Hierarchical Network Games with Large Numbers of Users

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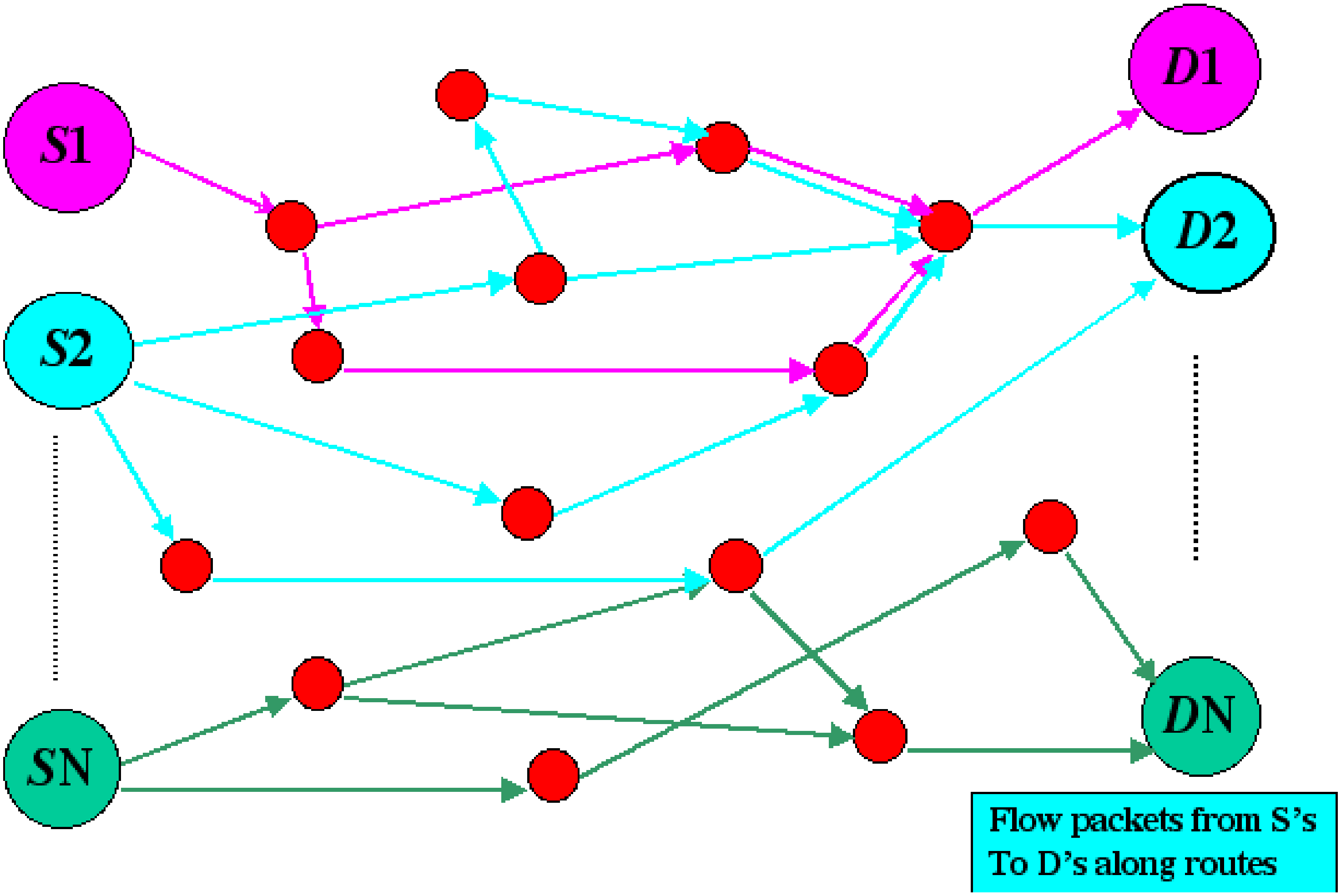
Network Game

non-cooperative users



Flow packets from S's
To D's along routes

Nash equilibrium (N'50, N'51)

(N'50) John Forbes Nash, Jr., Equilibrium points in n-person games, PNAS, 36:48-49

(N'50) Nash, Non-cooperative games, Annals of Math., 54:286-295

(N'98) Sylvia Nasar, A Beautiful Mind, Simon & Schuster

Network

Directed graph $\{V, L, J\}$

V **: set of nodes**

L **: set of directed arcs (links)**

$\mathcal{N}$**: set of users** $(1, \ldots, N)$

J **: vector of costs** $\{J^i\}_{i\in\mathcal{N}}$

λ^i_ℓ**: flow of user** i **on link** ℓ**;** $\lambda^i = \{\lambda^i_\ell\}_{\ell\in L^i}$

`For example :`

$$J^i(\lambda^i_\ell, \lambda^{-i}_\ell, \ell \in L) \equiv J^i(\lambda^i, \lambda^{-i}) \equiv \sum_\ell J^i_\ell(\lambda^i_\ell, \lambda^{-i}_\ell)$$

`A common structure:` $J^i_\ell(\lambda^i_\ell, \lambda_{\ell t}) = \lambda^i_\ell f_l(\lambda_{\ell t})$

`e.g.` $f_l(\lambda_{\ell t}) = 1/(c_\ell - \lambda_{\ell t})$

Conservation law **(for all** $v \in V$**)**

$$r_i(v) + \sum_{\ell\in\mathrm{In}(v)} \lambda^i_\ell = \sum_{\ell\in\mathrm{Out}(v)} \lambda^i_\ell$$

$$\begin{aligned} r_i(v) &= \Lambda^i \quad \text{if} \quad v = s(i) \\ &= -\Lambda^i \quad \text{if} \quad v = d(i) \\ &= 0 \quad \text{else} \end{aligned}$$

`Capacity constraints:` $\lambda_{\ell t} := \sum_{i\in\mathcal{N}} \lambda^i_\ell \le c_\ell,\ \ell \in L$

$\Rightarrow$ $\lambda^i \in \Omega^i$ **Routing; Flow control; Combined**

Game

PLAYERS $i = 1, 2, \ldots, N$

SOLUTION: **Nash equilibrium**

$$\lambda = \{\lambda^i\}_{i=1}^N$$

$$J^i(\lambda^i, \lambda^{-i}) = \min_{\lambda^i \in \Omega^i} J^i(\lambda^i, \lambda^{-i})$$

ROUTING: Total flow of each player fixed

FLOW CONTROL: Routes fixed

COMBINED: Both routes and flows to be determined

ISSUES

- **Existence**
- **Uniqueness**
- **Computation** **(Stability)**
- **Pricing** $J^i = J^i(\lambda^i, \lambda^{-i}; p) \to \lambda(p)$

  $$\max_p R(\lambda(p); p)$$
- **Efficiency** **(Braess paradox !)**

  Architecting networks
- **Asymptotics** **(as $N \to \infty$)**

  Relationship with Wardrop equilibrium

Routing: Nash Equilibrium

(General Results)

Existence: Ω^i's compact, J^i convex in λ^i, continuous in λ

Standard routing game with (additive) link costs has a NE (a convex game).

