); Hansen and Sargent (1995); James and Baras (1996); Pan and Başar (1996, 1999)* . In this paper we provide an overview of these latest advances as developed in the context of control theory, which however also have direct applications in economic systems. Our treatment here will involve continuous-time finite-dimensional models only, but in a general 'nonlinear' framework, with explicit results for linear-quadratic models also presented. The emphasis will be placed on methodological development rather than presentation of full details of proofs, where the later can be found in the references cited.

We address in the next section the deterministic problem (under the criterion (4)), and in Section 3 the risk-sensitive design problem, where we also establish connections with the results of Section 2. Section 4 is devoted to robust control of piecewise deterministic systems under deterministic uncertainty, again first for general nonlinear and subsequently for the special class of linear systems. Section 5 discusses robust filtering, for deterministic as well as stochastic (risk-sensitive) systems, and the paper ends with the concluding remarks of Section 6.

2 Deterministic Robust Control with State Feedback

Adopting a state-space representation for the deterministic input-output map alluded to in Section 1, we have the dynamics

$$\dot{x} = f(t;x,u,w), \quad x(0)=x_0, \quad t \geq 0 \tag{6a}$$

$$z = h(t,x,u,w) \tag{6b}$$

where u is the input, x is the state, z is the (controlled) output, and w is the disturbance. These are all vector-valued quantities, the dimensions of which we do not specify here. They are all assumed to be square integrable on an interval $[0,t_f)$, where t_f will also be allowed to be ∞ (viewed as the limit $t_f \to \infty$); for each fixed $t_f < \infty$, we denote the Hilbert space of square-integrable vector-valued functions generically by $\mathcal{L}_2^{t_f}$, regardless of the dimension of the function. We let $\|\cdot\|_{t_f}$ denote the norm on $\mathcal{L}_2^{t_f}$, i.e.,

$$\|w\|_{t_f}^2 := \int_0^{t_f} w(t)'w(t)dt$$

where *prime* ($'$) denoted transpose. The uncertainty ω is the pair (w,x_0), as we do not have parametric uncertainty, $\mathcal{X}$, here. We take as the performance index (counterpart of (1))

$$L(t_f;w,x_0;u) = |x(t_f)|_{Q_f}^2 + \|z\|_{t_f}^2 \tag{7}$$

where $|\cdot|_{Q_f}$ denotes Euclidean semi-norm, weighted by the nonnegative-definite matrix Q_f. Control is allowed to be a function of the state; that is, for some function μ, $u(t)=\mu(t,x)$.

The robust control design problem then is, in view of (4), finding a state-feedback controller μ° such that the following inequality holds for all $\omega=(w,x_0)$:

$$J_\gamma^{t_f}(\mu^\circ;\omega) := |x^\circ(t_f)|_{Q_f}^2 + \|h(t,x^\circ,u^\circ,w)\|^2 - \gamma^2(\|w\|_{t_f}^2 + |x_0|_{Q_0}^2) \leq 0 \tag{8}$$

where $Q_0>0$ is taken as a weighting on x_0, $u^\circ(t)=\mu^\circ(t,x)$, and $x^\circ(\cdot)$ is generated by (6a) with $u=u^\circ$.

We now present a set of sufficient conditions for the solution to this robust design problem to satisfy. Let $\bar{V}(t;x)$ be a scalar function, continuously differentiable in the pair (t,x), and satisfying the Hamilton-Jacobi-Isaacs (HJI) inequality:

$$\begin{aligned} -\bar{V}_t &\geq \inf_u \sup_w [\bar{V}_x \cdot f(t,x,u,w) + |h(t,x,u,w)|^2 - \gamma^2|w|^2] & \text{(9a)} \\ \bar{V}(t_f;x) &\geq |x|^2_{Q_f} & \text{(9b)} \end{aligned}$$

where subscripts t and x of V denote partial derivatives. Let $\bar{V}$ further satisfy:

$$\sup_x \left\{ \bar{V}(0;x) - \gamma^2|x|^2_{Q_0} \right\} = 0. \tag{10}$$

Then, we have the following:

Theorem 2.1. *Let $\bar{V}$ be defined through (9a)-(9b), and (10) above, and $u = \overline{\mu}_\gamma(t,x)$ be a controller that minimizes the RHS of (9a). Then,*

$$J^{t_f}_\gamma(\overline{\mu}_\gamma; w, x_0) \leq 0 \qquad \forall \omega = (w, x_0). \tag{11}$$

Thus, $\overline{\mu}_\gamma$ provides a robust controller, corresponding to the disturbance attenuation level γ.

Proof. Let

$$\begin{aligned} \bar{f}(t;x,w) &:= f(t,x,\overline{\mu}_\gamma(t,x),w) \\ \bar{g}(t;x,w) &:= g(t,x,\overline{\mu}_\gamma(t,x),w) \\ g(t;x,u,w) &:= |h(t,x,u,w)|^2, \end{aligned}$$

and note the following sequence of equalities and inequalities (where we drop the subscript "t_f" in $\|w\|_{t_f}$ to ease the notation)

$$\begin{aligned} J^{t_f}_\gamma(\overline{\mu}_\gamma;\omega) &= |x(t_f)|^2_{Q_f} + \int_0^{t_f} \bar{g}(t;x,w)dt - \gamma^2\|w\|^2 - \gamma^2|x_0|^2_{Q_0} \\ &\leq \bar{V}(t_f;x) + \int_0^{t_f} \bar{g}(t;x,w)dt - \gamma^2\|w\|^2 - \gamma^2|x_0|^2_{Q_0} \\ &= \int_0^{t_f} (\bar{V}_t + \bar{V}_x \cdot \bar{f}(t;x,w) + \bar{g}(t;x,w) - \gamma^2|w|^2)dt + \bar{V}(0;x_0) - \gamma^2|x_0|^2_{Q_0} \\ &\leq \int_0^{t_f} \left(-\sup_w \left\{\bar{V}_x \cdot \bar{f} + \bar{g} - \gamma^2|w|^2\right\} + V_x \cdot \bar{f} + \bar{g} - \gamma^2|w|^2\right) dt + \bar{V}(0;x_0) - \gamma^2|x_0|^2_{Q_0} \\ &\leq \bar{V}(0;x_0) - \gamma^2|x_0|^2_{Q_0} \leq 0, \end{aligned}$$

where the inequality in the second line follows from (9b), the one in the fourth line follows from (9a), and the last inequality follows from (10). □

Remark 2.1. *If we have equalities in (9a) and (9b), then the proof of Theorem 2.1 leads to*

$$\inf_{\mu} \sup_{w} J_{\gamma}^{t_f}(\mu; w, x_0) \quad = \quad \bar{V}(0; x_0) - \gamma^2 |x_0|^2_{Q_0} \qquad \forall x_0, \tag{12}$$

where the quantity on the LHS is the upper value of the underlying zero-sum game between the controller and the disturbance. □

Now consider again (9a) and (9b) but with the direction of the inequalities reversed, and inf and sup interchanged, that is: Let $\underline{V}(t; x)$ be a scalar function, continuously differentiable in (t, x), and satisfying the HJI inequality:

$$-\underline{V}_t \quad \leq \quad \sup_{w} \inf_{u} \left[\underline{V}_x \cdot f(t; x, u, w) + g(t; x, u, w) - \gamma^2 |w|^2 \right] \tag{13a}$$

$$\underline{V}(t_f; x) \quad \leq \quad |x|^2_{Q_f}. \tag{13b}$$

Then, the counterpart of Theorem 2.1 is the following, which we state here without a proof since it parallels that of Theorem 2.1.

Theorem 2.2. *Let $\underline{V}$ be defined through (13a)-(13b) above, let x_0° maximize $\underline{V}(0; x_0) - \gamma^2 |x_0|^2_{Q_0}$, with maximum value zero, and $w = \nu_\gamma^\circ(t, x)$ maximize the RHS of (13a). Then*

$$J_{\gamma}^{t_f}(u; \nu_\gamma^\circ, x_0^\circ) \geq 0 \qquad \forall u. \tag{14}$$

Thus, the uncertainty pair $(\nu_\gamma^\circ, x_0^\circ)$ guarantees a level of disturbance attenuation not falling below γ, regardless of the controller used. □

Remark 2.2. *As in the case of Remark 2.1, if we have equalities in (13a) and (13b), then*

$$\sup_{\nu} \inf_{u} J_{\gamma}^{t_f}(u; \nu(t, x), x_0) \quad = \quad \underline{V}(0; x_0) - \gamma^2 |x_0|^2_{Q_0} \qquad \forall x_0, \tag{15}$$

where the quantity on the LHS is the lower value of the underlying game.

If the order in which the inf *and* sup *operations apply in the HJI equation is immaterial (i.e., if Isaacs condition holds), then*

$$\underline{V}(t;x) \;\;\equiv\;\; \bar{V}(t;x) \;\;=:\;\; V(t;x)$$

where $V(t,x)$ is the value function for a differential game starting at the time-state pair (t,x). The pair $(\overline{\mu}_\gamma, \nu_\gamma^\circ)$ introduced in Theorems 2.1 and 2.2 would then constitute a saddle point for the differential game.