Uniqueness:

Rosen's diagonal strict convexity (R'65)

Contraction arguments (LB'87)

Standard routing game with additive link costs, parallel links, $J^i_\ell(\lambda^i_\ell, \lambda_{\ell t}) \uparrow (\lambda^i_\ell, \lambda_{\ell t})$, **and**

$$K^i_\ell(\lambda^i_\ell, \lambda_{\ell t}) := \frac{\partial J^i_\ell(\lambda^i_\ell, \lambda_{\ell t})}{\partial \lambda^i_\ell} \uparrow (\lambda^i_\ell, \lambda_{\ell t})$$

has a unique NE (ORS'93). **Further,**

$$\Lambda^i \geq \Lambda^i \quad \Rightarrow \quad \lambda^i_\ell \geq \lambda^i_\ell$$

(R'65) Rosen, "Existence and uniqueness of equilibrium points for concave n-person games," Econometrica, 33:520-34

(LB'87) Li and Başar, "Distributed algorithms for the computation of noncooperative equilibria," Automatica, 23(4):523-33

(ORS'93) Orda, Rom, Shimkin, "Competitive routing in multiuser communication networks," IEEE/ACM Trans Networking, 1:510-21

General network routing game with polynomial link costs, has a unique NE (ABJS'01)

$$J^i = \sum_{\ell \in L} J^i_\ell(\lambda^i_\ell, \lambda_{\ell t}) = \lambda^i_\ell \, f_\ell(\lambda_{\ell t})$$

$$f_\ell(\lambda) = a_\ell \lambda^{p(\ell)} + b_\ell \,, \;\; 0 < p(\ell) < p^*$$

$$p^* = (3N-1)/(N-1)$$

Proof uses Rosen's diagonal strict convexity:

$$\sum_{\ell \in L} \big(\lambda_\ell - \tilde{\lambda}_\ell\big)^T \big(g_\ell(\lambda_\ell) - g_\ell(\tilde{\lambda}_\ell)\big) > 0$$

$$g_\ell(\lambda_\ell) := [\, \nabla_1 J^1_\ell(\lambda^1_\ell, \lambda_{\ell t}), \ldots, \nabla_N J^1_\ell(\lambda^N_\ell, \lambda_{\ell t}) \,]^T$$

Jacobian of g_ℓ →

$$G_\ell(\lambda_l) = a_\ell \, p(\ell) \, \big(\lambda_{\ell t}\big)^{p(\ell)-2} \big(q_\ell 1^T + \lambda_{\ell t} I\big)$$

$$q^i_\ell = \lambda_{\ell t} + \big(p(\ell) - 1\big) \, \lambda^i_\ell \,, \qquad i\text{'th entry of } q_\ell$$

Can show that $G_\ell(\lambda_\ell) + G_\ell(\lambda_\ell)^T > 0 \;\; \forall \; \lambda^i_\ell \geq 0$

⇒ sufficient condition

(ABJS'01) Altman, Başar, Jiménez, Shimkin, "Competitive routing in networks with polynomial costs," IEEE TAC, January 2002.

Efficiency: **General network routing game with homogenous polynomial link costs ($b_\ell = 0$) has an efficient NE** (ABJS'01)

It minimizes $J(\lambda) = \sum_\ell \lambda_{\ell t}\, f_\ell(\lambda_{\ell t}) \quad \rightarrow \quad \lambda^*$

NE solution: $\lambda^i_\ell = \alpha_i\, \lambda^*_{\ell t}\,, \quad \alpha_i := \Lambda^i / \sum_j \Lambda^j$

$$J^i_{NE} = \alpha_i\, J(\lambda^*)$$

Computation: **For the same routing game, with M parallel links, and with affine link cost, $f_\ell(\lambda_{\ell t}) = a_\ell \lambda_{\ell t} + b_\ell$, the unique NE is :**

$$\lambda^i_\ell = \frac{(1/a_\ell)}{\sum_j (1/a_j)} \left(\Lambda^i + \frac{1}{N+1} \sum_{j=1}^{M} \frac{b_j - b_\ell}{a_j} \right)$$

provided that each user has positive flow on each link, i.e.

$$\min_i R(i) > \left[\max_\ell b_\ell\right]/(N+1)$$

$$R(i) := \frac{1}{\sum_j (1/a_j)} \left(\Lambda^i + \frac{1}{N+1} \sum_{j=1}^{M} \frac{b_j}{a_j} \right)$$

Flow Control and Pricing (BS'02a, BS'02b)

G **groups of users, with user** i **of group** g **having utility** $F_{gi}(x_{gi}; \bar{x}_j, j \in \mathcal{G}_l, l \in \mathcal{L}_g)$**:**

$$F_{gi} = w_{gi}\,\log(1 + x_{gi}) - \sum_{l \in \mathcal{L}_g} \frac{1}{N_l c_l - \sum_{j \in \mathcal{G}_l} \bar{x}_j} \\ - p\, x_{gi}\ \mathrm{card}\mathcal{L}_g$$

$\mathcal{L}_g$**: subsets of links used by group** g

$\mathcal{G}_l$**: subset of groups using link** l

$\mathcal{N}_l = \sum_{j \in \mathcal{G}_l} n_j$**;** n_j**: total # users in group** j

c_l**: unit capacity of link** l

$\bar{x}_g = \sum_{k=1}^{n_g} x_{gk}$**: total flow of group** g

$$L = p \sum_{g=1}^{G} \sum_{k=1}^{n_g} x_{gk}\, \mathrm{card}\mathcal{L}_g \quad \leftarrow \textbf{revenue of S.P.}$$

Given p**, obtain Nash eqm of the game** $\{F_{gi}\}$

$$\rightarrow x_{gi}(p),\ i = 1, \ldots, n_g;\ g = 1, \ldots, G \quad \mathbf{x}^*(p)$$

Then, $\max_{p \geq 0} L(p; \mathbf{x}^*(p)) \quad \rightarrow \quad p^*$

(BS'02a) Başar, Srikant, "Revenue-maximizing pricing and capacity expansion in a many-users regime," INFOCOM, June 2002.