We should also note that the value function V may not always be smooth (i.e., continuously differentiable in (t,x)), in which case the solution to the HJI equation will have to be interpreted in the viscosity sense ; see, for example, Lions (1982), Başar and Bernhard (1995), McEneaney (1995), Xiao and Başar (1999). □

The infinite-horizon case

We now discuss the counterpart of Theorem 2.1 for the infinite-horizon time-invariant case, where f and h are time-invariant (they do not explicitly depend on t), $Q_f = 0$, and $t_f \to \infty$. The counterpart of (9a) is

$$\inf_u \sup_w \left[\bar{V}_x \cdot f(x,u,w) + |h(x,u,w)|^2 - \gamma^2 |w|^2\right] \;\;\leq\;\; 0 \tag{16}$$

where $\bar{V}$ is also time invariant. Then, we have:

Theorem 2.3. *Let $\bar{V}(x)$ be a positive-definite function satisfying the HJI inequality (16), and further*

$$\sup_x \left[\bar{V}(x) - \gamma^2 |x|^2_{Q_0}\right] \;\;=\;\; 0.$$

Let $u = \overline{\mu}_\gamma(x)$ be the minimizer on the LHS of (16). Then,

$$J_\gamma^\infty(\overline{\mu}_\gamma; w, x_0) \;\;\leq\;\; 0 \qquad \forall \omega = (w, x_0), \tag{17}$$

where J_γ^∞ is defined as the lim sup *of (8) as $t_f \to \infty$, and with $Q_f \equiv 0$.*

Proof. Let $\bar{f}$ and $\bar{g}$ be defined as in the proof of Theorem 2.1, without the t-dependence. Then, for all ω,

$$
\begin{aligned}
J_\gamma^\infty(\overline{\mu}_\gamma;\omega) &= \limsup_{t_f\to\infty}\left\{\int_0^{t_f}\bar{g}(x,w)dt-\gamma^2\|w\|_{t_f}^2\right\}-\gamma^2|x_0|_{Q_0}^2\\
&= \limsup_{t_f\to\infty}\left\{\bar{V}(x_0)-\bar{V}(x(t_f))-\gamma^2|x_0|_{Q_0}^2+\int_0^{t_f}(\bar{V}_x\cdot\bar{f}(x,w)+\bar{g}(x,w)-\gamma^2|w|^2)dt\right\}\\
&\le \limsup_{t_f\to\infty}\left\{\bar{V}(x_0)-\gamma^2|x_0|_{Q_0}^2-\bar{V}(x(t_f))\right\}\\
&\le \bar{V}(x_0)-\gamma^2|x_0|_{Q_0}^2\le 0,
\end{aligned}
$$

which proves the desired result. □

Remark 2.3. *If the infinite-horizon cost is discounted, that is*

$$J_\gamma^\infty(\mu;\omega) = \int_0^\infty e^{-\rho t}\left(|h(x,u,w)|^2-\gamma^2|w|^2\right)dt-\gamma^2|x_0|_{Q_0}^2, \tag{18}$$

where $\rho>0$ is the discount factor, then the counterpart of (16) is

$$\frac{\rho}{2}\bar{V}(x) \ge \inf_u\sup_w\left[\bar{V}_x\cdot f(x,u,w)+|h(x,u,w)|^2-\gamma^2|w|^2\right], \tag{19}$$

and with this change, the statement of Theorem 2.3 remains intact. □

Special Structures

Affine-in-control and affine-in-disturbance

The statements of Theorems 2.1-2.3 could be strengthened if some specific structures are assumed for f and h. One such class is the one where f is affine in u and w, and h is affine in u and does not depend on w:

$$f(x,u,w) = a(x)+B_1(x)u+B_2(x)w \tag{20a}$$

$$h(x,u,w) = c(x)+Du \tag{20b}$$

where D is a constant matrix, with $D'D>0$. In the above, a, B_1, B_2, c could all also depend on t for the finite-horizon problem, but we have suppressed this dependence for convenience in notation.

We further assume (again for convenience, and without any loss of generality):

$$D'D = I, \qquad D'c(x) = 0, \tag{20c}$$

so that

$$|h(x,u)|^2 \;\; = \;\; |c(x)|^2 + |u|^2. \tag{20d}$$

If (20c) does not hold, then we can always apply a transformation to u (as a function of x) which will make (20d) hold, without changing the structure of f.

Under this structure, (9a) simplifies to

$$-\bar{V}_t \;\; \geq \;\; \bar{V}_x a + |c|^2 - \frac{1}{4}\bar{V}_x[B_1B_1' - \gamma^{-2}B_2B_2']\bar{V}_x', \tag{21}$$

which has to be solved subject to the boundary condition (9b). The corresponding robust controller $\overline{\mu}_\gamma$ can be computed explicitly in terms of $\bar{V}_x$, and is:

$$\overline{\mu}_\gamma(t,x) \;\; = \;\; -\frac{1}{2}B_1'(x)\bar{V}_x'(t;x). \tag{22}$$

For the infinite-horizon case, the counterparts of (21) and (22) are, respectively,

$$\bar{V}_x a + |c|^2 - \frac{1}{4}\bar{V}_x[B_1B_1' - \gamma^{-2}B_2B_2']\bar{V}_x \leq 0 \tag{23a}$$

$$\overline{\mu}_\gamma(x) = -\frac{1}{2}B_1'(x)\bar{V}_x'(x). \tag{23b}$$

Now, let there exist a positive-definite and radially unbounded function $\bar{V}(x)$ satisfying (23a) and the condition

$$\sup_x[\bar{V}(x) - \gamma^2|x|^2_{Q_0}] \;\; = \;\; 0. \tag{23c}$$

Consider the closed-loop system under the control (23b), and with $w \equiv 0$:

$$\dot{x} \;\; = \;\; a(x) - \frac{1}{2}B_1(x)B_1'(x)\bar{V}_x'(x), \quad x(0) = x_0. \tag{24}$$

Computing the time derivative of $\bar{V}$ on this trajectory, we have

$$\begin{aligned} \frac{d}{dt}\bar{V}(x) \;\; &= \;\; V_x(x)\left[a(x) - \frac{1}{2}B_1(x)B_1'(x)\bar{V}_x'(x)\right] \\ &\leq \;\; -|c(x)|^2 - \frac{1}{4}\bar{V}_x\left[B_1B_1' + \gamma^{-2}B_2B_2'\right]\bar{V}_x' \\ &\leq \;\; -|c(x)|^2 \end{aligned}$$

which is negative definite, provided that $|c(x)|^2$ is a positive-definite function. Hence, under this condition, $\bar{V}(x)$ serves as a Lyapunov function. This means that if $x = 0$ is an equilibrium of the closed-loop dynamics (24), then this equilibrium is globally asymptotically stable.

For the sake of completeness, let us note that for the discounted cost problem, and under the structure (20a)-(20d), the relevant HJI inequality (as the counterpart of (23a), and from (19)) is:

$$\frac{\rho}{2}\bar{V} \quad \geq \quad \bar{V}_x a + |c|^2 - \frac{1}{4}\bar{V}_x[B_1 B_1' - \gamma^{-2} B_2 B_2']\bar{V}_x', \tag{25a}$$

and the corresponding (disturbance attenuating) controller is (as the counterpart of (23b)):

$$\overline{\mu}_\gamma(x) \quad = \quad -\frac{1}{2}B_1'(x)\bar{V}_x'(x). \tag{25b}$$

In all cases, if the HJI inequalities are satisfied as equalities, then the upper and lower values of the underlying games become equal (see, Remark 2.2). Note that here the Isaacs condition is satisfied, as the u and w terms are additively separate, and hence the differential games actually admit saddle points. We now summarize these results for the special affine structure (20a)-(20d) in the following theorem.

Theorem 2.4. *Consider the special structure (20a)-(20d).*

(i) *If there exists $\bar{V}$ satisfying (21) along with (9b) and (10), then the controller (22) is disturbance attenuating for the finite-horizon design problem, with attenuation level no worse than γ.*

(ii) *If there exists a time-invariant $\bar{V}$ satisfying (23a) and (23c), then the controller (23b) is disturbance attenuating for the infinite-horizon problem, with attenuation level no worse than γ. Furthermore, if $|c(x)|^2$ is positive definite, $\bar{V}(x)$ is positive definite and radially unbounded, and $x = 0$ is an equilibrium state of (24), then the origin is globally asymptotically stable in the absence of disturbance w.*

(iii) *If there exists a time-invariant $\bar{V}$ satisfying (25a) and (23c), then the controller (25b) is disturbance attenuating for the discounted infinite-horizon problem, with attenuation level no worse than γ.* □

Remark 2.4. *The result above extends to the more general case where h is also affine in w, that is with (20b) replaced by*

$$h(x,u,w) \quad = \quad c(x) + D_1(x)u + D_2(x)w, \tag{26}$$

where $D_1'D_1 > \alpha I$, for some $\alpha > 0$ and $\forall x$. This formulation can be made equivalent to the earlier one by appropriate transformations on u and w. Define

$$\begin{aligned} \tilde{u} &= R^{\frac{1}{2}}[u + R^{-1}D_1\hat{Q}c(x)] \\ \tilde{w} &= N^{\frac{1}{2}}[w - \gamma^{-2}N^{-1}D_2'(c_1(x) + D_1u)] \end{aligned}$$

where

$$\begin{aligned} \widetilde{Q}(x) &:= c'(x)\hat{Q}[I - D_1'R^{-1}D_1\hat{Q}]c(x) \geq 0 \\ R &:= D_1'\hat{Q}D_1 > \alpha I, \quad N := I - \gamma^{-2}D_2'D_2 > 0 \\ \hat{Q} &:= I + D_2'N^{-1}D_2\,, \end{aligned}$$

where the condition $N > 0$ naturally places a lower threshold on γ: $\gamma^2 > \lambda_{\max}(D_2'D_2)$. In terms of the new control $\tilde{u}$ and new disturbance $\tilde{w}$, the system dynamics become:

$$\dot{x} \quad = \quad \tilde{a}(x) + \widetilde{B}_1\tilde{u} + \widetilde{B}_2\tilde{w}$$

where

$$\begin{aligned} \tilde{a} &= a(x) + B_2N^{-1}D_2'c_1(x) - (B_1 + B_2N^{-1}D_2'D_1)R^{-1}D_1'\hat{Q}c(x) \\ \widetilde{B}_2 &:= B_1N^{\frac{1}{2}} \\ \widetilde{B}_1 &:= (B_1 + B_2N^{-1}D_2'D_1)R^{-\frac{1}{2}}, \end{aligned}$$

and the term $|z|^2 - \gamma^2|w|^2$ becomes

$$|h(x,u,w)|^2 - \gamma^2|w|^2 \quad = \quad \tilde{g}(x) + |\tilde{u}|^2 - \gamma^2|\tilde{w}|^2$$

where

$$\tilde{g}(x) \quad := \quad x'\widetilde{Q}(x)x =: |\tilde{c}(x)|^2.$$

This is now in exactly the same form as the formulation that led to Theorem 2.4, with only tilde'd variables replacing the untilde'd ones. With this change, the statement of Theorem 2.4 equally *applies to this more general formulation. One caveat here is that the statement made just before Theorem 2.4, on the existence of a saddle point, does not necessarily apply unless $D_2'D_1 = 0$, which then decouples in the transformation $\tilde{w}$ from $\tilde{u}$. Otherwise, the disturbance has to be a function of u (informational advantage to the maximizing player), which means that the existence of a lower value function does not necessarily imply the existence of a bounded upper value function.* □