(BS'02b) Başar, Srikant, "A Stackelberg network game with a large number of followers," JOTA, September 2002.

Special case: Single group, single link

Followers/Users:

$$F_i(x_i, \mathbf{x}_{-i}; p) = w_i \log(1 + x_i) - \frac{1}{nc - \bar{\mathbf{x}}} - px_i$$

$$i = 1, \ldots, n, \quad \mathbf{x}_{-i} := \sum_{j \neq i} x_j, \quad \bar{\mathbf{x}} = x_i + \mathbf{x}_{-i}$$

Leader/Network: $L = p \sum_{j=1}^{n} x_j \quad \rightarrow \quad$ **max**

NE: An n-tuple $\{\mathbf{x}_i^*(p) \geq 0\}_{i=1}^{n}$**:**

$$\max_{0 \leq x_i < nc - \mathbf{x}_{-i}^*} F_i(x_i, \mathbf{x}_{-i}^*(p); p) = F_i(x_i^*, \mathbf{x}_{-i}^*; p)$$

$$\max_{p \geq 0} p\bar{\mathbf{x}}^*(p) \quad \rightarrow \quad p^*$$

Add $\sum_{j \neq i} \left(w_j \log(1 + x_j) - px_j \right)$ **to** F_i

$$\rightarrow F(x_1, \ldots, x_n; p) = \sum_j w_j \log(1 + x_j) - \frac{1}{nc - \bar{\mathbf{x}}} - p\,\bar{\mathbf{x}}$$

NE for $\{F_i\}$ $\iff$ **PBP optimal solution to** F**.**

F **is strictly concave in** $\{x_i \geq 0,\ \bar{\mathbf{x}} \leq nc\}$

$F \downarrow -\infty$ **as** $\bar{\mathbf{x}} \uparrow nc$

Unique maximum, also unique PBP opt sol

Unique maximum, also unique PBP optimal solution

$$F_i(x_i, \mathbf{x}_{-i}; p) = w_i \log(1 + x_i) - \frac{1}{nc - \bar{\mathbf{x}}} - px_i$$

$$\frac{w_i}{1 + x_i} - \frac{1}{(nc - \bar{\mathbf{x}})^2} = p \quad x_i > 0 \quad \textbf{positive NE}$$

$$< p \quad x_i = 0 \quad \textbf{not admitted}$$

$$x_i, x_j > 0 \ \Rightarrow \ \frac{w_i}{1 + x_i} = \frac{w_j}{1 + x_j} \equiv \frac{w_j}{y_j}$$

If all flows are positive: $y_i = \frac{w_i}{\bar{\mathbf{w}}} \bar{\mathbf{y}}$

$$g(\bar{\mathbf{y}}) = \frac{\bar{\mathbf{w}}}{\bar{\mathbf{y}}} - \frac{1}{(nc + n - \bar{\mathbf{y}})^2} - p = 0$$

Unique $\bar{\mathbf{y}}^* \in (n, nc + n)$ **iff** $p < w_{av} - \frac{1}{(nc)^2} =: \hat{p}$

NE is inner iff $\frac{w_i}{\bar{\mathbf{w}}} \bar{\mathbf{y}}^*(p) > 1$

Network's problem: $\max\limits_{0<p<\hat{p}} p\big(\bar{\mathbf{y}}^*(p) - n\big)$

$$\iff \quad \max_{n<\bar{\mathbf{y}}<(c+1)n} \left[\bar{\mathbf{w}}\Big(1 - \frac{n}{\bar{\mathbf{y}}}\Big) - \frac{\bar{\mathbf{y}} - n}{\big(n(c+1) - \bar{\mathbf{y}}\big)^2}\right]$$

$n^2c^2w_{av} > 1$ $\Rightarrow$

$(w_i/\bar{\mathbf{w}})\bar{\mathbf{y}} > 1$ unique sol $\frac{n\bar{\mathbf{w}}}{\bar{\mathbf{y}}^2} - \frac{n(c-1) + \bar{\mathbf{y}}}{\big(n(c+1) - \bar{\mathbf{y}}\big)^3} = 0$

Special case: $c = 1$

$$\bar{\mathbf{y}}^* = 2n(n^2 w_{av})^{1/3} / \left[1 + (n^2 w_{av})^{1/3}\right]$$

condition: $\left(2\,\frac{w_i}{w_{av}} - 1\right)\left(n^2 w_{av}\right)^{1/3} > 1$ **(iff)**

holds if n is large

$$p^* = \frac{w_{av}}{2}\left(1 + (n^2 w_{av})^{-\frac{1}{3}}\right) - \frac{1}{4n^2}\left(1 + (n^2 w_{av})^{\frac{1}{3}}\right)^2 < \hat{p}$$

$$x_i^* = \frac{w_i}{w_{av}}\left(x_{av}^* + 1\right) - 1\,; \qquad x_{av}^* = 1 - \frac{2}{1 + (n^2 w_{av})^{1/3}}$$

Asymptotically, for large n $\left(w_i > (1/2)\, w_{av}\right)$ **:**

$$p^* \sim \frac{w_{av}}{2} + \frac{1}{4}(w_{av})^{2/3} \cdot \mathbf{n}^{-2/3} \qquad \downarrow \mathbf{n}$$

$$x_{av}^* \sim 1 - 2(w_{av})^{-1/3} \cdot \mathbf{n}^{-2/3} \qquad \uparrow \mathbf{n}$$

Revenue / bw $\sim \frac{w_{av}}{2} - \frac{3}{4}(w_{av})^{2/3} \cdot \mathbf{n}^{-2/3} \qquad \uparrow \mathbf{n}$

$$\mathbf{D}^* = \frac{1}{n(1 - x_{av}^*)} \sim \frac{1}{2}(w_{av})^{1/3} \cdot \mathbf{n}^{-1/3} \qquad \downarrow \mathbf{n}$$

$$\mathbf{F}_\mathbf{i}^* = w_i \log \frac{2w_i}{w_{av}} + \frac{1}{2} w_{av} - w_i - \frac{1}{2}(w_{av})^{1/3} \cdot \mathbf{n}^{-1/3} \qquad \uparrow \mathbf{n}$$

Dropping/Admission Scheme:

Critical condition: $\left(2\,\frac{w_i}{w_{av}} - 1\right)\left(n^2 w_{av}\right)^{1/3} > 1$

`SATISFIED`

- if n is sufficiently large, provided $2w_i > w_{av}$
- or, if $n = 1$, need $w_1 > 1$