Linear-quadratic problems

A further special structure is the one where both f and h are linear in x as well. In (20a)-(20b), let

$$a(x) = Ax, \qquad c(x) = Cx \tag{27}$$

where A and C are x-independent matrices (possibly with time varying entries), with (in view of (20c)) $D'C = 0$. Also let B_1 and B_2 be independent of x, and again possibly time varying for the finite-horizon problem. Then, the solution to (16), whenever it exists, is in quadratic form, $x'Zx$, and satisfies (16) as an equality. The matrix $Z(t)$ here is nonnegative definite, and is obtained as the unique solution of the generalized Riccati differential equation (GRDE):

$$\begin{aligned} \dot{Z} + A'Z + ZA - Z(B_1B_1' - \gamma^{-2}B_2B_2')Z + C'C &= 0 \\ Z(t_f) &= Q_f. \end{aligned} \tag{28}$$

The counterpart of this in the infinite-horizon case is the generalized algebraic Riccati equation (GARE):

$$A'Z + ZA - Z(B_1B_1' - \gamma^{-2}B_2B_2')Z + C'C = 0, \tag{29}$$

and in the discounted cost case we again have a GARE:

$$\left(A - \frac{\rho}{2}I\right)' Z + Z\left(A - \frac{\rho}{2}I\right) - Z(B_1B_1' - \gamma^{-2}B_2B_2')Z + C'C = 0. \tag{30}$$

The counterpart of Theorem 2.4 for this linear structure (in which case the robust control problem is known as the H^∞ control problem) is the following [*Başar and Bernhard (1995)*]:

Theorem 2.5. *Consider the special structure (20a)-(20d) along with (27) and with $x_0 = 0$, which constitutes the linear-quadratic H^∞ optimal control problem.*

(i) *There exists a* $\gamma^* > 0$ *such that for all* $\gamma > \gamma^*$*, GRDE (28) admits a unique nonnegative-definite solution,* Z_γ*, and for each such* γ *the controller*

$$\mu_\gamma^*(t;x) \quad = \quad -B_1' Z_\gamma(t) x, \qquad t \geq 0 \tag{31}$$

guarantees (for the finite-horizon problem) a level of disturbance attenuation no worse than γ*. If* $\gamma < \gamma^*$*, the GRDE (28) does not admit a solution, and there is no control that guarantees a disturbance attenuation level of* γ*, or equivalently the upper value of the underlying differential game is infinite.*

(ii) *For the time-invariant infinite-horizon problem, let* (A, B_1) *be a stabilizable*[1] *pair, and* (A, C) *be a detectable*[2] *pair. Then, there exists a* $\gamma_\infty^* > 0$ *such that for all* $\gamma > \gamma_\infty^*$ *the GARE (29) admits a unique minimal nonnegative-definite solution,* Z_γ^+*, and the upper value of the underlying differential game is finite (in fact zero). For each such* γ*, the time-invariant controller*

$$\mu_\gamma^\infty(x) \quad = \quad -B_1' Z_\gamma^+ x \tag{32}$$

ensures a level of disturbance attenuation no worse than γ*, and leads to a bounded input-bounded state stable system; in particular, if* $w = 0$*, then the origin* ($x = 0$) *is asymptotically stable. If* $\gamma < \gamma^*$*, the GARE (29) does not admit any nonnegative-definite solution, and the upper value of the associated game is infinite.*

(iii) *For the discounted infinite-horizon problem, let* (A_ρ, B_1) *be stabilizable, and* (A_ρ, C) *be detectable, where* $A_\rho := A - (\rho/2)I$*. Then again there exists* $\gamma_\rho^* > 0$ *such that for all* $\gamma > \gamma_\rho^*$ *the GARE (30) admits a minimal nonnegative-definite solution,* $Z_{\gamma,\rho}^+$*. For each such* γ*, the controller (32) with* Z_γ^+ *replaced by* $Z_{\gamma,\rho}^+$ *guarantees a level of disturbance attenuation no worse than* γ*. If* $\gamma < \gamma_\rho^*$*, the upper value of the infinite-horizon discounted-cost differential game is infinite.* □

---

[1] (A, B) is *stabilizable* if there exists a constant matrix K such that all eigenvalues of the matrix $A + BK$ have negative real parts.

[2] (A, C) is detectable, if (A', C') is stabilizable.

Remark 2.5. *Part (iii) of the theorem above does not allude to any stability result, because with* $\rho > 0$ *one can only guarantee that the real part of the eigenvalues of the closed-loop system matrix* $A - B_1 B_1' Z^+_{\gamma,\rho}$ *is less than* $\rho/2$. *For the "positively discounted" case, however, that is when* $\rho < 0$, *there is a guaranteed stability margin of* $\rho/2$. □

3 Risk-Sensitive Design

We now consider a stochastic formulation and bring in *robustness* through exponentiation of the performance index. Without specifically identifying the dimensions of various processes involved, we have the state dynamics evolving according to the vector Itô stochastic differential equation [*Karatzas and Shreve(1991)*]

$$dx_t = a(t, x_t)dt + B_1(t, x_t)u_t dt + \sqrt{\epsilon} B_2(t, x_t)db_t, \quad x_{t|t=0} = x_0. \tag{33a}$$

$$z_t = c(t, x_t) + Du_t \tag{33b}$$

where $\{b_t, t \geq 0\}$ is a standard vector-valued Brownian motion process [*Karatzas and Shreve(1991)*], with $b_0 = 0$ with probability 1; $u = \{u_t, t \geq 0\}$ is the control process adapted to the sigma-field generated by $x = \{x_t, t \geq 0\}$ with the underlying probability space being $(\Omega, \mathcal{F}, \mathcal{P})$; ϵ is a positive parameter; $B_2 B_2' > \alpha I$, for some $\alpha > 0$; D is a constant matrix, with $D'D = I$, $D'c = 0$. Introducing the counterpart of the cost (7) in this case as

$$L(t_f; x, u) = |x_{t_f}|^2_{Q_f} + \|c(x)\|^2_{t_f} + \|u\|^2_{t_f}, \tag{34a}$$

we take as the performance index

$$J^{t_f}_\theta(u) = \frac{2}{\theta} \ln E\left\{\exp\left[\frac{\theta}{2} L(t_f; x, u)\right]\right\} \tag{34b}$$

which is to be minimized over all admissible control laws. Here θ is the risk-sensitivity parameter (already introduced in Section 1), with $\theta < 0$ corresponding to a *risk-seeking* mode, $\theta > 0$ to a *risk-averse* mode, and $\theta = 0$ (as a limit) to the *risk-neutral* mode of behavior.

Let $\psi(t; x)$ be the value function (minimum cost-to-go function) associated with the cost function

$$E\left\{\exp\frac{\theta}{2}\left[|x_f|^2_{Q_f} + \int_t^{t_f} (|c(x)|^2 + |u|^2)dt\right]\right\}, \tag{35}$$

and $V(t;x)$ be the value function associated with (34b), and note the relationship

$$V(t;x) \quad = \quad \frac{2}{\theta} \ln \psi(t;x). \tag{36}$$

If $\psi(t;x)$ is continuously differential in t and twice continuously differentiable in x (this can be assured under some regularity conditions on a, B_1, ρ_1, B_2, and c; see *Başar and Bernhard (1995)*), then dynamic programming and Itô differentiation rule [*Karatzas and Shreve(1991)*] leads to the following PDE satisfied by ψ (where we suppress the arguments of ψ, a, B_1, B_2, and c):

$$\inf_u \left\{ \psi_t + \psi_x \cdot (a + B_1 u) + \frac{\epsilon}{2} \mathrm{Tr}\left[\psi_{xx} B_2 B_2'\right] + \frac{\theta}{2}(|c|^2 + |u|^2)\psi \right\} = 0; \quad \psi(t_f;x) = \exp \frac{\theta}{2}|x|^2_{Q_f}. \tag{37}$$

Using the relationship (36), and carrying out the quadratic function minimization with respect to u, we arrive at the following corresponding PDE for V:

$$-V_t \quad = \quad V_x \cdot a + |c|^2 - \frac{1}{4} V_x (B_1 B_1' - \epsilon^2 \theta^2 B_2 B_2') V_x' + \frac{\epsilon}{2} \mathrm{Tr}[V_{xx} B_2 B_2']; \quad V(t_f;x) = |x|^2_{Q_f}, \tag{38a}$$

with the corresponding optimal (risk-sensitive) control being

$$u_t \quad = \quad \mu^*(t,x) \quad = \quad -\frac{1}{2} B_1'(x) V_x'(t;x). \tag{38b}$$

Now note that if we let $\theta^2 \epsilon^2 = \gamma^{-2}$ (a constant) in (38a) and let $\epsilon \to 0$, then assuming that the limit is well-defined and no singularities emerge, the limiting version of (38a) becomes equivalent to (21) with equality, and thereby the optimal risk-sensitive controller becomes identical with the robust (deterministic) controller (22). With the parameterization $\theta^2 \epsilon^2 = \gamma^{-2}$, the limit $\epsilon \to 0$ is actually a "large deviation limit" where roughly speaking as the system (33a) becomes 'more deterministic' the performance index (34b) places (through positive exponentiation) relatively more weight on sample paths with larger costs, so that in the limit a 'robustness' property is retained for the deterministic system.

To see this from a different perspective, consider the following stochastically perturbed version of the deterministic dynamics (6a)-(6b) with f and h given by (20a) and (20b):

$$dx_t \quad = \quad (a + B_1 u_t + B_2 w_t) dt + \sqrt{\epsilon} B_2 db_t, \quad x_{t|t=0} = x_0, \tag{39a}$$

$$z_t \quad = \quad c(t, x_t) + D u_t \tag{39b}$$

where a, B_1, B_2 are functions of t and x_t, and b is the standard vector-valued Brownian motion, as before. As the counterpart of the soft-constrained performance index $J_\gamma^{t_f}$ introduced through (8), consider the one:

$$\bar{J}_{\gamma}^{t_f}(\mu,\nu) \quad = \quad E\left\{|x_{t_f}|^2_{Q_f} + \|c(x)\|^2_{t_f} + \|u\|^2_{t_f} - \gamma^2\|w\|^2_{t_f}\right\} \tag{40}$$

where E denotes expectation with respect to the statistics of the Brownian motion process, μ is the feedback control law, and ν is the feedback policy of the disturbance w. Instead of the deterministic differential game of Section 2, we now have a *stochastic* differential game, with (40) to be minimized by μ and maximized by ν, both subject to (39a). Let this game have a smooth value function, $W(t;x)$, whose existence can be guaranteed again under some regularity conditions on a, B_1, B_2, and c. For this stochastic differential game, the Hamilton-Jacobi-Isaacs (HJI) equation satisfied by the value function W is:

$$\begin{aligned} \inf_u \sup_w \left\{W_t + W_x\cdot(a+B_1u+B_2w) + |c|^2 + |u|^2 - \gamma^2|w|^2 + \frac{\epsilon}{2}\mathrm{Tr}[W_{xx}B_2B_2']\right\} &= 0; \\ W(t_f;x) &= |x|^2_{Q_f}. \end{aligned} \tag{41}$$

Since the minimization and maximization operations are separable here (i.e., Isaacs conditions holds), and assuming (as we have done all along) that u and w are unconstrained, the HJI equation becomes

$$\begin{aligned} -W_t &= W_x\cdot a + |c|^2 - \frac{1}{4}W_x(B_1B_1' - \gamma^{-2}B_2B_2')W_x' + \frac{\epsilon}{2}\mathrm{Tr}[W_{xx}B_2B_2']; \\ W(t_f;x) &= |x|^2_{Q_f}, \end{aligned} \tag{42a}$$

and the minimizing control in (41) is

$$u_t \quad = \quad \mu^*(t,x) \quad = \quad -\frac{1}{2}B_1'(x)W_x'(t;x). \tag{42b}$$

Note that the two PDEs (38a) and (42a) are identical (under the correspondence $\gamma^{-2} = \epsilon^2\theta^2$), and so are the controls (38b) and (42b). Hence, the optimal controller in the risk-sensitive control problem is the minimizer's saddle-point policy in a particular stochastic differential game, which itself is obtained from the deterministic robust control problem of Section 2 by perturbing its state dynamics by an appropriately weighted Brownian motion process. Conversely, as $\epsilon \to 0$ in the stochastic differential game, we clearly capture the original deterministic robust control problem, which explains why the large deviation limit of the risk-sensitive optimal control problem is the deterministic robust control problem of Section 2.

The above relationship between the risk-sensitive optimal control problem and a stochastic differential game also leads to a robustness property of the solution of the former, which is that

$$E\left\{|x^*_{t_f}|^2_{Q_f} + \|c(x^*)\|^2_{t_f} + \|u^*\|^2_{t_f}\right\} \quad \leq \quad \gamma^2\|w\|^2_{t_f} + V(0;x_0) \tag{43}$$

where x^* is generated by (39a) under $u = u^* = \mu^*(\cdot,\cdot)$, and is a function of w. This says that the solution to the risk-sensitive optimal control problem is *robust* to additive perturbations in the state dynamics, which enter through the same channels as the Brownian motion process does, in the sense that the resulting perturbation in the cost function is bounded by a scalar multiple of the square of the norm of the disturbance.

For the linear-quadratic risk-sensitive control problem, where

$$a(x) = Ax, \qquad c(x) = Cx,$$

the solution of the PDEs (38a) and (42a) (whenever it exists) is in the form:

$$V(t;x) \;=\; W(t;x) \;=\; x'Z(t)x + m(t) \tag{44a}$$

where Z satisfies the GRDE (28), and m is given by

$$m(t) \;=\; \epsilon\int_t^{t_f} \mathrm{Tr}[Z(s)B_2(s)B_2'(s)]ds. \tag{44b}$$

Note that the corresponding controls (38b) or (42b) are still given by (31), and hence the robust (H^∞) optimal control of Section 2 and the risk-sensitive optimal control of this section are identical in the linear-quadratic case, even without going to the limit as $\epsilon \to 0$. The corresponding costs, however, are different in the two cases, with the difference being the positive term (44b). Note that this positive term grows with t_f, and hence for the infinite-horizon time-invariant linear-quadratic (LQ) risk-sensitive stochastic control problem (and also for the infinite-horizon time-invariant stochastic differential game) the cost is infinite for all controllers. To make the problem meaningful we have to consider the time-average risk-sensitive cost:

$$\bar{J}_\theta^\infty(u) \;:=\; \limsup_{t_f\to\infty}\frac{1}{t_f}J_\theta^{t_f}(u) \tag{45}$$

whose minimum is given by (using (44a) and (44b))

$$m^* \;=\; \epsilon\mathrm{Tr}[\bar{Z}B_2B_2']$$

provided that

$$\bar{Z} \;:=\; \lim_{t_f \to \infty} Z(t_f;t)$$

exists, where $Z(t_f;t)$ is the nonnegative-definite solution of the GRDE (28), with $\gamma^{-2} = \epsilon^2\theta^2$, and $Z(t_f;t_f) = 0$. $\bar{Z}$ clearly satisfies the GARE (29), and it is, in fact, its unique minimal nonnegative-definite solution (Z^+) as per Theorem 2.5(ii)). If (A, B_1) is stabilizable, and (A, C) is detectable, we know (again from Theorem 2.5(ii)) that such a solution exists, and all eigenvalues of the closed-loop matrix $A - B_1B_1'Z^+$ have negative real parts (i.e., the matrix is Hurwitz), provided that γ exceeds a certain threshold γ^*, or equivalently for this case (and for each fixed $\epsilon > 0$) $\theta < \theta^*$, for some positive θ^*. Because of the equivalence between the risk-sensitive LQ control problem and the H^∞ control problem at the control design stage (as mentioned earlier, for the finite-horizon case), the optimal controller for the infinite-horizon time-average risk-sensitive control problem is still given by (32), provided that $\theta < \theta^*$ (equivalently, $\gamma > \gamma^*$).

4 Piecewise-Deterministic Systems

We now consider systems with piecewise deterministic dynamics, that is systems whose dynamics experience abrupt changes (jumps) at a finite or countably infinite number of time instants, with both the timings and nature of the jumps governed by a probability law. Specifically, we have (6a) and (6b) generalized in this case to

$$\dot{x} \;=\; f(t;x,u,w;\theta), \qquad x(0) = x_0, \quad t \geq 0 \tag{46a}$$

$$z \;=\; h(t,x,u,w;\theta), \tag{46b}$$

where $\theta(t)$, $t \geq 0$, is a Markov jump process[3] with right-continuous sample paths, with initial distribution π_0, and with rate matrix $\Lambda = \{\lambda_{ij}\}_{i,j=1,\ldots,s}$, where λ_{ij}'s are real numbers such that for $i \neq j$, $\lambda_{ij} \geq 0$, and $\forall i \in \mathcal{S} := \{1,\ldots,s\}$, $\lambda_{ii} = -\sum_{j\neq i}\lambda_{ij}$. What we in fact have is that, $\forall i,j \in \mathcal{S}$,

$$\text{Prob}\;\{\theta(t+h) = j \mid \theta(t) = i\} \;=\; \begin{cases} \lambda_{ij}h + o(h), & j \neq i \\ 1 + \lambda_{ii}h + o(h), & j = i \end{cases} \tag{47}$$

where $o(h)/h \to 0$ as $h \to 0$.

[3] Even though we had already used θ to denote the risk sensitivity parameter in Section 3, the context in which it is used here is so different that this abuse of notation should not create any ambiguity.

The control u, at time t, is allowed to depend not only on $x(\tau)$, $\tau \leq t$, but also on $\theta(\tau)$, $\tau \leq t$, and so is the disturbance w. This means that both the control and the disturbance are aware of the past and present (if any) structural changes (jumps) the system experiences.