For the general case: $w_1 \geq w_2 \geq \ldots \geq w_n\,;\; w_1 > 1$

Let $\mathbf{n}^*$ be the largest integer $\tilde{n}$ satisfying

$$\left(2\,\frac{w_{\tilde{n}}}{\tilde{w}_{av}} - 1\right)\left(\tilde{n}^2 \tilde{w}_{av}\right)^{1/3} > 1$$

Unique Solution: exists if $w_1 > 1$

$x_i = x_i^*\,,\; i \leq n^*$ as before with $\mathbf{n} = \mathbf{n}^*$

$x_j = 0\,,\; j > n^*$

Example: $w_1 = 3,\ w_i = 1\ \forall\, i \geq 2$

$$n^* = 1 \quad \text{if} \;\; n \leq 4; \qquad n^* = n \quad \text{for} \;\; n \geq 5\,.$$

$$n = 1 \Rightarrow x^*_{av} = 0.1811,\ p^* = 1.0489,\ p^* x^*_{av} = 0.1899,$$

$$F^*_1 = -1.00428$$

$$n = 5 \Rightarrow x^*_{av} = 0.5317,\ p^* = 0.7316,\ p^* x^*_{av} = 0.3890,$$

$$x^*_1 = 2.2823,\ x^*_j = 0.0941, j \geq 2,$$

$$F^*_1 = -0.548245,\ F^*_j = -0.456863, j \geq 2$$

$$n = 10 \Rightarrow x^*_{av} = 0.6629,\ p^* = 0.6337,$$

$$p^* x^*_{av} = 0.4200,\ x^*_1 = 3.15722,\ x^*_j = 0.3857, j \geq 2,$$

$$F^*_1 = -0.440802,\ F^*_j = -0.399363, j \geq 2$$

$$n \to \infty \Rightarrow x^*_{av} = 1,\ p^* = 0.5,\ p^* x^*_{av} = 0.5,$$

$$x^*_1 = 5,\ x^*_j = 1, j \geq 2,$$

$$F^*_1 = -0.165546,\ F^*_j = -0.19897, j \geq 2\,.$$

General c case:

$$\frac{n\bar{\mathbf{w}}}{\bar{\mathbf{y}}^2} - \frac{n(c-1)+\bar{\mathbf{y}}}{\left(n(c+1)-\bar{\mathbf{y}}\right)^3} = 0\,; \quad n^2c^2w_{av} > 1,\ \frac{w_i}{\bar{\mathbf{w}}}\bar{\mathbf{y}} > 1$$

- **w_i's ordered as before,** $c^2 w_1 > 1$**,**
- **n* smallest integer satisfying conditions**

⇒ **Unique $\bar{\mathbf{y}}^*$, p^*, $x_i^* = (w_i/\bar{\mathbf{w}}^*)\,\bar{\mathbf{y}}^* - 1\,,\ i \le \mathbf{n}^*$**

Dynamic distributed computation:

For a fix p, needed to solve: for $i = 1, \dots, n$

$$f_i(x_i, \bar{\mathbf{x}}; p) := w_i - p - px_i - (1+x_i)\frac{1}{(nc-\bar{\mathbf{x}})^2} = 0$$

Decentralized congestion control algorithm:

$$\dot{x}_i = \kappa_i\, f_i(x_i, \bar{\mathbf{x}}; p)\,, \quad i = 1, \dots, n$$

GAS in + orthant. Use as Lyapunov function:

$$F(x_1, \dots, x_n; p) = \sum_j w_j \log(1+x_j) - \frac{1}{nc-\bar{\mathbf{x}}} - p\,\bar{\mathbf{x}}$$

Details:

$$F(x_1, \ldots, x_n; p) = \sum_j w_j \log(1 + x_j) - \frac{1}{nc - \bar{\mathbf{x}}} - p\,\bar{\mathbf{x}}$$

$$\begin{aligned}
\dot{F} &= \sum_{j=1}^{n} \left[\frac{w_j}{1 + x_j} - p - \frac{1}{(nc - \bar{\mathbf{x}})^2}\right] \\
&\qquad \cdot \left[w_j - p - p x_j - (1 + x_j)\frac{1}{(nc - \bar{\mathbf{x}})^2}\right] \\
&= \sum_{j=1}^{n} \kappa(1 + x_j)\left[\frac{w_j}{1 + x_j} - p - \frac{1}{(nc - \bar{\mathbf{x}})^2}\right]^2 > 0
\end{aligned}$$

Many-Users Regime

Main equation:

$$\frac{w_{av}(n)}{(x_{av}(n)+1)^2} = \frac{c + x_{av}(n)}{n^2(c - x_{av}(n))^3}$$

Assume $\lim_{n\to\infty} w_{av}(n) = w_{av}$ and $\min_i w_i > \frac{w_{av}}{c+1}$

Let $\alpha := \frac{2c(c+1)^2}{w_{av}}$ Then, asymptotically,

$$p^*(n) \sim \frac{w_{av}}{c+1} + (2c-1)\,\alpha^{-2/3} \cdot \mathbf{n}^{-2/3} \qquad \downarrow\uparrow \mathbf{n}$$

$$x^*_{av}(n) \sim c - \alpha^{1/3} \cdot \mathbf{n}^{-2/3} \qquad \uparrow \mathbf{n}$$

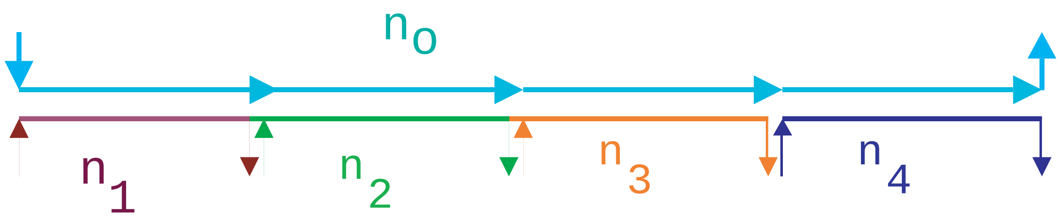
$$x^*_i(n) \sim (c+1)\frac{w_i}{w_{av}} - 1 - \frac{w_i}{w_{av}}\alpha^{1/3} \cdot \mathbf{n}^{-2/3} \qquad \uparrow \mathbf{n}$$

$$\text{Revenue/bw} = \frac{p(n)\,x_{av}(n)}{c} \sim \frac{w_{av}}{c+1} - 3\alpha^{-2/3} \cdot \mathbf{n}^{-2/3} \uparrow$$

$$\mathbf{D}^*(\mathbf{n}) = \frac{1}{n(c - x_{av}(n))} \sim \alpha^{-1/3} \cdot \mathbf{n}^{-1/3} \qquad \downarrow \mathbf{n}$$

$$\mathbf{F}^*_{\mathbf{i}}(\mathbf{n}) = w_i \log \frac{(c+1)w_i}{w_{av}} + \frac{w_{av}}{c+1} - w_i - \alpha^{-1/3}\mathbf{n}^{-1/3} \uparrow \mathbf{n}$$

Multiple Links



N **links;** n_k **users of class** $k, k = 0, \ldots, N$

Class 0 **users use all** N **links**

Class ℓ **user uses only link** $\ell, \ell = 1, \ldots, N$

$$F_{0i} = w_0 \log(1 + x_{0i}) - N p\, x_{0i} - \sum_{\ell=1}^{N} \frac{1}{c_\ell - \bar{x}_0 - \bar{x}_\ell}$$

$$F_{kj} = w_k \log(1 + x_{kj}) - p\, x_{kj} - \frac{1}{c_k - \bar{x}_0 - \bar{x}_k}$$

$$L(p; \bar{x}_k(p), 0 \le k \le N) = N p \bar{x}_0 + p \sum_{k=1}^{N} \bar{x}_k\,.$$

$\Rightarrow$ **Common objective function**

$$F(x_0, \ldots, x_N; p) = \sum_{k=0}^{N} w_k \sum_{j=1}^{n_k} \log(1 + x_{kj}) - N p \bar{x}_0$$

$$- p \sum_{\ell=1}^{N} \bar{x}_\ell - \sum_{\ell=1}^{N} \frac{1}{c_\ell - \bar{x}_0 - \bar{x}_\ell}$$

Unique NE for each p:

$$\frac{w_0}{1+x_{0i}} - \sum_{\ell=1}^{N} \frac{1}{(c_\ell - \bar{x}_0 - \bar{x}_\ell)^2} - Np = 0,$$
$$\frac{w_k}{1+x_{kj}} - \frac{1}{(c_k - \bar{x}_0 - \bar{x}_k)^2} - p = 0\,,\ k \geq 1$$

$\Rightarrow \quad x_{kj} = \bar{x}_k / n_k\,, \quad k \geq 0$

Assume: w_k **and** n_k **independent of** k **for** $k \geq 1$

`AND` $c_\ell = (n_0 + n_1)c =: nc \ \forall\ \ell$

$\Rightarrow$ **unique positive NE iff** $\exists\ \bar{y}$ **satisfying:**

$$g(\bar{y}) := \frac{\bar{w}}{\bar{y}} - \frac{N}{(nc + n - \bar{y})^2} - Np = 0$$

such that $\quad \min\left(w_0, Nw_1\right) \dfrac{\bar{y}(p)}{\bar{w}} > 1$

$$\bar{y}_0 := n_0 + \bar{x}_0\,, \quad \bar{y}_1 := n_1 + \bar{x}_1\,, \quad \bar{y} := \bar{y}_0 + \bar{y}_1$$

$$\bar{w} := n_0 w_0 + N n_1 w_1\,,\ w_{av} := (\bar{w}/n)$$

$$g(\bar{y}) = 0 \quad \Leftrightarrow \quad p < \frac{w_{av}}{N} - \frac{1}{(nc)^2} =: \hat{p}$$

Leader's problem

Maximize over $(n,(c+1)n)$**:**

$$\tilde{L}(\bar{y}) = \bar{w}\big(1-\frac{n}{\bar{y}}\big) - \frac{N(\bar{y}-n)}{(n(c+1)-\bar{y})^2}$$

Unique solution under $n^2c^2w_{av} > N$**:**

$$\frac{n\bar{w}}{\bar{y}^2} - \frac{N[n(c-1)+\bar{y}]}{(n(c+1)-\bar{y})^3} = 0$$

$$p^* = \frac{\bar{w}}{N\bar{y}^*} - \frac{1}{(nc+n-\bar{y}^*)^2}$$

also require: $\min\big(w_0, Nw_1\big)\,\bar{y}^* > \bar{w}$

Special case: $c = 1$

$$\bar{y}^* = \frac{2n(n\bar{w})^{\frac{1}{3}}}{N^{\frac{1}{3}} + (n\bar{w})^{\frac{1}{3}}} \quad \text{if} \quad n\bar{w} > N$$

Positivity : $\left(\frac{2\min(w_0, Nw_1)}{w_{av}} - 1\right)\left(n^2 w_{av}\right)^{\frac{1}{3}} > 1$

$$p^* = \frac{w_{av}}{2N}\big(1+N^{\frac{1}{3}}\left(n^2w_{av}\right)^{-\frac{1}{3}}\big) - \frac{1}{4n^2}\big(1+N^{\frac{1}{3}}\left(n^2w_{av}\right)^{\frac{1}{3}}\big)^2$$

Asymptotic Behavior

With $x_{av}(n) := \frac{1}{n}(\bar{x}_0 + \bar{x}_1)$, main equation:

$$\frac{w_{av}(n)}{N(x_{av}(n)+1)^2} = \frac{c + x_{av}(n)}{n^2(c - x_{av}(n))^3}$$

Assume $w_{av}(n) \to w_{av}$

$$p^* \sim \frac{w_{av}}{N(c+1)} + \frac{2c-1}{\alpha^{2/3} n^{2/3}}; \qquad \alpha := \frac{2c(c+1)^2 N}{w_{av}}$$

$$x_{av}(n) \sim c - \alpha^{1/3} n^{-2/3}$$

$$\text{Positivity}: \quad x_{av}(n) > \max\left(\frac{w_{av}}{w_0} - 1, \frac{w_{av}}{N w_1} - 1\right)$$

If $\alpha_0 := n_0/n < 1$, positivity $\Rightarrow$

$$\frac{\alpha_0}{c + \alpha_0} < \frac{N w_1}{w_0} < \frac{c}{1 - \alpha_0} + 1$$

$$\text{Revenue/bw/link} = \frac{p x_{av}}{c} \sim \frac{w_{av}}{(c+1)} - 3\alpha^{-\frac{2}{3}} n^{-\frac{2}{3}}$$