The performance index is the expected value of (7), with Q_f also allowed to depend on θ; that is, as a special case of (1), and in view of (7), we have

$$L(t_f; w, x_0, u) \quad = \quad E_\theta\{|x(t_f)|^2_{Q_f(\theta(t_f))} + \|z\|^2_{t_f}\}, \tag{48}$$

where $Q_f(\cdot) \geq 0$. The robust control problem then (as the counterpart of (8) in this case) is that of finding a controller $u^\circ(t) = \mu^\circ(t, x, \theta)$, with causal dependence on x and θ, such that for all $w(t) = \nu(t, x, \theta)$, and for all x_0,

$$J^{t_f}_\gamma(\mu^\circ; \nu, x_0) \quad := \quad L(t_f; w, x_0, \mu^\circ) - E_\theta\{\gamma^2\|w\|^2\} - \gamma^2|x_0|^2_{Q_0} \leq 0. \tag{49}$$

Note that we again have a zero-sum differential game between the controller, μ, and the disturbance (ν, x_0). To obtain the counterpart of Theorem 2.1 in this case, we let $\bar{V}^i(t; x)$ stand for the upper value of a similar differential game starting at time t, in system state x, and with $\theta(t) = i$. Let $\bar{V}^i$ be continuously differentiable in the pair (t, x), and this be true $\forall i \in \mathcal{S}$. Then, the counterparts of (9a)-(9b) are (with $Q^i_f := Q_f(\theta)\,|_{\theta=i}$):

$$-\bar{V}^i_t \quad \geq \quad \inf_u \sup_w \left[\bar{V}^i_x \cdot f(t, x, u, w; i) + |h(t, x, u, w; i)|^2 - \gamma^2|w|^2 + \sum_{j=1}^{s} \lambda_{ij}\bar{V}^j(t, x)\right] \tag{50a}$$

$$\bar{V}^i(t_f; x) \quad \geq \quad |x|^2_{Q^i_f}, \quad i \in \mathcal{S}, \tag{50b}$$

which constitute a set of linearly coupled partial differential inequalities. Let $\bar{V}^i$ further satisfy

$$\sup_x \left\{\bar{V}^i(0; x) - \gamma^2|x|^2_{Q_0}\right\} \quad = \quad 0, \quad i \in \mathcal{S}. \tag{51}$$

Then, we have the following.

Theorem 4.1. *Let $\bar{V}^i$, $i \in \mathcal{S}$, exist, satisfying (50a)-(50b), with property (51). Let $u = \overline{\mu}_\gamma(t, x, \theta)$ be a controller that minimizes the RHS of (50a). Then,*

$$J^{t_f}_\gamma(\overline{\mu}_\gamma; \nu, x_0) \quad \leq \quad 0 \qquad \forall(\nu, x_0). \tag{52}$$

Thus, $\overline{\mu}_\gamma$ provides a robust controller for the piecewise deterministic system, guaranteeing a disturbance attenuation level of γ.

Proof. Let us first note that if $V(t,x;\theta)$ is continuously differentiable in (t,x) for each fixed (t,x,θ), then its infinitesimal generator on the sample paths of (46a), and in view of (47) is [*Fleming and Soner (1993)*]:

$$\begin{aligned} \mathcal{L}V(t,x;\theta)|_{\theta=i} &:= \lim_{h\downarrow 0} E\left\{V(t+h,x(t+h);\theta(t+h)) - V(t,x(t),\theta(t)) \mid x(t)=x, \theta(t)=i\right\} \\ &= V_t(t,x;i) + V_x(t,x;i)f(t,x,u,w;i) + \sum_{j=1}^{s} \lambda_{ij} V(t,x;j) \end{aligned} \tag{53a}$$

where u and w are chosen to be memoryless functions of x and θ. Furthermore, again from *Fleming and Soner (1993)* or *Rishel (1975)*,

$$E_\theta\left[V(t_f,x(t_f);\theta(t_f))\right] - E_\theta\left[V(0,x_0;\theta(0))\right] = E_\theta \int_0^{t_f} \mathcal{L}V(t,x(t);\theta(t))dt. \tag{53b}$$

Let us introduce, as in the proof of Theorem 2.1,

$$\begin{aligned} \bar{f}(t,x,w;\theta) &:= f(t,x,\overline{\mu}_\gamma(t,x,\theta),w;\theta), \\ \bar{g}(t,x,w;\theta) &:= |h(t,x,\overline{\mu}_\gamma(t,x,\theta),w;\theta)|^2, \end{aligned}$$

and note the following sequence of equalities and inequalities:

$$\begin{aligned} J_\gamma^{t_f}(\overline{\mu}_\gamma;w,x_0) &= E_\theta\left\{|x(t_f)|^2_{Q_f(\theta)} + \int_0^{t_f} \bar{g}(t,x,w;\theta)dt - \gamma^2\|w\|^2 - \gamma^2|x_0|^2_{Q_0}\right\} \\ &\le E_\theta\left\{\bar{V}^{\theta(t_f)}(t_f;x(t_f)) + \int_0^{t_f} \bar{g}(t,x,w;\theta)dt - \gamma^2\|w\|^2 - \gamma^2|x_0|^2_{Q_0}\right\} \\ &= E_\theta\left\{\int_0^{t_f} (\mathcal{L}\bar{V}^{\theta(t)} + \bar{g} - \gamma^2|w|^2)dt + \bar{V}^{\theta(0)}(0,x_0)\right\} - \gamma^2|x_0|^2_{Q_0} \\ &\le E_\theta\left\{\int_0^{t_f} \left(-\sup_w\left\{\bar{V}_x^{\theta(t)}\cdot\bar{f} + \bar{g} - \gamma^2|w|^2\right\} + \bar{V}_x^{\theta(t)}\cdot\bar{f} + \bar{g} - \gamma^2|w|^2\right)dt \right. \\ &\qquad \left. + \bar{V}^{\theta(0)}(0,x_0)\right\} - \gamma^2|x_0|^2_{Q_0} \\ &\le E_{\theta(0)}\bar{V}^{\theta(0)}(0,x_0) - \gamma^2|x_0|^2_{Q_0} \le 0. \end{aligned}$$

In the above, the first inequality (on the second line) follows from (50b) by averaging over the statistics of $\theta(t_f)$, the equality on the third line follows from (53b), the inequality on the fourth line follows from (50a) by taking expectations with respect to θ, the next inequality is just a property of maximization, and the final inequality follows from (51). □

Remark 4.1. *If we have equalities in (50a) and (50b), then the proof of Theorem 4.1 leads to*

$$\inf_{\mu} \sup_{\nu} J_\gamma^{t_f}(\mu;\nu,x_0) \;=\; \sum_{i=1}^{s} \bar{V}^i(0,x_0)\pi_0(i) - \gamma^2|x_0|^2_{Q_0} \quad \forall x_0, \tag{54}$$

where $\pi_0(i) = \text{Prob}\ (\theta(0) = i)$, $i \in \mathcal{S}$. *The quantity on the LHS of (54) is the upper value of the underlying zero-sum game between the controller and the disturbance.* □

A special structure

To strengthen the result of Theorem 4.1, we now consider special structures for f and h, as counterparts of (20a)-(20d). Let

$$f(x,u,w;\theta)|_{\theta=i} \;=\; a^i(x) + B_1^i(x)u + B_2^i(x)w, \quad i \in \mathcal{S}, \tag{55a}$$

$$|h(x,u,w;\theta)|^2\Big|_{\theta=i} \;=\; |c^i(x) + Du|^2 = |c^i(x)|^2 + |u|^2. \tag{55b}$$

Under this structure, (50a) simplifies to

$$-\bar{V}_t^i \;\geq\; \bar{V}_x^i a^i + |c^i|^2 - \frac{1}{4}\bar{V}_x^i[B_1^i B_1^{i'} - \gamma^{-2}B_2^i B_2^{i'}]\bar{V}_x^{i'} - \sum_{j=1}^{s} \lambda_{ij}\bar{V}^j \tag{56}$$

which has to be solved subject to (50b). This structure also allows for an explicit construction of the robust controller $\overline{\mu}_\gamma$, which is

$$\overline{\mu}_\gamma(t,x,\theta)\Big|_{\theta=i} \;=\; -\frac{1}{2}B_1^{i'}(x)\bar{V}_x^{i'}(t;x), \quad i \in \mathcal{S}. \tag{57}$$

Note that the robust controller involves a bank of s controllers, one for each state of the Markov chain, and in the actual implementation the controller switches accordingly as the Markov chain switches from one state to another.

Remark 4.2. *If we have equalities in (56) and (50b), then the quantity in (54) is the actual value of the zero-sum differential game, since the Isaacs condition is satisfied (i.e., the minimization and maximization operations separate out) under the structure (55a)-(55b). The maximizing policy for the disturbance, corresponding to the (minimizing) robust controller (57) is*

$$\overline{\nu}_\gamma(t,x,\theta)|_{\theta=i} \;=\; \frac{1}{2}\gamma^{-2}B_2^{i'}(x)\bar{V}_x^{i'}(t;x), \quad i \in \mathcal{S}. \tag{58}$$

If the value function is not continuously differentiable in (t,x), but only continuous, then the solution to the set of coupled HJI equations will have to be interpreted in the viscosity sense [Fleming and Soner (1993), Sethi and Zhang (1994)]. □

Linear-quadratic structure

Even more explicit results can be obtained, and the infinite-horizon case can be studied, for the further special case where in (55a)-(55b), $a^i(x) = A^i x$, $c^i(x) = C^i x$, with A^i, C^i, B_1^i, B_2^i not depending on x, but possibly depending on t. Then, the solution to the equality version of (56) subject to (50b) as an equality, is in the quadratic form $x' Z^i x$, $i \in \mathcal{S}$, whenever it exists. The Z^i's here satisfy the linearly coupled GRDEs;

$$\dot{Z}^i + A^{i'} Z^i + Z^i A^i - Z^i (B_1^i B_1^{i'} - \gamma^{-2} B_2^i B_2^{i'}) Z^i + C^{i'} C^i + \sum_{j=1}^{s} \lambda_{ij} Z^j = 0; \quad Z^i(t_f) = Q_f^i. \tag{59}$$

The counterpart of this in the infinite-horizon case (i.e., as $t_f \to \infty$, and as an extension of (29)), with all matrices being constants, and $Q_f^i = 0 \ \forall i \in \mathcal{S}$, is the set of linearly coupled GAREs:

$$A^{i'} Z^i + Z^i A^i - Z^i (B_1^i B_1^{i'} - \gamma^{-2} B_2^i B_2^{i'}) Z^i + C^i C^{i'} + \sum_{j=1}^{s} \lambda_{ij} Z^j \quad = \quad 0, \quad i \in \mathcal{S}. \tag{60}$$

This now leads to the following theorem, which constitutes an extension of Theorem 2.5 to switching systems (or systems with Markov jump parameters). A rigorous proof of this result is rather involved, and can be found in *Pan and Başar (1995)*.

Theorem 4.2. *Consider the special structure (55a)-(55b), along with $a^i(x) = A^i(x)$, $c^i(x) = C^i x$, $i \in \mathcal{S}$; A^i, C^i, B_1^i, B_2^i all matrices possibly depending on t, and with $x_0 = 0$, which constitutes the linear-quadratic H^∞ optimal control problem with switching dynamics.*

(i) *There exists a $\gamma^* > 0$ such that for all $\gamma > \gamma^*$, the set of GRDEs (59) admits a nonnegative-definite solution set Z_γ^i, $i \in \mathcal{S}$, and for each such γ the controller*

$$\mu_\gamma^{*i}(t,x) \quad = \quad -B_1^{i'} Z_\gamma^i(t) x, \qquad t \geq 0, \qquad i \in \mathcal{S},$$

guarantees (for the finite-horizon problem) a level of disturbance attenuation no worse than γ. If $\gamma < \gamma^*$, the set of GRDEs (59) does not admit a solution, and there is no control that guarantees a disturbance attenuation level of γ.

(ii) *For the time-invariant infinite-horizon problem, let the pair* $(A(\theta(t)), B_1(\theta(t)))$ *be stochastically stabilizable,*[4] *where* $A(\theta)|_{\theta=i} = A^i$, $B_1(\theta)|_{\theta=i} = B_1^i$, *and the pair* (A^i, C^i) *be observable for each* $i \in \mathcal{S}$. *Further, let the Markov chain* $\theta(t)$, $t \geq 0$, *be irreducible. Then, there exists* $\gamma_\infty^* > 0$ *such that for each* $\gamma > \gamma_\infty^*$ *there exists a set of minimal positive-definite solutions,* $\bar{Z}_\gamma^i$, $i \in \mathcal{S}$, *to the set of GAREs (60), and the upper value of the underlying differential game is zero. For each such* γ, *the time-invariant switching controller*

$$\mu_\gamma^{\infty i}(x) = -B_1^{i\prime} \bar{Z}_\gamma^i x, \quad i \in \mathcal{S},$$

ensures a level of disturbance attenuation no worse than γ, *and leads to a mean-square stable system dynamics in the absence of disturbance, i.e.,* $E[|x(t)|^2] \to 0$ *as* $t \to \infty$, *where* x *is generated by*

$$\dot{x} = [\, A(\theta(t)) - B_1(\theta(t)) B_1'(\theta(t)) \bar{Z}(\theta(t)) \,] x$$

where

$$\bar{Z}(\theta)|_{\theta=i} = \bar{Z}^i, \quad i \in \mathcal{S}.$$

If $\gamma < \gamma_\infty^*$, *the set of GAREs (60) does not admit any set of nonnegative-definite solutions, and the upper value of the associated differential game is infinite.* □

5 Robust Filtering for Deterministic Systems

We now turn to problems where full state is not available, but only a noise-corrupted version of the state is measured, and it is desired to deduce its true value. We consider the control-free version of the dynamics (6a), which we write by a slight abuse of notation as

$$\dot{x} = f(x, w), \quad x(0) = x_0, \quad t \geq 0, \tag{61}$$

where dependence on t has been suppressed. Furthermore, we take the measurement equation as

$$y = h(x, w). \tag{62}$$

[4] $(A(\theta), B(\theta))$ is *stochastically stabilizable* if there exists a matrix-valued function of θ, $K(\theta)$, such that the dynamic system $\dot{x} = [A(\theta(t)) - B(\theta(t))K(\theta(t))]x$, $x(0) = x_0$, is *mean-square stable*, i.e., $E[|x(t)|^2] \to 0$ as $t \to \infty$, $\forall x_0$ [*Ji and Chizeck (1990)*].

Let $\hat{x}(t)$ denote an estimate for x at time t, using $y_{[0,t)} := \{y(\tau), 0 \leq \tau < t\}$, which we write as

$$\hat{x}(t) \quad = \quad \delta_t(y_{[0,t)}), \quad t \geq 0, \tag{63}$$

where δ_t, $t \geq 0$, is the function to be determined. Consistent with the objective at hand, the error term, e, introduced in Section 1 is now, at time τ,

$$e(\omega; \delta) \quad = \quad x(\tau) - \delta_\tau(y_{[0,\tau)})$$

where the uncertainty ω is $\omega = (w, x_0)$. The counterpart of the performance index $J_\gamma^{t_f}$ of Section 2 is, for each $t \geq 0$,

$$J_\gamma^t(\delta; w) \quad = \quad \int_0^t \left\{ |x(\tau) - \delta_\tau(y_{[0,\tau)})|^2 - \gamma^2 |w(\tau)|^2 d\tau - \gamma^2 |x_0 - \bar{x}_0|^2_{Q_0} \right\}, \tag{64}$$

where $\bar{x}_0$ is some initial estimate of x_0. The robust filtering problem is to find δ_t^*, $t \geq 0$, such that for each $t \geq 0$,

$$\sup_\omega J_\gamma^t(\delta^*; \omega) \quad = \quad 0. \tag{65a}$$

Note that this implies the disturbance attenuation property

$$\|x - \delta^*\|_t^2 \quad \leq \quad \gamma^2 \|w\|_t^2 + \gamma^2 |x_0 - \bar{x}_0|^2_{Q_0}, \quad \forall w, x_0. \tag{65b}$$

Such a δ^* can be obtained using *cost-to-come* methods [*Başar and Bernhard (1995), Didinsky, Başar, and Bernhard (1993a, b)*], which employ the concept of forward dynamic programming. Toward this end, introduce *worst-case conditional cost*:

$$W(\delta; t, x; y_{[0,t)}) \quad = \quad \sup_{\omega | x(t) = x, y_{[0,t)}} J_\gamma^t(\delta; \omega) \tag{66}$$

which is the maximum value of (64) over all uncertainty ω up to time t, consistent with the measurements $y_{[0,t)}$ made up to that time, and under the side information that $x(t) = x$ as given and fixed. If W is smooth in t and x, it satisfies, for each fixed δ, the forward dynamic programming (Hamilton-Jacobi-Bellman) equation

$$\sup_{w | y_{[0,t)}} \left\{ -W_t - W_x \cdot f(x, w) + |x - \delta|^2 - \gamma^2 |w|^2 \right\} = 0; \qquad W(\delta; 0, x) \; = -\gamma^2 |x - \bar{x}_0|^2. \tag{67}$$

The robust filter, δ^*, is then obtained by minimizing (over δ) the maximum of W over x. Since only the difference of x and δ appear in W, this optimization is in fact much simpler: δ^* is simply the $\hat{x}(t)$ that solves the maximization problem

$$W(\hat{x}(t); t, \hat{x}(t); y_{[0,t)}) = \max_x W(\hat{x}(t); t, x; y_{[0,t)}). \tag{68}$$

To obtain more concrete results, we now invoke some structure on the functions f and h. In particular, let f and h depend linearly on w:

$$f(x, w) = a(x) + D(x)w \tag{69a}$$

$$h(x, w) = h_0(x) + Ew. \tag{69b}$$

Further, to avoid singularity in the measurement equation, assume that E is surjective, and write w as

$$w = P \begin{pmatrix} \bar{w} \\ v \end{pmatrix} \tag{70a}$$

with $PP' = I$, $EP = [0 \;\; E_2]$, and E_2 of the same dimension as y, and invertible. Further let

$$E_2 E_2' =: N > 0, \qquad DP =: [D_1 \;\; D_2]. \tag{70b}$$

Then, (61) and (62) can be written as

$$\dot{x} = a(x) + D_1(x)\bar{w} + D_2(x)v \tag{71a}$$

$$y = h_0(x) + E_2 v \tag{71b}$$

where $(\bar{w}, v)$ are equivalent disturbances. The implication of this is that part of the original disturbance, namely v, can be expressed in terms of y, due to invertibility of E_2:

$$v = E_2^{-1}y - E_2^{-1}h_0(x), \tag{72a}$$

and hence (in view of also orthonormality of P):

$$|w|^2 = |\bar{w}|^2 + |v|^2 = |\bar{w}|^2 + |y - h_0(x)|^2_{N^{-1}}. \tag{72b}$$

Furthermore, using (72a) in (69a), we have in view of (71a):

$$f(x, w) = a(x) + D_2(x)E_2^{-1}y - D_2(x)E_2^{-1}h_0(x) + D_1(x)\bar{w}.$$

Using all this in (67) leads to a quadratic maximand in $\bar{w}$, which admits the unique maximizing solution

$$\bar{w} \quad = \quad -\frac{\gamma^{-2}}{2} D_1'(x) W_x',$$

and substituting this back into (67) finally leads to the forward PDE:

$$W_t + W_x \cdot (a - D_2 E_2^{-1} h_0 + D_2 E_2^{-1} y) - |x - \delta|^2 + \gamma^2 |y - h_0(x)|^2_{N^{-1}} - \frac{\gamma^{-2}}{4} W_x D_1 D_1' W_x' = 0;$$
$$W(\delta; 0, w) = -\gamma^2 |x - x_0|^2. \quad (73)$$

Under the assumption that this PDE admits a smooth solution, the robust filter is obtained through (68), that is by solving for $\hat{x}(t)$ from:

$$\frac{d}{dx} W(\hat{x}(t); t, x; y_{[0,t)}) \Big|_{x=\hat{x}(t)} \quad = \quad 0. \quad (74)$$

To obtain further insight into the solution, let us total differentiate (74) with respect to g, under the assumption that (68) has a unique maximum and W_{xx} is negative definite at $x = \hat{x}(t)$:

$$W_{xt}'(t, \hat{x}) + W_{xx}(t, \hat{x}) \cdot \dot{\hat{x}}(t) \quad = \quad 0$$

$\Rightarrow$

$$\dot{\hat{x}}(t) \quad = \quad -W_{xx}^{-1}(t, \hat{x}) \cdot W_{xt}'(t, \hat{x}) \quad (75)$$

where by a slight abuse of notation we have written $W(t, x)$ for $W(\hat{x}(t)' t,\, x' y_{[0,t)})$. Now, an explicit expression can be obtained for $W_{xt}(t, \hat{x})$ by differentiating (73) with respect to x, interchanging the two partial differentiations with respect to t and x, and by making use of the fact that $W_x(t, \hat{x}) = 0$:

$$W_{xt}' + W_{xx} \cdot (a - D_2 E_2^{-1} h_0 + D_2 E_2^{-1} y) + 2\gamma^2 h_{0x} N^{-1}(h_0 - y) \quad = \quad 0.$$

Substituting this expression for W_{xt} into (75) finally leads to the robust filter dynamics

$$\dot{\hat{x}}(t) \quad = \quad a(\hat{x}) + [D_2(\hat{x}) E_2^{-1} N - 2\gamma^2 W_{xx}^{-1}(t, \hat{x}) h_{0x}(\hat{x})] N^{-1} [y - h_0(\hat{x})]; \quad \hat{x}(0) = \bar{x}_0, \quad (76)$$

where the initial condition for the filter follows from maximization of $W(\delta; 0, x)$ with respect to x.