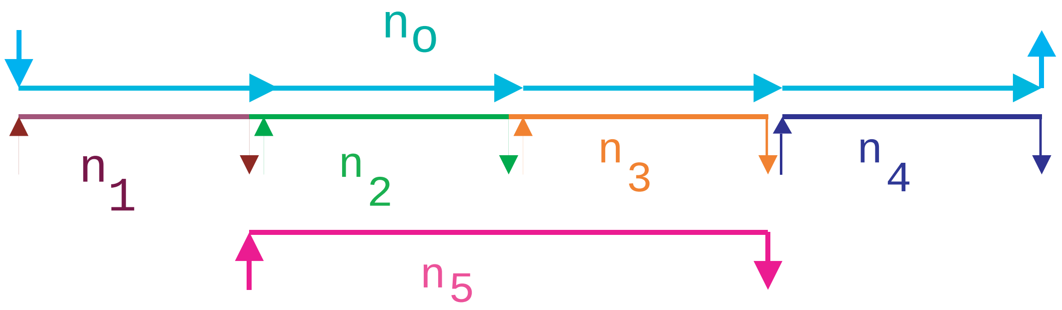
$$\text{Congestion cost} = \frac{1}{n(c - x_{av}(n))} \sim \alpha^{-\frac{1}{3}} n^{-\frac{1}{3}}$$

$$F_{0i}^* = w_0 \log \frac{N(c+1)w_0}{w_{av}} - N w_0 + \frac{w_{av}}{c+1} - N\alpha^{-\frac{1}{3}} n^{-\frac{1}{3}}$$

$$F_{kj}^* = w_1 \log \frac{N(c+1)w_1}{w_{av}} - w_1 + \frac{w_{av}}{c+1} - \alpha^{-\frac{1}{3}} n^{-\frac{1}{3}}$$

More General Topologies

For example:



Qualitatively similar results

Asymptotics similar

Details different

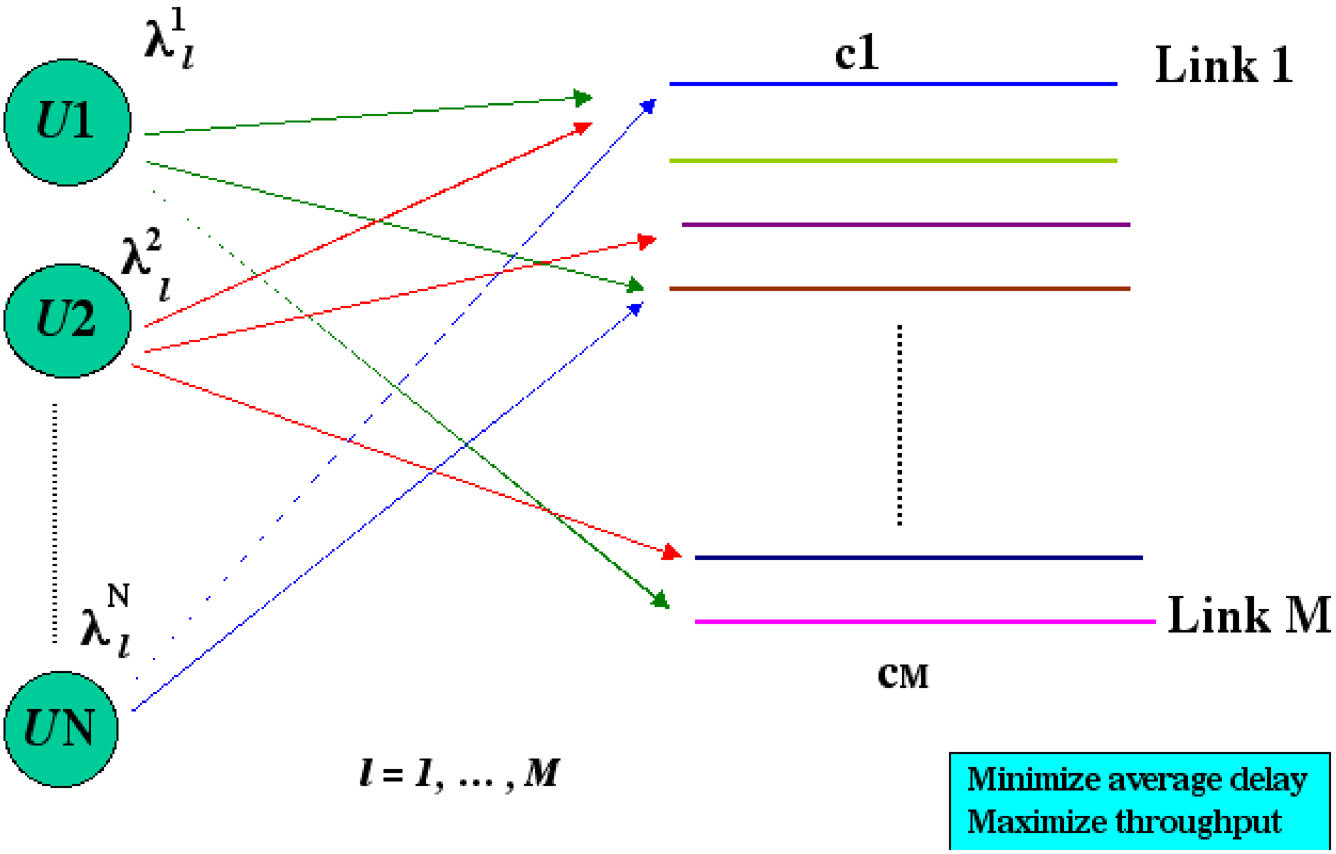
For each p, still unique NE

For network problem use Lagrange multipliers, and asymptotics for them

The relative rates at which n_i's grow determine admission, flow rates, and optimum price

Combined Routing and Flow Control

non-cooperative users



Minimize average delay
Maximize throughput

(ABS'02) Altman, Başar, Srikant, "Nash equilibria for combined flow control and routing in networks: Asymptotic behavior for a large number of users," IEEE TAC, May 2002

Combined Routing and Flow Control

M parallel links and N users

Total flow on link ℓ: $\lambda_\ell = \sum_i \lambda_\ell^i$

Total throughput of user i: $\lambda^i = \sum_\ell \lambda_\ell^i$

Average delay for user i:

$$d^i(\lambda) = \frac{1}{\lambda^i} \sum_\ell \frac{\lambda_\ell^i}{c_\ell - \lambda_\ell}$$

Utility function of user i: $\beta \in (0, 1)$

$$U^i(\lambda) = (\lambda^i)^\beta / d^i(\lambda) = (\lambda^i)^{\beta+1} / \sum_\ell \tfrac{\lambda_\ell^i}{c_\ell - \lambda_\ell}$$

to be maximized **(not concave !)**

Nash eqm: $U^i(\{\lambda_\ell^i\}, \lambda^{-i}) \geq U^i(\{\lambda_\ell^i\}, \lambda^{-i}) \ \forall \{\lambda_\ell^i\}_{\ell \in \mathcal{N}}$