The following theorem now summarizes the preceding result.

Theorem 5.1. *Consider the robust filtering problem formulated in this section, with f and h given by (69a)-(69b) and with a nonsingular measurement process (i.e., with E surjective in (69b)). Let there exist a smooth function $W(t,x)$ satisfying the forward PDE (73) and that the maximum in (68) be unique and W_{xx} be negative definite at $x = \hat{x}$. Then, $\delta_t^*(y_{[0,t)}) = \hat{x}(t)$, generated by (76), is a robust filter for the state x, having the disturbance attenuation property:*

$$\|x - \hat{x}\|_t^2 \quad \leq \quad \gamma^2 \|w\|_t^2 + \gamma^2 |x_0 - \bar{x}_0|^2, \qquad \forall (t, w, x_0). \tag{77}$$

Remark 5.1. *The robust filter of Theorem 5.1 is in general infinite-dimensional, as it depends on the solution of a PDE which is driven by the measurement process. As such it is not easily implementable, unless some further structural assumptions are made, as discussed below. In this form, it is similar to the Mortensen filter [Mortensen (1968)] derived using different methods.* □

Linear systems

For the special class of linear systems, (73) can be solved explicitly, leading to a finite-dimensional robust filter. Toward showing this, let f and h be structured as

$$f(x,w) \quad = \quad Ax + Dw \tag{78a}$$

$$h(x,w) \quad = \quad Hx + Ew, \tag{78b}$$

where A, D, H, and E are matrices, possibly dependent on t. Let D_2, E_2, and N be as defined earlier. Then, whenever it exists, the solution to (73) is in the general quadratic form

$$W(t,x) \quad = \quad -\gamma^2 x' \Sigma^{-1} x - 2\gamma^2 x' \ell - m$$

where Σ is a positive-definite matrix obtained from the GRDE:

$$\dot{\Sigma} \quad = \quad \tilde{A}\Sigma + \Sigma \tilde{A}' - \Sigma (H' N^{-1} H - \gamma^{-2} I) \Sigma + D_1 D_1' \,; \quad \Sigma(0) = Q_0^{-1}, \tag{79a}$$

where

$$\tilde{A} \quad := \quad A - D_2 E_2^{-1} H \tag{79b}$$

and ℓ and m satisfy some linear differential equations, whose precise expressions are not relevant to the development below (but being linear DEs they always admit unique solutions).

Now, since $W_{xx} = -2\gamma^2\Sigma^{-1} < 0$, all conditions of Theorem 5.1 are satisfied, and hence the robust filter is finite-dimensional (since W_{xx} does not depend on y), and is generated by

$$\dot{\hat{x}} = A\hat{x} + (D_2E_2^{-1}N + \Sigma H')N^{-1}(y - H\hat{x}); \qquad \hat{x}(0) = \bar{x}_0. \tag{80}$$

Since

$$\begin{aligned} D_2E_2^{-1}N &= DE' =: L && (81a)\\ D_1D_1' &= DD' - LN^{-1}L, && (81b) \end{aligned}$$

the solution (80), (79a)-(79b) can be rewritten in terms of the original system parameters in (78a)-(78b):

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + (L + \Sigma H')N^{-1}(y - H\hat{x}), \quad \hat{x}(0) = \hat{x}_0 && (82a)\\ \dot{\Sigma} &= \tilde{A}\Sigma + \Sigma\tilde{A}' - \Sigma(H'N^{-1}H - \gamma^{-2}I)\Sigma + DD' - LN^{-1}L'; \quad \Sigma(0) = Q_0^{-1} && (82b)\\ \tilde{A} &= A - LN^{-1}H. && (82c) \end{aligned}$$

If the system and measurement noises are "independent," i.e., $DE' = 0$, then $\tilde{A} = A$, and (82b) becomes the dual of the GRDE (28), with the correspondences $t \to t_f - t$, $A \to A'$, $N^{-\frac{1}{2}}H \to B_1'$, $I \to B_2$, $D \to C'$, $Q_0^{-1} \to Q_f$. Hence, as in Theorem 2.5(i), (82b) admits a unique positive-definite solution for all $\gamma > \gamma^*$, for some $\gamma^* > 0$. This is made precise in the theorem below. But first we provide below, for the sake of completeness, the GARE version of (82b), which is the relevant equation for the infinite-horizon case:

$$\tilde{A}\Sigma + \Sigma\tilde{A}' - \Sigma(H'N^{-1}H - \gamma^{-2}I)\Sigma + DD' - LN^{-1}L = 0. \tag{83}$$

Theorem 5.2. *Consider the robust filtering problem for the linear system*

$$\dot{x} = Ax + Dw \tag{84a}$$

with measurement equation

$$y = Hx + Ew \tag{84b}$$

with $EE' =: N > 0$.

(i) There exists a $\gamma^* > 0$ such that for all $\gamma > \gamma^*$, GRDE (82b) admits a unique positive-definite solution, Σ_γ, and for each such γ the filter (82a) guarantees a level of disturbance attenuation no larger than γ, i.e.,

$$\|x - \hat{x}\|_{t_f}^2 \quad \leq \quad \gamma^2 \|w\|_{t_f}^2 + \gamma^2 |x_0 - \hat{x}_0|_{Q_0}^2, \qquad \forall (w, x_0). \tag{85}$$

If $\gamma < \gamma^*$, the GRDE (82b) does not admit a solution, and there is no filter that delivers an attenuation level of γ.

(ii) For the time-invariant infinite-horizon problem, let $(\tilde{A}, D_1)$ be controllable and $(\tilde{A}, H)$ be detectable. Then, there exists a $\gamma_\infty^* > 0$ such that for all $\gamma > \gamma_\infty^*$ the GARE (83) admits a unique minimal positive-definite solution, Σ_γ^+, and the time-invariant filter (82a) with $\Sigma = \Sigma_\gamma^+$ delivers an attenuation no worse than γ. If $\gamma < \gamma_\infty^*$, the GARE (83) does not admit any nonnegative-definite solution, and there is no filter with an attenuation level of γ.

(iii) For $\gamma > \gamma_\infty^*$, the time-invariant filter error dynamics

$$\dot{e} \quad = \quad (\tilde{A} - \Sigma_\gamma^+ H' N^{-1} H) e + (D - (L + \Sigma_\gamma^+ H') N^{-1} E) w$$

is asymptotically stable, i.e., all eigenvalues of the matrix $\tilde{A} - \Sigma_\gamma^+ H' N^{-1} H$ have negative real parts. □

Remark 5.2. *The linear filter of Theorem 5.2 (also called the H^∞ filter) is in the form of a Kalman filter as $\gamma \to \infty$, and hence admits a stochastic interpretation in this limit. In that case the disturbance w is treated as a standard vector white noise, and the initial state x_0 as a Gaussian random vector with mean zero and covariance Q_0^{-1}. The filter state $\hat{x}(t)$ then becomes a minimum mean square estimate of $x(t)$.* □

We conclude this section by providing a stochastic interpretation for the H^∞ filter of Theorem 5.2 not only as $\gamma \to \infty$, but for all $\gamma > \gamma^*$ (or $\gamma > \gamma_\infty^*$ in the infinite-horizon case). This is obtained through a risk-sensitive formulation of the filtering problem as presented below.

Risk-sensitive filter

Consider the linear stochastic signal model

$$dx_t \quad = \quad Ax_t dt + D db_t, \tag{86a}$$

where $\{b_t, t \geq 0\}$ is a standard vector-valued Brownian motion process as in Section 3, and x_0 is a Gaussian random variable with mean $\bar{x}_0$ and covariance $\Sigma_0 > 0$. Also consider the measurement process

$$dy_t \quad = \quad Hx_t dt + E db_t. \tag{86b}$$

Let $\mathcal{Y}_t$ be the sigma-field generated by $\{y_\tau, 0 \leq \tau \leq t\}$, and $\hat{x}_t$ be an estimate for x_t, adapted to $\mathcal{Y}_t$. The risk-sensitive filtering problem is one of minimizing the performance index

$$J \quad = \quad \frac{2}{\theta} \ln E \left\{ \exp \frac{\theta}{2} \int_0^{t_f} |x_t - \hat{x}_t|^2 dt \right\} \tag{87}$$

over all such estimates $\hat{x}_t$, where $\theta > 0$ is a risk-sensitivity parameter.