- **no general theory (existence and uniqueness)**
- **Asymptotic ($O(1/N)$) NE (with exponent κ)**

$$L_i^N(\{\lambda_\ell^{i*}(N)\}_{\ell \in \mathcal{M}}, \{\lambda_\ell^{k*}(N)\}_{k \in \mathcal{N}, k \neq i, \ell \in \mathcal{M}})$$

$$= \max_{\{\lambda_\ell^i\}_{\ell \in \mathcal{M}}} L_i^N(\{\lambda_\ell^i\}_{\ell \in \mathcal{M}}, \{\lambda_\ell^{k*}(N)\}_{k \in \mathcal{N}, k \neq i, \ell \in \mathcal{M}})$$

$$+ \tfrac{\kappa}{N} + o(1/N) \quad \forall \{\lambda_\ell^i\}_{\ell \in \mathcal{M}}$$

Single User

Inner solution: $\lambda_\ell = c_\ell - \mu\sqrt{c_\ell} \; > \; 0$

$$(\beta+1)M\mu^2 - \bar{c}_{sq}(\beta+2)\mu + \bar{c} = 0$$

$$\bar{c} := \sum_{\ell=1}^{M} c_\ell \,, \qquad \bar{c}_{sq} := \sum_{\ell=1}^{M} \sqrt{c_\ell}$$

An inner solution may not always exist

Drop links

Can restrict w/o.l.o.g. to sets

$$\mathcal{S}_m = \{1, 2, \ldots, m\} \qquad (c_\ell \geq c_{\ell+1})$$

$$\bar{c}_m := \sum_{\ell=1}^{m} c_\ell \,, \qquad \bar{c}_{sq,m} := \sum_{\ell=1}^{m} \sqrt{c_\ell}$$

There exists an optimal solution:

$$\mathcal{M}_f := \left\{ m : \frac{\bar{c}_{sq,m}^2}{m\bar{c}_m} \geq \frac{4(\beta+1)}{(\beta+2)^2} \;\; \& \;\; \mu_m^- < \sqrt{c_m} \right\}$$

$$m^* = \arg \max_{m \in \mathcal{M}_f} \frac{(\bar{c}_m - \mu_m^- \bar{c}_{sq,m})^{\beta+1}}{\bar{c}_{sq,m} - m\mu_m^-} \mu_m^-$$

$$\lambda_\ell = \begin{cases} c_\ell - \mu_{m^*}^- \sqrt{c_\ell} & \ell \in \mathcal{S}_{m^*} \\ 0 & \ell \notin \mathcal{S}_{m^*} \end{cases}$$

Details of Derivation

Single user

$$L(\lambda) = (\beta+1)\log\Big(\sum_{\ell=1}^{M}\lambda_\ell\Big) - \log\sum_{\ell=1}^{M}\frac{\lambda_\ell}{c_\ell-\lambda_\ell}$$

$$\frac{\beta+1}{\displaystyle\sum_{\ell=1}^{M}\lambda_\ell} - \frac{c_m}{\displaystyle(c_m-\lambda_m)^2\sum_{\ell=1}^{M}\frac{\lambda_\ell}{c_\ell-\lambda_\ell}} = 0\,.$$

$$\mu := (c_m-\lambda_m)/\sqrt{c_m} \qquad \Rightarrow$$

$$\mu^{\pm} = \frac{(\beta+2)\bar{c}_{sq} \pm \sqrt{(\beta+2)^2\bar{c}_{sq}^2 - 4(\beta+1)\bar{c}M}}{2(\beta+1)M}\,.$$

Important property

$$U(\lambda^-) > U(\lambda^+)$$

(ABS'02) Altman, Başar, Srikant, "Nash equilibria for combined flow control and routing in networks: Asymptotic behavior for a large number of users," IEEE TAC, May 2002

Equal link capacities (c) **:** $\lambda_\ell = \frac{\beta}{\beta+1} c \; \forall \, \ell$ **unique**

- **Robustness: still an inner solution if**

$$c_l = c + \delta_\ell, \quad |\delta_\ell| < \delta \text{ for some } \delta > 0$$

- **Additional links of lower capacity:** $\exists \, \beta^* > 0$ **such that** $\forall \beta \in (0, \beta^*)$, **zero flow on new links.**

Example 1:

$$M = 10, \; c_\ell = 100 - 10(\ell - 1), \; \beta = 0.6$$

$$\mathcal{M}_f = \{1, \ldots, 7\} \quad \Rightarrow \quad m^* = 7$$

$\lambda_1 = 44.9, \; \lambda_2 = 37.73, \; \lambda_3 = 30.72, \; \lambda_4 = 23.9,$

$\lambda_5 = 17.32, \; \lambda_6 = 11.04, \; \lambda_7 = 5.15,$ **the rest zero**

Example 2:

$$M \geq 5\,; \; c_1 = 2.29, c_2 = \cdots = c_4 = 1,$$

$$c_\ell = 0.25, \ell \geq 5\,; \quad \beta = 0.5$$

$$\mathcal{M}_f = \{1, 4\} \quad \Rightarrow \quad m^* = 4$$

$$\lambda_1 = 0.9113, \; \lambda_2 = \lambda_3 = \lambda_4 = 0.00889.$$

Two other types of utility functions

Link-additive

$$\bar{U}(\lambda) = \sum_{\ell=1}^{M} \lambda_\ell^\beta (c_\ell - \lambda_\ell)\,, \qquad \beta \in (0,1)$$

Unique optimal solution: $\lambda_\ell = \frac{\beta}{\beta+1}\, c_\ell$

Flow-additive

$$\bar{V}(\lambda) = \Big(\sum_{\ell=1}^{M} \lambda_\ell\Big)^\beta \Big(\bar{c} - \sum_{\ell=1}^{M} \lambda_\ell\Big)\,, \qquad \beta \in (0,1)$$

Solution: $\sum_{\ell=1}^{M} \lambda_\ell = \frac{\beta}{\beta+1}\, \bar{c}$

Back to the multiuser case

Scaling of β: $N\beta \to \alpha$ **a constant**

This scaling ensures finiteness of delay

e.g. $M = 1 \quad \Rightarrow \quad \lambda_1^i(N) = \dfrac{\beta}{N\beta + 1} c_1$

Conditions for symmetric NE (across users)

$\lambda_\ell^i = \lambda_\ell / N$ **in** $O(1/N)$ **NE** **if**

$\lambda_\ell = c_\ell - \frac{1}{M}(\bar{c} - \bar{\lambda}) \geq 0\,, \;\; \ell = 1, \ldots, M$

$(\alpha + 1)\bar{\lambda}^2 - \alpha \bar{c} \bar{\lambda} + M\overline{c^2} - \bar{c}^2 = 0$

$0 \;\leq\; \bar{\lambda} \;\leq\; \bar{c}$

exponent:

$$\Rightarrow \qquad \kappa = \alpha \log \frac{2\bar{\lambda}}{\sqrt{\gamma \overline{Q}} + \alpha M \gamma} < 0\,,$$

$$\overline{Q} := \frac{\bar{\lambda}^2 - M\overline{\lambda^2}}{\gamma} + \alpha^2 M^2 \gamma > 0\,,$$

$$\gamma := \overline{\lambda^2} / (\alpha \bar{\lambda})\,.$$

- **Uses all links**
- **Delay equalized on all links**

An inner $O(1/N)$ NE solution may not exist

Drop links: $\mathcal{S}_m = \{1, \ldots, m\}$

Let $\{\lambda^i_\ell(N) = \lambda^{(m)}_\ell / N\}$ be an $O(1/N)$ NE with exponent $\kappa^{(m)}$ for the network game on $\mathcal{S}_m$.

$$\lambda^i_\ell(N) = \begin{cases} \lambda^{(m)}_\ell / N, & \ell \le m \\ 0 & \ell > m \end{cases}, \quad \ell \in \mathcal{M},\ i \in \mathcal{N},$$

provides an $O(1/N)$ NE for the M-link game iff

$$c_{m+1} < c_1 - \lambda^{(m)}_1 .$$

The exponent is still $\kappa^{(m)}$.

Still delay-equalizing.

- **There could be multiple $O(1/N)$ SNE**
  - **but can be strictly ordered**
- **There could be no $O(1/N)$ SNE !**
- **Nonsymmetric NE ?**

Details of Derivation

Multiple users

$$L_i^N(\lambda) = (\beta_N + 1)\log \sum_{\ell\in\mathcal{M}} \lambda_\ell^i - \log \sum_{\ell\in\mathcal{M}} \frac{\lambda_\ell^i}{c_\ell - \lambda_\ell}$$

$$\frac{\beta_N + 1}{\sum_{\ell\in\mathcal{M}} \lambda_\ell^i(N)} - \frac{c_k - \lambda_k(N) + \lambda_k^i(N)}{(c_k - \lambda_k(N))^2 \sum_{\ell\in\mathcal{M}} \frac{\lambda_\ell^i(N)}{c_\ell - \lambda_\ell(N)}} = 0,$$

$i \in \mathcal{N},\ k \in \mathcal{M}$ **As $N \to \infty$,**

$$c_j - \lambda_j(N) = c_1 - \lambda_1(N) + \frac{\mu_j}{N}, \qquad j > 1 \quad \Rightarrow$$

$$\alpha = \frac{1}{c_k - \lambda_k}\Big[\lambda_k + \frac{1}{\bar\lambda}\sum_{j\in\mathcal{M}} \lambda_j(\mu_j - \mu_k)\Big].$$

$$(\alpha + 1)\bar\lambda^2 - \alpha\bar c\bar\lambda + M\overline{c^2} - \bar c^2 = 0\,, \Rightarrow$$

$$\bar\lambda^{\pm} = \frac{\alpha \pm \sqrt{(\alpha+2)^2 - 4(\alpha+1)M\nu}}{2(\alpha+1)}\,\bar c$$

where $\nu = \overline{c^2}/\bar c^2$.

At NE

Average delay per user: $d^i(\lambda) = \dfrac{1}{c_j - \lambda_j}$

Throughput per user: $T^i(\lambda) = \dfrac{1}{N}\sum_j \lambda_j$

Utility per user:

$$U^i(\lambda) = \frac{\left(T^i\right)^\beta}{d^i(\lambda)} = (c_\ell - \lambda_\ell)\Big(\frac{1}{N}\sum_j \lambda_j\Big)^{\frac{\alpha}{N}} \to c_\ell - \lambda_\ell$$

Cooperative solution

$U^i(\lambda) \to$ **max** **by all users (symmetric)**

equiv single user problem

$$U^i(\lambda) = \frac{\left(\sum_j \lambda_j\right)^{\beta+1}}{\sum_j \lambda_j/(c_j - \lambda_j)}\Big(\frac{1}{N}\Big)^\beta$$

Maximization → **use only link 1, for N large**

$$\mu^- = \frac{1}{1+\beta}\sqrt{c_1}\,, \quad \lambda_1 = \frac{\beta}{1+\beta}\,c_1$$

$$\max U^i(\lambda) = (\beta/N)^\beta (c_1)^{\beta+1} \to c_1$$

An earlier work on Nash equilibrium →

Wardrop equilibrium (W'52) is (HM'85)

Each player's cost (convex)=

transportation cost - total value of sales on his market

(W'52) Wardrop, "Some theoretical aspects of road traffic research," Proc. Inst. Civil Engi. II 1:325-378

(HM'85) Haurie and Marcotte, "On the relationship between Nash-Cournot and Wardrop equilibria," Networks, 15:295-308

Example 3: $M = 10,\ c_m = 100 - 10(m-1),$

$\alpha = 0.6.$ **only one $O(1/N)$ SNE**

$\lambda_1 = 55.49,\ \lambda_2 = 45.49,\ \lambda_3 = 35.49,\ \lambda_4 = 25.49,$

$\lambda_5 = 15.49,\ \lambda_6 = 5.49,\ \lambda_i = 0,\ i > 6$

$\bar{\lambda}^+ = 182.95\,, \quad \kappa = -0.0875\,, \quad U^i_{NE} = 44.51$

Delay equalization over active links

Example 4: $M = 10,\ c_1 = 100, c_m = 50, m \geq 2,$

$\alpha = 0.9$ **2 $O(1/N)$ SNE using all links**

- $\bar{\lambda}^+ = 201.9 \quad \Rightarrow \quad \kappa^+ = -0.1472$

  $\lambda_1 = 65.19, \quad \lambda_\ell = 15.19, \quad \ell \geq 2$

- $\bar{\lambda}^- = 58.7 \quad \Rightarrow \quad \kappa^- = -0.7773$

  $\lambda_1 = 50.87, \quad \lambda_\ell = 0.87, \quad \ell \geq 2$

Both delay equalizing over active links

One $O(1/N)$ SNE that uses only one link

- $\lambda_1 = 47.37 \quad \kappa = -0.6904$ **The best one!**

$U^i_{NE} = 52.63$

No other $O(1/N)$ SNE

RELATED ISSUES

- **Dynamic adjustment process toward NE**
- **Pricing mechanisms to achieve efficiency**
  – additional ‘cost for throughput’ term in U^i’s –
  — Stackelberg-Nash game —
- **User-dependent β’s**
- **Small number of users**
- **Hierarchy in design**
- **Building in Priorities**
- **Restrictions on routes and flows**
- **Combined routing and flow control on general topology networks**

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