It has been shown in *Pan and Başar (1996)* that this is a meaningful problem if $\theta < \theta^*$, for some positive scalar θ^*, and for $\theta > \theta^*$ the value of (87) is infinite regardless of the choices of $\hat{x}_t$. For $\theta < \theta^*$, the solution is unique (almost surely), and is generated by

$$d\hat{x}_t \quad = \quad A\hat{x}_t dt + (L + \Sigma H')N^{-1}(dy_t - H\hat{x}_t dt), \quad \hat{x}_0 = E[x_0], \tag{88a}$$

where L and N are as defined earlier in Theorem 5.2, $\Sigma > 0$ is the unique solution of the GRDE

$$\dot{\Sigma} \quad = \quad \tilde{A}\Sigma + \Sigma\tilde{A}' - \Sigma(H'N^{-1}H - \theta I)\Sigma + DD' - LN^{-1}L'; \qquad \Sigma(0) = \Sigma_0, \tag{88b}$$

and $\tilde{A}$ is as defined by (82c).

Note that (88a) and (88b) are identical (in form) with (82a) and (82b), under the correspondences $dy_t \to y(t)$, $\Sigma_0 \to Q_0^{-1}$, $\theta \to \gamma^{-2}$. Hence, the counterpart of Theorem 5.2 can be stated (almost verbatim) for the risk-sensitive filtering problem.

This result now parallels the one in Section 3, where it was shown that the optimal controller for the risk-sensitive stochastic LQ control problem is identical with the H^∞-optimal controller for the deterministic LQ robust control problem. There was, in fact, a more general correspondence covering also nonlinear systems, between the risk-sensitive stochastic control problem and a class of zero-sum stochastic differential games. This more general correspondence had allowed us in Section 3 to ascribe a robustness interpretation to the solution of the risk-sensitive stochastic control problem, which was that the controller features a stochastic disturbance attenuation property with respect to unmodeled perturbations in the system dynamics (see (43)). It would be interesting to explore whether such a feature is also shared by the risk-sensitive filtering problem (86a)-(86b), and this has in fact been done in *Başar (1999b)*. Briefly, if instead of (86a) we have the perturbed (by w_t) signal model:

$$d\zeta_t = A\zeta_t dt + D db_t + C w_t dt, \qquad \zeta_0 = x_0,$$

and instead of (86b) we have the perturbed (by p_t) measurement model:

$$d\tilde{y}_t = H\zeta_t dt + E db_t + G p_t dt, \quad \tilde{y}_0 = 0,$$

where C and G are appropriately picked, and w_t, p_t are general dynamic perturbations, possibly correlated with the measurement process, then the filter (88a), with y replaced by $\tilde{y}$, has the property that

$$E\{\|\zeta - \hat{x}\|^2_{t_f}\} \leq \frac{2\gamma^2}{\theta}\left(\|w\|^2_{t_f} + \|p\|^2_{t_f}\right) + k$$

for some positive constants γ and k. This says that the error in estimating x through a risk-sensitive filter does not grow faster than the norms of the unmodeled perturbations in the signal and measurement processes. Hence, the risk-sensitive filter (88a)-(88b) is *robust* to such perturbations, just as the risk-sensitive controller of Section 3 was robust to perturbations in the system dynamics.

6 Conclusions

We have provided in this paper an overview of robustness issues that arise in controller and filter designs for dynamic systems, and have presented some explicit designs with build-in robustness features. For the controller designs, we have restricted the coverage to state feedback designs, and have not discussed the (noisy) output feedback case. An extensive discussion of the latter would have required the development of additional mathematical and conceptual frameworks, resulting in a much longer paper, which is the reason why we have not included it here. We should remark that for linear-quadratic systems, robust (H^∞ as well as risk-sensitive) controllers feature *certainty* *equivalence*, which makes it possible to combine the results of Sections 2, 3, 4 with those of 5, to come up with designs compatible with output feedback; see *Başar and Bernhard (1995)* for H^∞ control with output feedback, *Pan and Başar (1996)* for risk-sensitive optimal control with output feedback, and *Pan and Başar (1995)* for output feedback control of jump linear systems. Extending these results to nonlinear systems, however, generally meets with formidable difficulties, mainly because certainty equivalence generally does not hold in nonlinear systems, and when it does it holds in a much higher dimensional space, requiring the solution of a state feedback robust control problem where the state is that of the cost-to-come function (information state) which is infinite dimensional [*Başar and Bernhard (1995)* , *James and Baras (1996)*].

References

[1] Ball, J.A. , J.W. Helton, and M.L. Walker (1993) H^∞ control for nonlinear systems via output feedback. *IEEE Transactions on Automatic Control*, 38, 546–558.

[2] Başar, T. (1995) Minimax control of switching systems under sampling, *Systems & Control Letters*. 25(5), 315–325.

[3] Başar, T. (1999a) Nash equilibria of risk-sensitive nonlinear stochastic differential games, *J. Optimization Theory and Applcations*, 100(3), 479–498.

[4] Başar, T. (1999b) Risk-averse designs: From exponential cost to stochastic games. in T. Djaferis (ed.), *Advances in System Theory*. Kluwer (in press)

[5] Başar, T. and P. Bernhard (1995) *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, 2nd ed. Boston, MA: Birkhäuser.

[6] Didinsky, G., T. Başar, and P. Bernhard (1993a) Structural properties of minimax controllers for a class of differential games arising in nonlinear H^∞-control. *Systems & Control Letters*, 21, 433–441.

[7] Didinsky, G., T. Başar, and P. Bernhard (1993b) Structural properties of minimax controllers for a class of differential games arising in nonlinear H^∞-control and filtering. in *Proceedings of the IEEE 32nd Conference on Decision and Control*, pp. 184–189.

[8] Fleming, W.H. and W.M. McEneaney (1992) Risk sensitive control and differential games. *Lecture Notes in Control and Information Sciences*, vol. 184, pp. 185–197. Berlin: Springer.

[9] Fleming, W.H. and W.M. McEneaney (1995) Risk sensitive control on an infinite time horizon. *SIAM Journal on Control and Optimization*, 33, 1881–1915.

[10] Fleming, W.H. and H.M. Soner (1993) *Controlled Markov Processes and Viscosity Solutions*, New York: Springer-Verlag.

[11] Green, M. and D.J.N. Limebeer (1995) *Linear Robust Control* New Jersey: Prentice Hall.

[12] Hansen, L.P. and T.J. Sargent (1995) Discounted linear exponential quadratic Gaussian control. *IEEE Transactions on Automatic Control*, 40(5), 968–971.

[13] Hassibi, B., A. H. Sayed, and T. Kailath (1999) *Indefinite-Quadratic Estimation and Control: A Unified Approach to H^2 and H^∞ Theory*. Philadelphia: SIAM Studies in Applied and Numerical Mathematics.

[14] Isidori, A. and W. Kang (1995) H_∞ control via measurement feedback for general nonlinear systems. *IEEE Transactions on Automatic Control*, 40(3), 466–472.

[15] James, M.R. and J.S. Baras (1996) Partially observed differential games, infinite dimensional HJI equations, and nonlinear H_∞ control. *SIAM Journal on Control and Optimization*, 34,1342–1364.

[16] James, M.R., J. Baras, and R.J. Elliott (1994) Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems. *IEEE Transactions on Automatic Control*, 39(4), 780–792.

[17] Ji, Y. and H.J. Chizeck (1990) Controllability, stabilizability, and continuous-time Markovian jump linear systems. *IEEE Transactions on Automatic Control*, 35, 777–788.

[18] Karatzas, I. and S.E. Shreve (1991) *Brownian Motion and Stochastic Calculus*, 2nd ed., New York: Springer-Verlag.

[19] Lions, P. (1982) *Generalized Solution of Hamilton-Jacobi Equations*, Pitman Publishing Inc.

[20] McEneaney, W. (1995) Uniqueness for viscosity solutions of nonstationary Hamilton-Jacobi-Bellman equations under some a priori conditions (with applications). *SIAM Journal on Control and Optimization*, 33, 1560–1576.

[21] Mortensen, R.E. (1968) Maximum likelihood recursive nonlinear filtering. *Journal of Optimization Theory and Applications*, 2, 386–394.

[22] Pan, Z. and T. Başar (1995) H^∞-control of Markovian jump systems and solutions to associated piecewise-deterministic differential games. In G. J. Olsder (ed.), *Annals of the International Society of Dynamic Games*, vol. 3, pp. 61–92. Boston, MA: Birkhäuser.

[23] Pan, Z. and T. Başar (1996). Model simplification and optimal control of stochastic singularly perturbed systems under exponentiated quadratic cost. *SIAM Journal on Control and Optimization*, 34(5), pp. 1734–1766.

[24] Pan, Z. and T. Başar (1999) Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criteria. *SIAM Journal on Control and Optimization*, 37(3): 957–995.

[25] Rishel, R. (1975) Dynamic programming and minimum principles for systems with jump Markov disturbances. *SIAM Journal on Control and Optimization*, 13, 338–371.

[26] Runolfsson, T. (1994) The equivalence between infinite-horizon optimal control of stochastic systems with exponential-of-integral performance index and stochastic differential games. *IEEE Transactions on Automatic Control*, 39, 1551–1563.

[27] Sethi, S.P. and Q. Zhang (1994) *Hierarchical Decision Making in Stochastic Manufacturing Systems*. Boston, MA: Birkhäuser.

[28] Van der Schaft, A. J. (1992) L_2 gain analysis of nonlinear systems and nonlinear H_∞ control. *IEEE Transactions on Automatic Control*, 37, 770–784.

[29] Van der Schaft, A. J. (1993) Nonlinear state space H_∞ control theory," in H.L. Trentelman and J.C. Williams (eds.), *Essays on Control*, pp. 153–190. Boston, MA: Birkhäuser.

[30] Whittle, P. (1990) *Risk Sensitive Optimal Control*. Wiley.

[31] Xiao, M.-Q. and T. Başar (1999) Nonlinear H^∞ controller design via viscosity supersolutions of the Isaacs equation," in W.M. McEneaney, G. G. Yin, Q. Zhang (eds.), *Stochastic Analysis, Control, Optimization and Applications*, pp. 151-170. Boston, MA: Birkhäuser.